A Theory of Vortex Merger

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This paper discusses a 2D vortex merger for the particular case of a weak point vortex and an extended vortex of nearly circular cross section. A Hamiltonian analysis describes the interaction of the point vortex with surface waves on the extended vortex. Critical layers around the extended vortex are introduced where "point vortex–surface wave" resonances occur. Vortex merger occurs only when there is an overlap of neighboring critical layers. In this case the point vortex cascades through a sequence of resonances as it spirals in and merges with the extended vortex. [S0031-9007(97)03812-X]

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Two-dimensional (2D) vortex dynamics in an ideal fluid has received extensive theoretical and experimental attention (see, e.g., survey [1] and cited literature). The merger of like-signed vortices is particularly important, since it is a crucial element in the decay of turbulence in 2D flows [2]. Most previous work [3–9] has considered the merger of two vortices of comparable size, where the merger process involves the shearing apart of each vortex in the velocity field of the other. The gross distortions of the vortex boundaries complicate the problem, and no simple analytical treatment has been possible. In this Letter we focus on a limit where the merger takes place without such distortions. In particular, we consider the limit where one of the two vortices may be approximated by a weak point vortex during the merger process. This simplifies the analysis and provides useful insight into the dynamics of vortex merger. The applicability limits for the "weak point vortex" approximation will be defined below.

The 2D motion of an incompressible ideal fluid is governed by the set of equations [10]

$$\nabla \cdot \boldsymbol{v} = 0, \quad \zeta = \hat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{v}, \quad (\partial_t + \boldsymbol{v} \cdot \nabla)\zeta = 0, \tag{1}$$

where $\boldsymbol{v}$ is the fluid velocity, ζ is the vorticity, and $\hat{\boldsymbol{z}}$ is the axis normal to the plane of motion. Equations (1) are equivalent to the drift-Poisson equations for the 2D $\boldsymbol{E} \times \boldsymbol{B}$ drift evolution of a non-neutral plasma column in a uniform magnetic field [11]. The problem of vortex merger can be described in the language of fluid dynamics or plasma physics [12], but we follow the more traditional language of fluid dynamics.

Three approximations define our model. (i) The total circulation (vorticity integrated over the area) Γ_* of the point vortex is assumed to be small compared to the total circulation $\Gamma_{\rm ext}$ of the extended vortex,

$$\Gamma_* \ll \Gamma_{\rm ext}\,. \tag{2}$$

(ii) The extended vortex is treated in the "vortex patch" approximation, where vorticity is constant within a boundary curve and zero outside. (iii) The smaller vortex, which is approximated as a point vortex, must be intense enough so as not to be sheared apart during the merger process.

To see that approximations (i) and (iii) can be satisfied simultaneously, let us consider two uniform density, circular vortices that have vorticities ζ_* and $\zeta_{\rm ext}$ and radii r_* and $R_{\rm ext}$, where $r_* \ll R_{\rm ext}$. As before, the subscripts $(*)$ and (ext) refer to the small ("point") and extended vortices, respectively. Approximation (i) [i.e., inequality (2)] may be written as

$$\zeta_* r_*^2 \ll \zeta_{\rm ext} R_{\rm ext}^2\,. \tag{3}$$

To put condition (iii) in quantitative form, we first note that the self-velocity field of the point vortex near its location is of the order of $\zeta_* r_*$. The shear in the velocity field of the extended vortex produces a velocity difference across the small vortex that is of the order

$$\frac{\zeta_{\rm ext} R_{\rm ext}^2}{d}\,\frac{r_*}{d}, \tag{4}$$

where d is the separation of the vortex centers. For the small vortex to remain intact (nearly circular), it is necessary that the self-velocity field dominate, so we obtain the condition

$$\zeta_* r_* \gg \frac{\zeta_{\rm ext} R_{\rm ext}^2}{d}\,\frac{r_*}{d}\,. \tag{5}$$

Combining inequalities (3) and (5) and using $d \sim R_{\rm ext}$ yields the desired applicability conditions,

$$\zeta_{\rm ext} \ll \zeta_* \ll \zeta_{\rm ext}\left(\frac{R_{\rm ext}}{r_*}\right)^2. \tag{6}$$

In contrast, most previous work considered the case where $\zeta_{\rm ext} \sim \zeta_*$.

In the limit (2), the point vortex orbits around the extended vortex, whose displacement as a whole is negligible. While orbiting, the point vortex excites Kelvin waves on the surface of the extended vortex (diocotron modes in the language of non-neutral plasmas), and these waves, in turn, influence the dynamics of the point vortex. The energy of the interaction between like-signed vortices is positive and increases as the separation decreases (by analogy with the interaction energy of two like-signed charges), while the energy of the surface waves can be negative [11]. The difference in sign of these energies makes possible an instability in which the point vortex moves radially inward while a surface wave grows, with the total energy conserved. In what follows we will see that such an instability plays a crucial role in the vortex merger.

Since the point vortex is assumed to be weak (2), the amplitudes of the excited Kelvin waves are small. Consequently, the surface of the extended vortex stays nearly circular during the vortex interaction, at least until the point vortex comes very close to the surface of the extended vortex, $\delta r \sim R_{\rm ext}(\Gamma/\Gamma_{\rm ext})$.

The most efficient excitation of a surface wave takes place when the orbital frequency $\omega_{\rm rot}$ of the point vortex around the extended vortex satisfies the resonance condition,

$$\omega_m - m\omega_{\rm rot}(r) = 0\,, \tag{7}$$

where ω_m is the frequency of the surface wave and m is the azimuthal wave number. The frequency $\omega_{\rm rot}$ is a function of the radial position r of the point vortex, where the center of the coordinate frame is the unperturbed center of the extended vortex. It is convenient to introduce units where distance is normalized to the radius $R_{\rm ext}$ and time to the inverse of the self-rotation frequency $\Gamma_{\rm ext}/2\pi R_{\rm ext}^2$ of the extended vortex. In these units,

$$\omega_{\rm rot} \simeq \frac{1}{r^2}\,, \tag{8}$$

and ω_m is given by [13]

$$\omega_m = m - 1\,. \tag{9}$$

Equations (7)–(9) determine a set of critical radial positions r_m,

$$r_m = \left(\frac{m}{m-1}\right)^{1/2}, \tag{10}$$

near which the point vortex resonantly excites a surface wave.

We start by considering a simplified model that describes the interaction of the point vortex with linear surface waves. The dynamics for this model is governed by the Hamiltonian,

$$H = -\gamma \ln p - \sum_{m=1}^{\infty} \omega_m p_m + 2\gamma \sum_{m=1}^{\infty} \frac{\gamma^{m/4}}{\sqrt{m}} \sqrt{\frac{p_m}{p^m}} \cos\left[\frac{m}{\sqrt{\gamma}}\, q - q_m\right]. \tag{11}$$

Here, $\gamma \ll 1$ is the total circulation of the point vortex normalized to the total circulation of the extended vortex; the conjugate coordinate and momentum of the point vortex are defined as

$$q = -\sqrt{\gamma}\,\theta\,, \qquad p = \sqrt{\gamma}\, r^2, \tag{12}$$

where (r,θ) are the polar coordinates of the point vortex position; and the conjugate coordinate and momentum of the mth surface wave are defined as

$$q_m = -\psi_m\,, \qquad p_m = \frac{4}{m}\,\zeta_m^2\,, \tag{13}$$

where $\zeta_m \exp(-im\psi_m)$ is the complex amplitude of this wave. In the vortex patch approximation, the linearly perturbed vorticity distribution for the extended vortex can be written as a superposition of surface waves

$$\delta\zeta(r,\theta) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \delta(r-1)\zeta_m \exp(im\theta - i\psi_m)\,, \tag{14}$$

where it is necessary that $\zeta_{-m} = \zeta_m, \psi_{-m} = -\psi_m$ for $\delta\zeta$ to be real, and the $m = 0$ term is excluded because the flow is incompressible. Note that, for incompressible flow acting on a uniform vorticity patch, a perturbation $\delta\zeta(r,\theta)$ can develop only at the surface. The surface waves form a complete set.

The first term in Eq. (11) is the energy of the point vortex in the logarithmic potential of the extended vortex, the second is the energy of the surface waves, and the third is the interaction energy of the point vortex and the surface waves. One can check that Hamilton's equations of motion,

$$\dot{q} = \frac{\partial H}{\partial p}\,, \qquad \dot{p} = -\frac{\partial H}{\partial q}\,, \tag{15}$$

$$\dot{q}_m = \frac{\partial H}{\partial p_m}\,, \qquad \dot{p}_m = -\frac{\partial H}{\partial q_m}\,, \tag{16}$$

are the same as the equations of motion that follow from Eqs. (1) in the limit where the extended vortex is treated in the vortex patch approximation, and the surface waves are of small enough amplitude that nonlinear wave-wave coupling is negligible. Of course, the self-field of the point vortex does not contribute to the center of mass motion of the point vortex, that is, Eqs. (15) describe the motion of the point vortex in the field of the extended vortex. At the end of this paper, the results of this simple Hamiltonian model will be compared to a numerical integration of Eqs. (1) using contour dynamics, and we will see that the model captures the essential physics.

Since Hamiltonian (11) is invariant under rotation (i.e., $\theta \to \theta + \Delta\theta, \psi_m \to \psi_m + m\Delta\theta$), the total angular momentum L is conserved:

$$L = \sqrt{\gamma}\, p + \sum_{m=1}^{\infty} m p_m = \text{const}\,. \tag{17}$$

One can easily check that $[L,H] = 0$. It is useful to note that Eqs. (12), (13), and (17) impose an upper bound on the separation distance between the point and extended vortices for a given value of L.

In the vicinity of the critical radius r_m defined by (10), the interaction between the point vortex and the mth surface wave dominates. Retaining only this wave in the Hamiltonian yields a system with two degrees of freedom (the point vortex and the resonant wave). Since there are two degrees of freedom and two constants of motion, H and L, Hamilton's equations are integrable in this approximation.

Suppose that at the initial moment the point vortex is located at a radius r_0 close to r_m,

$$\frac{r_0 - r_m}{r_m} = \epsilon \ll 1\,, \tag{18}$$

and that the resonant surface wave has zero amplitude, $\zeta_m(t = 0) = 0$. Constraint (17) then yields the result,

$$L = \gamma r_0^2\,, \qquad 4\zeta_m^2 = \gamma(r_0^2 - r^2)\,. \tag{19}$$

By using Eq. (19) and expanding Hamiltonian (11) in $(r_0 - r)$, one obtains

$$h = \tilde{r}^2 - \alpha\tilde{r} + \sqrt{\tilde{r}}\,\cos(\varphi)\,, \tag{20}$$

where $\varphi = m\theta - \psi_m$, the constant h marks the value of the energy, the constant α is related to ϵ as

$$\alpha = \epsilon\,\frac{2^{2/3}m^{2/3}}{\gamma^{1/3}}\left(\frac{m}{m-1}\right)^{(m-1)/3}, \tag{21}$$

and $\tilde{r}$ measures the radial displacement of the point vortex,

$$r_0 - r = \tilde{r}\,\delta r_m\,, \qquad \delta r_m = \gamma^{1/3}\,\frac{2^{1/3}}{m^{2/3}}\left(\frac{m-1}{m}\right)^{(2m-5)/6}. \tag{22}$$

Note that the constant α determines the value of L through Eqs.(18) and (19).

The phase-space trajectories determined by Eq. (20) describe the point vortex motion, which involves rotation around the extended vortex together with finite amplitude oscillations in the radial direction. A typical phase space portrait is shown in Fig. 1. The radial motion of the point vortex is trapped inside a "critical layer" of characteristic width δr_m [see Eq. (22)].

The picture of trapped motion inside the critical layer is valid when there is a single resonant wave that interacts with the point vortex. The latter assumption is justified in the limit where critical layers that correspond to different m numbers are well separated in space, so that there is no overlap between them:

$$r_m - r_{m+1} \gg \delta r_m\,. \tag{23}$$

In the limit (23), the point vortex undergoes small (the order of δr_m) radial oscillations and no merger occurs.

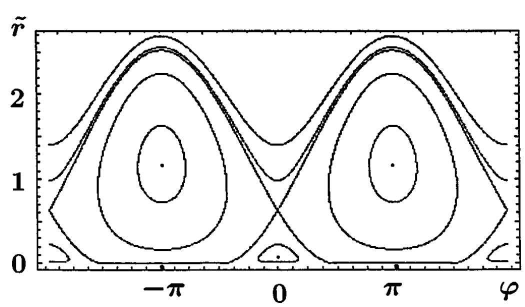

FIG. 1. Phase-space portrait of the point vortex motion inside the critical layer.

However, if γ exceeds a critical value γ_c, the width δr_m becomes sufficiently large that an overlap of neighboring critical layers occurs,

$$r_m - r_{m+1} \lesssim \delta r_m\,. \tag{24}$$

We will see that in this case the point vortex consecutively excites surface waves cascading from one critical layer to another until it merges with the extended vortex. Note that Eq. (24) coincides with the well-known Chirikov criterion [14] for the overlap of nonlinear resonances. The value of γ_c can be estimated by substituting Eqs. (10) and (22) into Eq. (24). For the most distant critical layer, which corresponds to a separation $r_2 = \sqrt{2}$ between the vortex centers, one finds γ_c to be of the order of 10^{-3}.

When there is an overlap and more than one wave participates in the dynamics, Hamilton's equations (15) and (16) must be integrated numerically. Figure 2 presents the results of a numerical integration for a case of a slight overlap between critical layers, $0 < (\gamma - \gamma_c)/\gamma_c \ll 1$. Initially, all 30 wave amplitudes included are set equal to zero, $\zeta_m(0) = 0$. Furthermore, the point vortex is placed near the critical radius r_2, so the $m = 2$ wave [curve 2(a)] grows rapidly and dominates the early evolution. The curve below [Fig. 2(b)] shows the radial position of the point vortex. Near the troughs of its early time oscillations, the point vortex penetrates the $m = 3$ critical layer and excites the $m = 3$ wave [curve 3(a)]. After several oscillations, the point vortex leaves the $m = 2$ critical layer, the $m = 2$ mode saturates, and the interaction with the $m = 3$ wave starts to dominate. Then the $m = 4$ wave [curve 4(a)] comes into play, and so forth. Such a cascade finally leads to vortex merger.

If $(\gamma - \gamma_c)/\gamma_c \gtrsim 1$, there is substantial overlap between critical layers, and from the very beginning the

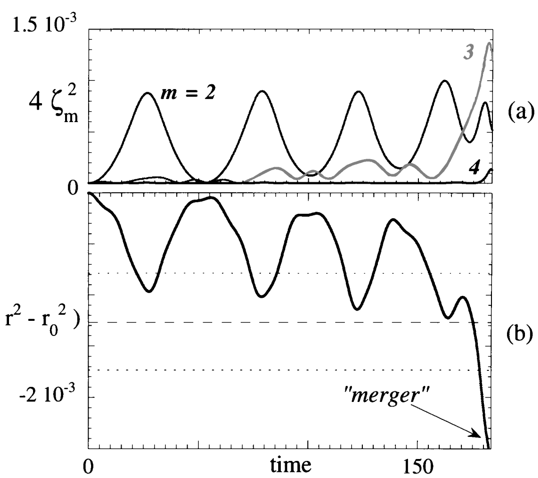

FIG. 2. Numerical integration of Hamilton's equations (15) and (16). Time dependence of the mode amplitudes (a) and radial position of the point vortex (b) in the case of slight resonance overlap, $0 < (\gamma - \gamma_c)/\gamma_c \ll 1$ [$r_0 = 1.418$, $\gamma = 2.5 \times 10^{-3}$, $\zeta_m(0) = 0$]. The dashed line corresponds to the $m = 3$ critical radius r_3. The dotted lines correspond to the characteristic boundaries $r_3 \pm \delta r_3$ of the critical layer [see Eq. (22)].

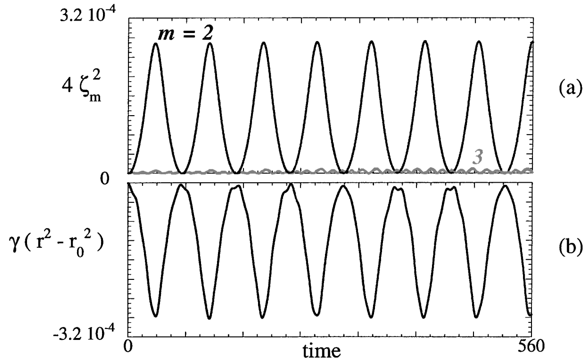

FIG. 3. Numerical integration of Hamilton's equations (15) and (16). Time dependence of the mode amplitudes (a) and radial position of the point vortex (b) in the case of the trapped motion inside the critical layer, $\gamma/\gamma_c < 1$ [$r_0 = 1.418$, $\gamma = 10^{-3}$, $\zeta_m(0) = 0$].

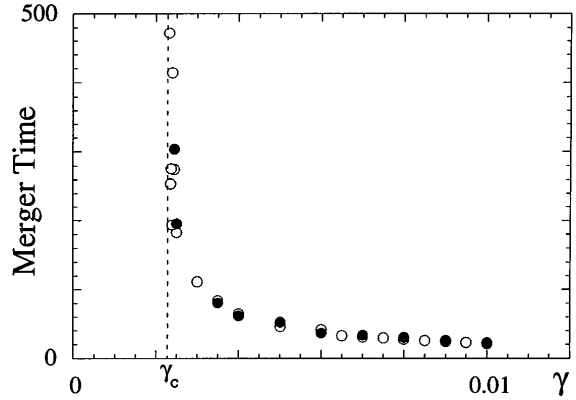

FIG. 4. The time to merge versus γ for $r_0 = 1.418$. Open circles result from the numerical integration of Eqs. (15) and (16), and solid circles result from the numerical integration using contour dynamics.

point vortex interacts resonantly with a whole set of waves. In this case the point vortex passes directly from layer to layer and merges with the extended vortex.

In contrast, Fig. 3 shows the results of a numerical integration for the same conditions as Fig. 2, except that the strength of the point vortex is below the critical value, $\gamma/\gamma_c < 1$. Only the $m = 2$ mode becomes excited to a significant level and there is no merger, even if the integration time is extended to 10^4 rotation periods of the extended vortex (i.e., $t \simeq 6.28 \times 10^4$). This justifies the "single wave–point vortex" analytic approximation used above for the analysis of the point vortex motion inside the critical layer.

We have compared our simplified model based on Eqs. (11), (15), and (16) to numerical integration of the full nonlinear equations (1) using contour dynamics with 240 nodes [15]. In the limit (23) of the trapped motion of the point vortex inside the critical layer, one finds near exact agreement between the models. Contour dynamics reproduces Fig. 3 with an accuracy of a line thickness. For the slightly supercritical case, $0 < (\gamma - \gamma_c)/\gamma_c \ll 1$, the contour dynamics results are very close to those of the simplified model [see Fig. 2] and support the scenario of cascading excitation of the resonant surface waves. For the case of substantial overlap between critical layers, there remains qualitative agreement between results of the models; however, distinctions due to the influence of nonlinear "wave-wave" interactions become apparent.

The dependence of the time to merge versus γ based on a numerical integration of Eqs. (15) and (16) (open circles) and contour dynamics (solid circles) is shown in Fig. 4. The point vortex is considered to be merged when its radial position becomes equal to $r = 1$, that is, to the initial radius of the extended vortex. If $\gamma < \gamma_c$ there is no resonance overlap and there is no merger. If γ exceeds γ_c, vortex merger occurs. As γ grows, the overlap becomes increasingly substantial, and the merger time decreases. One can see that the critical value is $\gamma_c \simeq 0.002$, which is near the estimated value.

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