Test Particle Diffusion and the Failure of Integration along Unperturbed Orbits

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This paper discusses a new effect in the kinetic theory of single species plasmas in the guiding-center regime $r_c \ll \lambda_D$. If two guiding centers collide once, their velocities parallel to the magnetic field are eventually reversed through interactions with surrounding particles, and the same two guiding centers then collide several times. These multiple collisions are ignored in the method of integration along unperturbed orbits (IUO) that is usually applied in plasma kinetic theory problems. This effect leads to a factor of 3 enhancement in the predicted rate of collisional test particle diffusion as compared to theory based on IUO. The new prediction is in agreement with recent experiments. [S0031-9007(97)04177-X]

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The collisional diffusion of test particles across a magnetic field has recently been measured in a single species ion plasma confined in a Penning-Malmberg trap in the regime $r_c \ll \lambda_D$ (where r_c is the cyclotron radius and λ_D is the Debye length) [1]. In this paper we present a calculation of the collisional test particle diffusion coefficient $\mathcal{D}$, and compare it to the measurement. The calculation involves a novel effect in plasma kinetic theory: an enhanced diffusion coefficient arising from multiple binary collisions between the same pair of particles. The multiple collisions are caused by velocity diffusion of the particle trajectories.

Before we can explain this new effect, we must first explain why previous theoretical calculations of the test particle diffusion coefficient fail to reproduce the experimental data. The well-known classical diffusion coefficient [2],

$$\mathcal{D}^{\text{class}} = \frac{5}{4}\,\nu_{\text{ii}} r_c^2 = \frac{4}{3}\sqrt{\pi}\,n\bar{v}b^2 \ln(r_c/b) r_c^2\,, \tag{1}$$

describes cross-field steps of average size r_c due to velocity-scattering collisions at rate ν_{ii}. Here, ν_{ii} is the ion-ion collision frequency, $\bar{v} = \sqrt{T/m}$ is the thermal speed, $b = e^2/T$ is the distance of closest approach, n is the density, and T is the temperature. This diffusion arises from interactions with impact parameter ρ in the interval $b \lesssim \rho \lesssim r_c$. Equation (1) predicts a diffusion rate approximately an order of magnitude below the measured diffusion [1].

Lifshitz and Pitaevskii recognized that when $r_c \ll \lambda_D$ there are collisions with impact parameters in the range $r_c \ll \rho \lesssim \lambda_D$ that are neglected in the derivation of Eq. (1) [3]. These relatively long-range collisions result in negligible velocity scattering, can be treated with guiding center theory, and provide the dominant transport mechanism when $r_c \ll \lambda_D$. However, this guiding-center theory of test particle diffusion also predicts a diffusion coefficient that is less than the experimental measurement (roughly by a factor of 3) [1]. We will show that this guiding-center theory fails because it employs the method of integration along unperturbed orbits (IUO), and this method misses the novel effect referred to above.

Under the approximation of IUO, two particles collide only once as they stream past one another along the magnetic field; in an infinite plasma they never have another encounter. However, this is not an accurate picture of the actual dynamics. Velocity-scattering collisions with surrounding particles cause the velocities of the two colliding particles to diffuse over time. Eventually the velocity diffusion causes the two particles to reverse their relative velocity parallel to the magnetic field, and they collide again; in fact, they may collide several times. This effect is neglected in IUO, and we will show that it leads to an increase in the test particle diffusion coefficient by a factor of 3, bringing it into agreement with the experimental measurement.

Surprisingly, this factor of 3 enhancement need not vanish even when the velocity diffusion becomes arbitrarily weak. This is because a decrease in the velocity diffusion rate merely increases the time one must wait before the relative parallel velocity is reversed. Provided that some other effect does not decorrelate the particles first, they will always suffer multiple collisions. Note that one normally expects that when perturbations to orbits are sufficiently weak, IUO will provide the right answer; here we have an example of a system for which this intuition fails. It fails because arbitrarily weak perturbations build up over time and eventually cause a large effect (multiple collisions between the same pair of particles) which is neglected in IUO.

The failure of IUO occurs because the collisions between particles are one dimensional: the particle guiding centers are constrained to follow the magnetic field, and therefore can encounter one another several times when their parallel velocities diffuse. However, if the particles could move freely in two or three dimensions, they would have negligible probability of colliding more than once, even if some velocity diffusion were added to their orbits.

Taking the magnetic field to be in the z direction, the basic step δx of the transport process occurs when the test particle $\mathbf{E} \times \mathbf{B}$ drifts across the magnetic field due to a Coulomb interaction with another particle:

$$\delta x = \int_{-\infty}^{\infty} dt \, \frac{c}{B} E_y(t), \tag{2}$$

where $E_y(t) = -\partial[e/|\Delta\mathbf{r}(t)|]/\partial y$ and $\Delta\mathbf{r}(t)$ is the relative displacement between the colliding particles. Under IUO, one neglects the effect of collisions on the relative displacement and takes $\Delta\mathbf{r}(t) = \Delta x \, \hat{\mathbf{x}} + \Delta y \, \hat{\mathbf{y}} + v_{\rm rel} t \, \hat{\mathbf{z}}$, where $v_{\rm rel}$ is the (constant) relative $\hat{z}$ velocity between the colliding particles, and $(\Delta x, \Delta y)$ is the (constant) transverse displacement between the particles. Performing the time integral in Eq. (2), one obtains the step $\delta x^{\rm IUO} = (2ec/B|v_{\rm rel}|)\Delta y/\rho^2$, where $\rho = \sqrt{\Delta x^2 + \Delta y^2}$. The test particle diffusion coefficient can then be obtained by integrating $(\delta x^{\rm IUO})^2$ over a flux of incident particles:

$$\mathcal{D}^{\rm IUO} = \int_{v_{\rm min}}^{\infty} dv_{\rm rel} \int_{r_c}^{\lambda_D} \rho \, d\rho \times \int_0^{2\pi} d\theta \, n v_{\rm rel} f(v_{\rm rel}) (\delta x^{\rm IUO})^2, \tag{3}$$

where $f(v_{\rm rel})$ is a Maxwellian distribution of relative velocities.

The allowed range of impact parameters ρ in Eq. (3) is determined by the fact that when $\rho < r_c$, the guiding-center approximation used to evaluate δx is no longer valid, while for $\rho > \lambda_D$ Debye shielding cuts off the interaction [3]. In the velocity integral, $v_{\rm min}$ is the minimum average relative z velocity during the collision. This cannot be set equal to zero, otherwise $\delta x^{\rm IUO}$ would be infinite, because particles could then interact for an infinite time [3]. Rather, velocity diffusion causes particles to move apart and sets $v_{\rm min} \sim (D_v \rho)^{1/3}$, where D_v is the relative velocity diffusion coefficient, due to velocity-scattering collisions with other particles [4]. Note that this implies that IUO does not entirely neglect velocity diffusion, since velocity diffusion is invoked here in order to limit the interaction time between particles. However, as we discussed in the introduction, IUO does ignore the multiple collisions caused by velocity diffusion. We will determine the effect of multiple collisions on $\mathcal{D}$ presently, but first we must finish the calculation of test particle diffusion using IUO.

Evaluation of the integrals in Eq. (3) can be easily carried out to logarithmic accuracy, observing that $v_{\rm min} \ll \bar{v}$ and $r_c \ll \lambda_D$:

$$\mathcal{D}^{\rm IUO} = 2\sqrt{\pi}\, n \bar{v} b^2 r_c^2 \ln\left(\frac{\lambda_D}{r_c}\right) \ln[\bar{v}/(D_v \sqrt{\lambda_D r_c})^{1/3}]. \tag{4}$$

This result is compared to the experimental measurements in Fig. 1 (the dashed line) [5]. Here we have taken $D_v = \nu_{\rm ii} \bar{v}^2$, and we have subtracted out of the experimental data the relatively small extra diffusion due to the classical process given by Eq. (1), so that we can directly compare the theory to measured diffusion due only to long-range collisions. Even with the classical diffusion removed

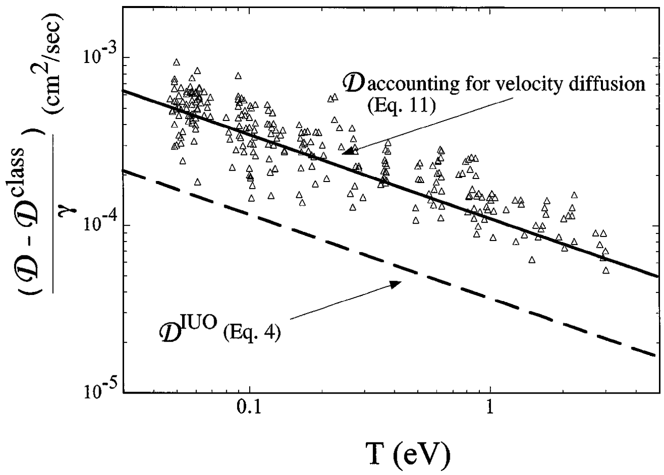

FIG. 1. Dots are the measured test particle diffusion coefficient $\mathcal{D}$ in cm^2/sec, from which the classical result for small-impact parameter collisions, $\mathcal{D}^{\rm class}$, has been subtracted. Data are plotted versus plasma temperature T. Diffusion is normalized by the dimensionless factor $\gamma \equiv [B/1\ {\rm T}]^{-2}[n/10^6\ {\rm cm}^{-3}] \ln[\lambda_D/r_c] \ln[\bar{v}/(D_v\sqrt{\lambda_D r_c})^{1/3}]$ in order to display data taken at several different magnetic field strengths and densities. Dashed line is the theoretical result for $\mathcal{D}$ using IUO; solid line is the improved theory accounting for velocity diffusion. There are no adjustable parameters.

from the experimental measurement, one can see that the theory still falls well below the scatter of the data.

However, if one properly accounts for velocity diffusion, one obtains a different (larger) answer for δx and $\mathcal{D}$. The relative z position of the guiding centers is more correctly described by

$$\Delta z(t) = v_{\rm rel} t + \frac{e}{m} \int_0^t dt' \int_0^{t'} dt'' [\delta E_1(t'') + \delta E_2(t'')], \tag{5}$$

where δE_1 and δE_2 are the z components of electric fields felt by the two colliding particles. These electric fields describe the effect of many uncorrelated small-impact parameter collisions, so we treat δE_1 and δE_2 as uncorrelated fluctuating fields with zero mean and white-noise statistics. The test particle transport step can then be determined by Fourier transforming Eq. (2) and averaging over the fluctuating electric fields:

$$\delta x = -\frac{4\pi ec}{B} \int \frac{d^3k}{(2\pi)^3} \frac{ik_y}{k^2} e^{ik_x \Delta x + ik_y \Delta y} \times \int_{-\infty}^{\infty} dt \, \langle e^{ik_z \Delta z(t)} \rangle. \tag{6}$$

This type of average is well understood from the theory of stochastic processes [6], and yields

$$\delta x = -\frac{4\pi ec}{B} \int \frac{d^3k}{(2\pi)^3} \frac{ik_y}{k^2} e^{ik_x \Delta x + ik_y \Delta y} \times \int_{-\infty}^{\infty} dt \, e^{ik_z v_{\rm rel} t - k_z^2 D_v |t|^3/3} \equiv -\frac{4\pi ec}{B} \int \frac{dk_x dk_y}{(2\pi)^2} ik_y e^{ik_x \Delta x + ik_y \Delta y} I, \tag{7}$$

where I is discussed below and

$$D_v = \left(\frac{e}{m}\right)^2 \int_0^\infty dt\,[\langle \delta E_1(t)\delta E_1(0)\rangle + \langle \delta E_2(t)\delta E_2(0)\rangle]$$

is the relative velocity diffusion coefficient. The $k^2 D_v t^3$ term in the exponent appearing in Eq. (7) is the usual way in which velocity diffusion enters in resonance broadening theory [7].

Surprisingly, the evaluation of Eq. (7) differs depending on whether D_v equals zero or D_v approaches zero. Consider the time and k_z integrals that appear in Eq. (7):

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dk_z}{k_\perp^2 + k_z^2} \int_{-\infty}^{\infty} dt\, e^{ik_z v_{\rm rel} - k_z^2 D_v |t|^3/3}$$
$$= 1/|v_{\rm rel}| k_\perp^2 \quad \text{for } D_v = 0\,. \qquad (8)$$

The second line follows because the time integral yields $2\pi\delta(k_z v_{\rm rel})$ when $D_v = 0$. Using Eq. (8) in Eq. (7) then leads back to $\delta x^{\rm IUO}$, as expected. However, if we take D_v small but nonzero and perform the coordinate transformations $\hat{t} = (k_z^2 D_v)^{1/3} t$, $u = (k_z/D_v)^{1/3} v_{\rm rel}$, we obtain

$$I = \frac{3}{2\pi}\int_{-\infty}^{\infty} \frac{du}{|v_{\rm rel}|}\, \frac{1}{k_\perp^2 + (D_v u^3/v_{\rm rel}^3)^2} \int_{-\infty}^{\infty} d\hat{t}\, e^{iu\hat{t} - |\hat{t}|^3/3}$$
$$= 3/|v_{\rm rel}| k_\perp^2 \quad \text{for } D_v \text{ small.} \qquad (9)$$

The second line follows because we can replace $[k_\perp^2 + (D_v u^3/v_{\rm rel}^3)^2]^{-1}$ by $k_\perp^{-2}$ when D_v is sufficiently small, in which case interchanging the u and $\hat{t}$ integrals yields $I = 3/|v_{\rm rel}| k_\perp^2$. Note that the factor of 3 that now appears in I does not appear when $D_v = 0$. We therefore obtain

$$\lim_{D_v \to 0} \delta x = 3\delta x^{\rm IUO} = \frac{6ec}{B|v_{\rm rel}|}\,\frac{\Delta y}{\rho^2}\,. \qquad (10)$$

Thus, the limit of δx as D_v approaches zero is unequal to the case $D_v = 0$, which corresponds to IUO. The evaluation of the basic transport step δx joins a small class of singular perturbation theory problems that are both analytically tractable and physically relevant.

Some physical insight into this surprising result can be obtained by considering the time behavior of $\exp(ik_z v_{\rm rel} t - k_z^2 D_v |t|^3/3)/(k_z^2 + k_\perp^2)$, integrated over k_z. Rescaling k_z to $\bar{k}_z = k_z/k_\perp$, the integral can be written as

$$J(\bar{t}, \bar{D}_v) \equiv \int_{-\infty}^{\infty} d\bar{k}_z \exp[i\bar{k}_z\bar{t} - \bar{D}_v \bar{k}_z^2 |\bar{t}|^3]/(\bar{k}_z^2 + 1)\,,$$

where $\bar{D}_v = D_v/(3k_\perp v_{\rm rel}^3)$ and $\bar{t} = k_\perp v_{\rm rel} t$. This integral can be expressed in terms of error functions, and the result is displayed in Fig. 2 for several values of $\bar{D}_v$. Since $I = \int_{-\infty}^{\infty} d\bar{t}\, J(\bar{t}, \bar{D}_v)/(2\pi k_\perp^2 |v_{\rm rel}|)$, we are chiefly interested in the area under the curves. When $\bar{D}_v = 0$, $J(\bar{t}, 0) = \pi \exp(-|\bar{t}|)$; this corresponds to IUO and is shown by the dots in the figure. However, for $\bar{D}_v$ sufficiently small but finite, there is a second peak at $\bar{t} \neq 0$. As $\bar{D}_v$ approaches zero, the peak at $\bar{t} \neq 0$ becomes broader and moves to larger $\bar{t}$, but the area under it does not vanish. The extra area provides the factor of 3 enhancement to δx.

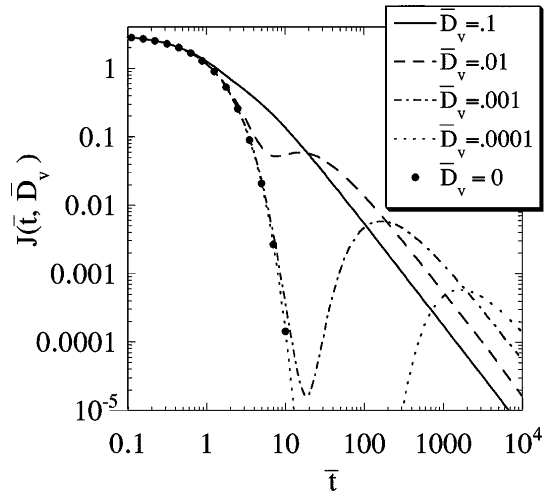

FIG. 2. Behavior with scaled time $\bar{t} = k_\perp v_{\rm rel} t$ of the integral $J(\bar{t}, \bar{D}_v)$ as the scaled velocity diffusion coefficient $\bar{D}_v \equiv D_v/3k_\perp v_{\rm rel}^3$ decreases toward zero. Dots correspond to IUO ($\bar{D}_v = 0$). Logarithmic axes are employed to display the broad peak at $\bar{t} = \bar{t}_p \sim \bar{D}_v^{-1}$.

One can see from Fig. 2 that the second peak occurs at a scaled time $\bar{t}_p \approx 1/\bar{D}_v$, or in unscaled units at a time $t_p \approx v_{\rm rel}^2/D_v$. In order to obtain the full factor of 3 enhancement, one must therefore integrate the interaction over a time of order t_p. This is the time required for $v_{\rm rel}$ to be changed through velocity diffusion. Particles may reverse their relative velocity and return to suffer a second collision with the test particle in a time t_p. The enhancement arises from particles which collide once, then through velocity diffusion return to collide again.

If one now naively attempts to evaluate the test particle diffusion coefficient $\mathcal{D}$ by employing the new expression for δx in Eq. (3), one obtains $\mathcal{D} = 9\mathcal{D}^{\rm IUO}$, because $\delta x = 3\delta x^{\rm IUO}$. However, this is also incorrect. In order to properly evaluate $\mathcal{D}$ one needs to know the rate at which steps δx occur. Since each step δx is now determined by multiple collisions, the rate at which these multiple collisions occur determines the diffusion.

Fortunately, the rate of the multiple collisions can be easily evaluated. Since δx is 3 times the result obtained from a single collision, one may conclude that on average there are 3 single collisions making up each multiple collision, so the rate of these multiple collisions is decreased by a factor of 3 compared to the rate of the single collisions. When combined with the fact that $\delta x = 3\delta x^{\rm IUO}$, this implies that

$$\mathcal{D} = 3\mathcal{D}^{\rm IUO}. \qquad (11)$$

This result can also be derived from a more rigorous approach that uses the general expression for $\mathcal{D}$ as a time integral over the correlation function of the $\mathbf{E} \times \mathbf{B}$ velocity fluctuations: $\mathcal{D} = (c/B)^2 \int_0^\infty dt\, \langle E_y(t) E_y(0)\rangle$. This approach will be presented in a separate more detailed paper [8].

The new value of $\mathcal{D}$ given by Eq. (11) is shown by the solid curve in Fig. 1. The theory now matches the experiment to within the scatter of the data.

In conclusion, we have shown that integration along unperturbed orbits fails to capture a significant new effect in plasma kinetic theory: velocity diffusion of particle trajectories causes multiple collisions between the same pair of particles. The velocity diffusion is due to collisions with surrounding particles. One normally thinks that collisions with surrounding particles cause spatial decorrelation of a particle pair. Here we observe that the collisions have the opposite effect, "caging" the two particles, making them interact more strongly than they would otherwise. This is an effect that is usually associated with the liquid or solid state, not with weakly correlated plasmas.

We have seen that this effect leads to an increase by a factor of 3 in the test particle diffusion coefficient $\mathcal{D}$ (as well as in the basic transport step δx) caused by long-range guiding-center collisions in a magnetized single-species plasma. Our analysis can be extended to cover diffusion caused by collisions between multiple species in a neutral plasma with no change in the factor of 3 enhancement, provided that each species is in the guiding-center regime $r_c \ll \lambda_D$ [8]. The factor of 3 resolves a discrepancy between an experimental measurement of the diffusion [1] and a previous theory [1,3] based on IUO. The factor of 3 does not vanish, even when the velocity diffusion coefficient D_v approaches zero, provided that some other effect does not first decorrelate the colliding particles before multiple collisions can occur. Thus, we have a system for which the limit as D_v approaches zero is unequal to the case $D_v = 0$.

Since IUO has been applied to many problems throughout physics, it is important to consider whether similar enhancements may occur when the effect of multiple collisions is taken into account. First, as we have already discussed, the collisional dynamics must be one dimensional so that velocity diffusion leads to a reversal of the trajectories and multiple collisions. In the problem considered here, the one-dimensional nature of the motion is assured by the application of a strong magnetic field. The strong confining fields encountered in some other problems such as 1D quantum wires might also lead to a similar effect.

Second, we note that any process that decorrelates the particles before they can reverse their trajectories and collide again will nullify the effect considered here, and IUO will again provide the correct answer. As we observed in connection with Fig. 2, the time for the multiple collisions is of order $t_p \sim v_{\rm rel}^2/D_v$. The decorrelation time must be shorter than t_p in order for IUO to be correct. Note that collisions with surrounding particles do not cause this decorrelation: over a time of $O(t_p)$ these collisions cage the particles and *increase* their spatial correlation. However, decorrelation can be caused by significant rotational shear in the background plasma, causing particles to move apart in the transverse (x, y) directions in a time of order $|\rho \partial\omega/\partial r|^{-1}$, where $\omega(r)$ is the $\mathbf{E} \times \mathbf{B}$ rotation frequency. Comparing this time to t_p implies that $|\partial\omega/\partial r| > D_v/\rho v_{\rm rel}^2$ must be satisfied in order for IUO to be correct for any given collision. [In the present experiments this inequality is not satisfied for the collisions of interest ($\rho < \lambda_D$, $v_{\rm rel} < \bar{v}$) because the plasma is nearly in a state of confined thermal equilibrium, with shears minimized by plasma viscosity.]

One previously published result which will be affected by this enhancement is the evaluation of the collisional viscosity η of a magnetized single species plasma [9]. Preliminary analysis indicates that the same factor of 3 enhancement that we encountered here will also occur in η. These issues will be considered in more detail in a future publication [8].

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[4] Other effects, such as velocity shear in the plasma, can also determine $v_{\rm min}$. However, in the experiments of Ref. [1] velocity diffusion is believed to be the dominant mechanism.
[5] The $\ln[\bar{v}/D_v\sqrt{\lambda_D r_c})^{1/3}]$ term in Eq. (4) is slightly different than the form in Ref. [1], because in Ref. [1] $v_{\rm min}$ was estimated incorrectly as $(D_v \lambda_D)^{1/3}$. However, in the experiments, where $r_c/\lambda_D \sim 0.1$, the difference in $D^{\rm IUO}$ is negligible.
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