Rotational pumping and damping of the $m=1$ diocotron mode

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An effect called rotational pumping by the authors (by analogy with magnetic pumping) causes a slow damping of the $m=1$ diocotron mode in non-neutral plasmas. In a frame centered on the plasma and rotating at the diocotron mode frequency, the end confinement potentials are nonaxisymmetric. As a flux tube of plasma undergoes $\mathbf{E}\times\mathbf{B}$ drift rotation about the center of the column, the length of the tube oscillates about some mean value, and this produces a corresponding oscillation in $T_\parallel$. In turn, the collisional relaxation of $T_\parallel$ toward $T_\perp$ produces a slow dissipation of electrostatic energy into heat and a consequent radial expansion (cross-field transport) of the plasma. Since the canonical angular momentum is conserved, the displacement of the column off axis must decrease as the plasma expands. In the limit where the axial bounce frequency of an electron is large compared to its $\mathbf{E}\times\mathbf{B}$ drift rotation frequency theory predicts the damping rate $\gamma=-2\kappa^2\nu_{\perp,\parallel}(r_p^2/R_w^2)(\lambda_D^2/L_0^2)/(1-r_p^2/R_w^2)$, where κ is a numerical constant, λ_D is the Debye length, R_w is the radius of the cylindrical conducting wall, r_p is the effective plasma radius, L_0 is the mean length of the plasma, and $\nu_{\perp,\parallel}$ is the equipartition rate. A novel aspect of this theory is that the magnetic field strength enters only through $\nu_{\perp,\parallel}$. As the field strength is increased, the damping rate is nearly independent of the field strength until the regime of strong magnetization is reached [i.e., $\Omega_c>\bar{v}/b=(kT)^{3/2}/\sqrt{m}e^2$], and then the damping rate drops off dramatically. This signature has been observed in recent experiments. For completeness, the theory is extended to the regime where the bounce frequency is comparable to the rotation frequency, and bounce-rotation resonances are included. © *1995 American Institute of Physics.*

I. INTRODUCTION

Recent experiments have involved the confinement of pure electron plasmas in Penning traps.[1–4] A schematic diagram for such a trap is shown in Fig. 1. A conducting cylinder is divided axially into three sections, the two end sections being held at a negative potential relative to the central section. There is a uniform magnetic field, B, directed along the axis of the cylinder. The electron plasma resides in the central section, with axial confinement provided by the negatively biased end sections and radial confinement by the magnetic field. The Larmor radius is typically small, so the cross-field motion may be described by $\mathbf{E}\times\mathbf{B}$ drift dynamics.[3,5]

The most commonly observed excitation of such a plasma is the diocotron mode of azimuthal wave number $m=1$.[3,6–8] One can think of this mode as a rigid displacement of the plasma column away from the central axis of the trap. The image charges induced in the conducting wall cause the column as a whole to $\mathbf{E}\times\mathbf{B}$ drift about the central axis of the trap, while the space charge field causes the column to rotate about an axis through its center of charge. The main reason that this mode plays such a prominent role in the dynamics of pure electron plasmas is that it is damped only weakly; in typical experiments, it is observed to survive 10^5 periods. In spite of this, Cluggish and Driscoll have been able to measure the damping rate and characterize its parameter dependence over a wide range.[9] In this paper we present a theory of the damping that agrees with the parameter dependence observed in the experiments. This theory is closely related to the work of Ryutov and Stupukov on transport in magnetic mirror traps.[10]

Diocotron modes of azimuthal wave number $m>1$ typically damp due to a wave–particle resonance.[11] The resonance is spatially localized at the resonant radius, r_s, defined by $m\omega_R(r_s)=\omega$, where ω_R is the single-particle $\mathbf{E}\times\mathbf{B}$ rotation frequency and ω is the mode frequency. For a monotonically decreasing density profile, the $m=1$ diocotron mode is special because $r_s=R_w$, the radius of the conducting wall.[11] There are no resonant particles, because the density is zero at the wall, and therefore a different mechanism is required to explain the observed damping of the $m=1$ diocotron mode.

Since the damping time scale is characteristic of collisional transport time scales, we look for an explanation that involves collisional transport. It is convenient to work in a frame that rotates with the mode so that the off-axis column is a stationary state (an equilibrium), except for the slow evolution on the transport time scale. The connection between damping and transport follows from the conservation of angular momentum. In guiding center theory, the canonical angular momentum of the plasma is approximately[12]

$$P_\theta\simeq\frac{eB}{2c}\sum_{j=1}^{N}R_j^2,\tag{1}$$

where $\mathbf{R}_j$ is the position of the jth particle measured from the

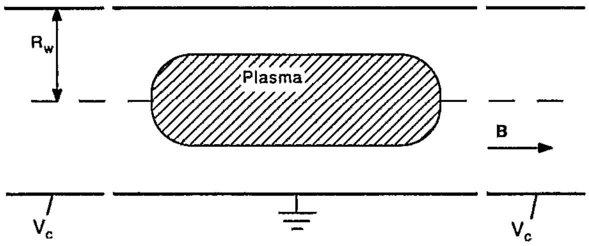

FIG. 1. The confinement geometry.

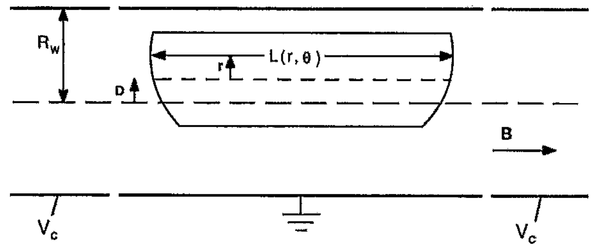

FIG. 3. Length of the off-axis plasma.

trap axis (see Fig. 2), and e carries a sign. If $\mathbf{D}$ is the displacement of the center of charge and $\mathbf{r}_j$ is measured from the center of charge ($\mathbf{R}_j=\mathbf{D}+\mathbf{r}_j$), the canonical angular momentum can be written as

$$P_\theta \simeq \frac{eB}{2c} N\left(D^2 + \frac{1}{N} \sum_{j=1}^{N} r_j^2 \right). \tag{2}$$

Note that the cross-term, $\Sigma_j 2\mathbf{r}_j \cdot \mathbf{D}$, vanishes because $\mathbf{D}$ defines the center of charge. Since the apparatus is cylindrically symmetric, P_θ is conserved. This implies a relation between plasma expansion and mode damping. Differentiating Eq. (2) with respect to time yields the relation

$$\frac{\partial}{\partial t}(D^2) = -\frac{\partial}{\partial t}\left(\frac{1}{N} \sum_{j=1}^{N} r_j^2 \right) = -\frac{\partial}{\partial t}\langle r^2 \rangle. \tag{3}$$

Given a transport theory that describes the radial expansion of the plasma, Eq. (3) can be used to calculate the damping rate.

The angular momentum calculated about an axis through the center of charge is not conserved. If it were, $\langle r^2 \rangle$ would be constant in time, and the mode would not damp. We therefore restrict our attention to transport processes that depend on the nonaxisymmetric nature of the confining fields in a frame centered on the plasma. In particular, we consider the effect of the end confinement potentials.

When the Debye length is small, the plasma has a well-defined edge. A displaced column sees nonaxisymmetric end potentials, and these cause the end shape of the plasma to be nonaxisymmetric. This asymmetry can be characterized by the length of the plasma parallel to the magnetic field,

$$L(r,\theta) = L_0(r) + \delta L(r,\theta), \tag{4}$$

where (r,θ) is a cylindrical coordinate system centered on an axis through the center of charge (see Fig. 3). A simple analytic theory, as well as numerical studies,[13] indicate that the asymmetric component is well represented by a uniform tilt at an angle proportional to D/R_w, where R_w is the radius of the conducting cylinder and D is the displacement of the center of charge off axis. Therefore, the asymmetric part may be written as

$$\delta L(r,\theta) = \kappa \frac{D}{R_w} r \sin\theta, \tag{5}$$

where κ is a numerical constant.

A simple transport equation can be derived by considering a single flux tube of plasma, as shown in Fig. 4. The flux tube has length $L(r,\theta)$, as given by Eq. (4), cross-section area δA, and contains δN particles. The dominant cross-field motion of the flux tube is the $\mathbf{E}\times\mathbf{B}$ drift $\mathbf{v}_D = (c/B)\hat{z}\times\nabla\Phi$. Assuming that the plasma column has a circular cross sec-

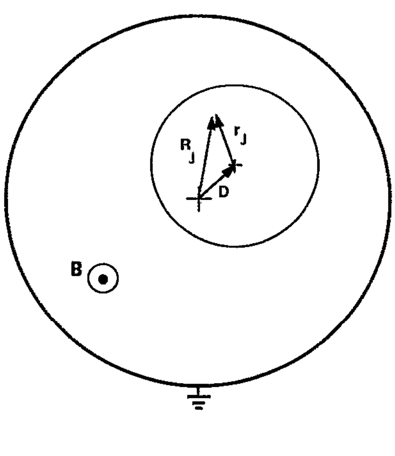

FIG. 2. Coordinate system for the off-axis plasma (end view).

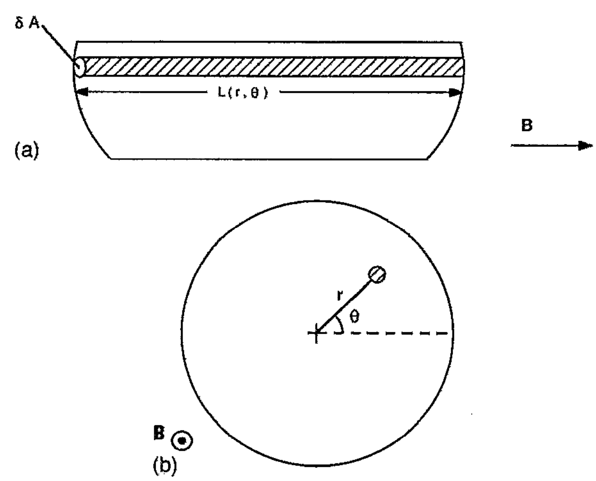

FIG. 4. (a) Side view of flux tube. (b) End view of flux tube.

tion, the electric potential is of the form $\Phi=\Phi(r)$, and the flux tube drifts in a circular orbit about the center of the column, with the frequency

$$\omega_R=\frac{c}{Br}\frac{\partial\Phi}{\partial r}. \tag{6}$$

Setting $\theta=\omega_R t$ in Eq. (4) then implies that the length of the flux tube varies temporally as $L(r,t)=L_0(r)+\delta L(r,\omega_R t)$. From Eq. (5), it then follows that the length of the flux tube undergoes a sinusoidal variation about the length $L_0(r)$. The cyclic axial compression and expansion produces a cyclic variation in the parallel temperature, and this is coupled collisionally to the perpendicular temperature. The full temperature evolution is governed by the equations

$$\frac{dT_\parallel}{dt}=-T_\parallel\frac{2}{L}\frac{dL}{dt}+2\nu_{\perp,\parallel}(T_\perp-T_\parallel) \tag{7}$$

and

$$\frac{dT_\perp}{dt}=-\nu_{\perp,\parallel}(T_\perp-T_\parallel), \tag{8}$$

where $\nu_{\perp,\parallel}$ is the collisional equipartition rate. We have used the fact that δN is constant in deriving these equations. The first term on the right-hand side of Eq. (7) describes the compressional heating (or expansion cooling) of the parallel degrees of freedom, and the second term describes the collisional coupling to the perpendicular degrees of freedom. The perpendicular degrees of freedom are not directly affected by the change in length, so the right-hand side (RHS) of Eq. (8) contains only the collisional coupling term. The factor of 2 difference in the collisional coupling term for Eq. (8) relative to Eq. (7) simply reflects the fact that there are two perpendicular degrees of freedom and one parallel.

A two time scale analysis of Eqs. (7) and (8) based on the frequency ordering $\omega_R\gg\nu_{\perp,\parallel}$ yields the result

$$\frac{d\langle T_\parallel\rangle}{dt}=8\nu_{\perp,\parallel}\frac{\langle T_\parallel\rangle}{L_0^2}\langle\delta L^2(r,t)\rangle+2\nu_{\perp,\parallel}(\langle T_\perp\rangle-\langle T_\parallel\rangle), \tag{9}$$

$$\frac{d\langle T_\perp\rangle}{dt}=-\nu_{\perp,\parallel}(\langle T_\perp\rangle-\langle T_\parallel\rangle), \tag{10}$$

where $\langle\cdot\rangle$ indicates an average over the fast time scale, that is, over one rotation period. In addition to the energy conserving terms, the first term on the right-hand side of Eq. (9) represents a secular increase in $T_\parallel$. Physically, this term arises because collisions cause a small phase shift in the parallel temperature fluctuations, so that the parallel temperature and pressure are slightly larger in the compression stage than in the expansion stage. More work is done on the plasma during compression than is done by the plasma during expansion. The result is that the plasma in the flux tube is heated. This effect is similar to magnetic pumping,[14] and, by analogy, we refer to it as rotational pumping.

Since the confinement potentials are time independent, the total energy in the plasma is conserved, and the increase in thermal energy must be balanced by a corresponding decrease in the electrostatic energy. The particle flux is found by equating the increase in the thermal energy to local Joule heating. That is,

$$n\frac{d}{dt}\left(\frac{1}{2}\langle T_\parallel\rangle+\langle T_\perp\rangle\right)=-e\frac{\partial\Phi}{\partial r}\Gamma_r, \tag{11}$$

where Γ_r is the radial particle flux and n is the density. The RHS of this equation is the Joule heating per unit volume, and again we have used the fact that $\delta N=\text{const}$. Equations (9)–(11) are solved for the flux, and yield

$$\Gamma_r=4\nu_{\perp,\parallel}n(r)\frac{T}{-e\,\partial\Phi/\partial r}\frac{\langle\delta L^2\rangle}{L_0^2}, \tag{12}$$

where

$$\langle\delta L^2\rangle=\frac{1}{2\pi}\int_0^{2\pi}d\theta\,\delta L^2(r,\theta). \tag{13}$$

Finally, the damping rate is calculated by using the conservation of angular momentum. After introducing the particle flux, Eq. (3) becomes

$$2D\frac{dD}{dt}=-\frac{\partial}{\partial t}\langle r^2\rangle=-\frac{1}{N}\int d^3\,r\,2r\Gamma_r. \tag{14}$$

Using Eq. (12) and the assumed form of the asymmetry from Eq. (5), we find the result

$$\frac{\partial}{\partial t}(D)=\left(-\frac{1}{N}\int d^3r\,2\nu_{\perp,\parallel}n(r)\times\frac{T}{-e\,\partial\Phi/\partial r}\frac{\kappa^2}{L_0^2R_w^2}r^3\right)D. \tag{15}$$

The quantity in parentheses is γ, the damping rate of the mode. In this equation, it should be remembered that Φ is the potential in a frame rotating at the mode frequency $\omega_{\rm D}$.[6] It is related to the potential in the lab frame through

$$\Phi=\Phi_L-\frac{BR^2}{2c}\omega_{\rm D}, \tag{16}$$

where the second term, $(BR^2/2c)\omega_{\rm D}$, arises from the motion of the plasma column through the magnetic field. The potential in the lab frame, Φ_L, is composed of two parts: the space charge potential, Φ_0, and the potential due to the image charges induced on the conducting wall, Φ_I. Changing variables to a coordinate system centered on the plasma using $\mathbf{R}=\mathbf{D}+\mathbf{r}$ yields the result

$$\Phi=\Phi_0-\frac{Br^2}{2c}\omega_{\rm D}+\left(\Phi_I-\frac{B}{c}\mathbf{r}\cdot\mathbf{D}\omega_{\rm D}\right)-\frac{BD^2}{2c}\omega_{\rm D}. \tag{17}$$

The two terms in parentheses cancel (this condition may be used to determine $\omega_{\rm D}$), and the last term may be dropped as it is just an additive constant. This leaves

$$\Phi=\Phi_0-\frac{Br^2}{2c}\omega_{\rm D}. \tag{18}$$

Of course, Φ_0 is related to the density through Poisson's equation.

For simplicity, we consider an isothermal, constant density plasma of radius r_p. The integral in Eq. (15) is then trivial, and yields the result

$$\gamma=-2\kappa^2\nu_{\perp,\parallel}\frac{\lambda_D^2 r_p^2}{L_0^2 R_w^2}\frac{1}{(1-r_p^2/R_w^2)}, \tag{19}$$

where $\lambda_D = \sqrt{T/4\pi e^2 n}$ is the Debye length. It is striking that γ depends on the magnetic field strength only through $\nu_{\perp,\parallel}$. In the regime of weak magnetization (i.e., $r_c \gg b$, where $r_c=\bar{v}/\Omega_c$ and $b=e^2/m\bar{v}^2$), this dependence is very weak, $\nu_{\perp,\parallel}\propto\ln(r_c/b)$. In the regime of strong magnetization (i.e., $r_c \ll b$), $\nu_{\perp,\parallel}$ becomes exponentially small,[15,16] and our theory predicts that γ becomes exponentially small. These unusual scalings agree with the observations of Cluggish and Driscoll.[9] In particular, the dramatic decrease in the observed damping rate when r_c becomes small compared to b is rather convincing evidence that our theory focuses on the relevant damping mechanism.

In Sec. II, we present a more rigorous calculation of the transport by solving the drift-kinetic Boltzmann equation in the limit that $\omega_B \gg \omega_R$, where ω_B is the single-particle bounce frequency parallel to the magnetic field and ω_R is the rotation frequency. In Sec. III we consider the effect of bounce-rotation resonances, and show that in some regimes resonant particles enhance the damping rate.

II. KINETIC TREATMENT IN THE ADIABATIC LIMIT

In this section and in Sec. III, we assume the following frequency ordering:

$$\Omega_c \gg \omega_R, \omega_B \gg \nu \gg \gamma, \tag{20}$$

where Ω_c is the cyclotron frequency, ω_B is the axial bounce frequency, ω_R is the rotation frequency, ν is the collision frequency, and γ is the damping rate. Since Ω_c is the largest frequency, we may describe the collisionless single-particle dynamics with a guiding center Hamiltonian of the form[16]

$$H=p_z^2/2m+\mu B+e\Phi(p_\theta)+e\Phi_e(\theta,p_\theta,z), \tag{21}$$

where $p_\theta=(eB/2c)r^2$ is the canonical angular momentum conjugate to θ, and (r,θ) is a cylindrical coordinate system centered on axis through the center of charge. We break up the potential into two parts: $\Phi(p_\theta)$ is the space charge potential in a frame rotating at the diocotron frequency and $\Phi_e(\theta,p_\theta,z)$ is the Debye-screened end potential. Since the Debye length is small, we let

$$e\Phi_e(\theta,p_\theta,z)=\begin{cases}0, & |z|<\frac{1}{2}L(\theta,p_\theta),\\ \infty, & \text{otherwise,}\end{cases} \tag{22}$$

where $L(\theta,p_\theta)$ is the length of the plasma parallel to the magnetic field, as discussed in the Introduction. The term $\mu B=\frac{1}{2}mv_\perp^2$ is the perpendicular kinetic energy of the particle. In the guiding center limit, μ=const, and since the magnetic field is assumed to be uniform, μB enters the Hamiltonian as an additive constant. We retain this term in the Hamiltonian because it is useful to write Maxwellian distribution functions as a function of H.

In the experiments, it is typically the case that the bounce frequency, $\omega_B=2\pi|v_z|/2L$, is much larger than the rotation frequency, $\omega_R=\partial/\partial p_\theta(e\Phi)$, for the vast majority of the particles. In this section, we assume that this is true for all the particles. (In Sec. III we allow for bounce-rotation resonances.) In the limit $\omega_B \gg \omega_R$, the bounce action,

$$\begin{aligned} I&=\frac{1}{2\pi}\oint p_z\,dz \\ &=\frac{1}{2\pi}\oint\sqrt{2m(H-B\mu-e\Phi-e\Phi_e)}dz, \end{aligned} \tag{23}$$

is a good adiabatic invariant. As noted by Taylor,[17] an equation of this form implicitly defines H in terms of I, θ, and p_θ. Given the simple form of the end potential, this equation is easily inverted to give

$$H(I,\theta,p_\theta)=\frac{\pi^2 I^2}{2mL^2(\theta,p_\theta)}+B\mu+e\Phi(p_\theta). \tag{24}$$

We represent the plasma with a distribution of guiding centers,

$$f=f(I,\psi,p_\theta,\theta,\mu,t), \tag{25}$$

where ψ is the angle conjugate to I and indicates the phase of a particle in its bounce motion (i.e., its position along the magnetic field). This distribution function evolves according to the drift-kinetic Boltzmann equation,

$$\frac{\partial f}{\partial t}+[f,H]=C(f), \tag{26}$$

where $C(\cdot)$ is the collision operator, and the Poisson bracket is given by

$$[f,H]=\frac{\partial f}{\partial\psi}\frac{\partial H}{\partial I}+\frac{\partial f}{\partial\theta}\frac{\partial H}{\partial p_\theta}-\frac{\partial f}{\partial p_\theta}\frac{\partial H}{\partial\theta}. \tag{27}$$

In the adiabatic limit, $\omega_B=\partial H/\partial I$ is large, and so $\partial f/\partial\psi$ must be small. Otherwise, $\partial f/\partial t$ would be large, and the distribution would evolve rapidly along the magnetic field. Physically, this corresponds to the fact that any initially large ψ variations are rapidly phase mixed by the bounce motion. The small ψ variations are uninteresting from the standpoint of cross-field transport, and may be eliminated by integrating Eq. (26) over ψ. The result is

$$\frac{\partial\bar{f}}{\partial t}+\frac{\partial\bar{f}}{\partial\theta}\frac{\partial H}{\partial p_\theta}-\frac{\partial\bar{f}}{\partial p_\theta}\frac{\partial H}{\partial\theta}=C(\bar{f}), \tag{28}$$

where

$$\bar{f}(I,p_\theta,\theta,\mu,t)=\int_0^{2\pi}d\psi\,f(I,\psi,p_\theta,\theta,\mu,t). \tag{29}$$

Rewriting Eq. (28) as

$$\frac{\partial\bar{f}}{\partial t}+\frac{\partial}{\partial\theta}\left(\frac{\partial H}{\partial p_\theta}\bar{f}\right)-\frac{\partial}{\partial p_\theta}\left(\frac{\partial H}{\partial\theta}\bar{f}\right)=C(\bar{f}), \tag{30}$$

and integrating over I, μ, and θ yields the transport equation

$$\frac{\partial N(p_\theta)}{\partial t}=\frac{\partial}{\partial p_\theta}\left(\int\frac{d\theta}{2\pi}\int dI\,d\mu\,\frac{\partial H}{\partial\theta}\bar{f}\right), \tag{31}$$

where

$$N(p_\theta)=\int \frac{d\theta}{2\pi}\int dI\, d\mu\, \bar{f}. \tag{32}$$

The integral over the collision operator vanishes because collisions conserve the number of particles.

To obtain a transport equation accurate to second order in $\partial H/\partial\theta$, we must obtain $\bar{f}$ accurate to first order in $\partial H/\partial\theta$. Thus, we look for the solution to Eq. (28) in the form

$$\bar{f}=\bar{f}_0(H,p_\theta)+\delta\bar{f}(I,p_\theta,\theta,\mu), \tag{33}$$

where $\delta\bar{f}/\bar{f}_0\sim \delta L/L_0$ and

$$\bar{f}_0=N(p_\theta)(2\pi T/m)^{-3/2}e^{e\Phi/T}e^{-H/T}. \tag{34}$$

Written in velocity variables, $\bar{f}_0$ is just a Maxwellian times the θ-averaged two-dimensional (2-D) density. Taking $\bar{f}_0$ in this form makes use of the frequency ordering, $\nu\gg\gamma$. Collisions are assumed to occur more rapidly than transport, so the zeroth-order distribution is a Maxwellian along the magnetic field.

Here $\delta\bar{f}$ is obtained from Eq. (28) written to first order in $\delta L/L_0$,

$$\frac{\partial(\delta\bar{f}\,)}{\partial t}+\bar{\omega}_R\frac{\partial(\delta\bar{f}\,)}{\partial\theta}+\left(\frac{\pi^2I^2}{2mL^2}\right)\frac{2}{L}\frac{\partial\delta L}{\partial\theta}\left(\frac{\bar{\omega}_R}{T}+\frac{1}{\bar{f}_0}\frac{\partial\bar{f}_0}{\partial p_\theta}\right)\bar{f}_0 =C[\bar{f}_0+\delta\bar{f}], \tag{35}$$

where

$$\bar{\omega}_R=\frac{\partial H}{\partial p_\theta}=\frac{\partial e\,\Phi}{\partial p_\theta}-\frac{\pi^2I^2}{2mL^2}\frac{2}{L}\frac{\partial L_0}{\partial p_\theta} \tag{36}$$

is the total bounce-averaged rotation frequency, and

$$\frac{1}{\bar{f}_0}\frac{\partial\bar{f}_0}{\partial p_\theta}=\frac{1}{N}\frac{\partial N}{\partial p_\theta}-\frac{3}{2}\frac{1}{T}\frac{\partial T}{\partial p_\theta}+\left(\frac{\pi^2I^2}{2mL^2}+B\mu\right)\frac{1}{T^2}\frac{\partial T}{\partial p_\theta} +\frac{1}{T}\left(\frac{\pi^2I^2}{2mL^2}\right)\frac{2}{L}\frac{\partial L_0}{\partial p_\theta}. \tag{37}$$

In general, the solutions to Eq. (35) consist of a sum of a driven response and free oscillations of the form

$$\delta\bar{f}=\sum_l \delta f_l e^{il(\theta-\bar{\omega}_R t)}. \tag{38}$$

Since the system has finite shear ($\partial\bar{\omega}_R/\partial p_\theta\neq 0$), these free oscillation terms become rapidly oscillating functions of p_θ at large t, and are rapidly damped by any diffusive transport processes. Since the driving terms vary on the slow transport time scale, the term $\partial\delta\bar{f}/\partial t$ may be dropped from Eq. (35), leaving

$$\bar{\omega}_R\frac{\partial\delta\bar{f}}{\partial\theta}+\left(\frac{\pi^2I^2}{2mL^2}\right)\frac{2}{L}\frac{\partial\delta L}{\partial\theta}\left(\frac{\bar{\omega}_R}{T}+\frac{1}{\bar{f}_0}\frac{\partial\bar{f}_0}{\partial p_\theta}\right)\bar{f}_0 =C[\bar{f}_0+\delta\bar{f}]. \tag{39}$$

Given the frequency ordering $\nu\ll\omega_R$, this equation may be solved perturbatively in the effective collision frequency. Dropping the collision operator term and integrating yields

$$\delta\bar{f}^{(0)}=-\frac{1}{\bar{\omega}_R}\left(\frac{\pi^2I^2}{2mL^2}\right)\left(\frac{2\ \delta L}{L_0}\right)\left(\frac{\bar{\omega}_R}{T}+\frac{1}{\bar{f}_0}\frac{\partial\bar{f}_0}{\partial p_\theta}\right)\bar{f}_0, \tag{40}$$

where the superscript indicates the ordering in collisions. The collisional response is obtained by inserting $\delta\bar{f}^{(0)}$ into the collision operator on the RHS of Eq. (39). This yields

$$\frac{\partial}{\partial\theta}(\partial\bar{f}^{(1)})=\frac{1}{\bar{\omega}_R}C\left\{\bar{f}_0\left[1-\frac{1}{\bar{\omega}_R}\left(\frac{\pi^2I^2}{2mL^2}\right)\left(\frac{2\ \delta L}{L_0}\right) \times\left(\frac{\bar{\omega}_R}{T}+\frac{1}{\bar{f}_0}\frac{\partial\bar{f}_0}{\partial p_\theta}\right)\right]\right\}. \tag{41}$$

Substituting $\bar{f}=\bar{f}_0+\delta\bar{f}^{(0)}+\delta\bar{f}^{(1)}$ into the transport equation [Eq. (31)] yields

$$\frac{\partial N(p_\theta)}{dt}=\frac{\partial}{\partial p_\theta}\left[\int\frac{d\theta}{2\pi}\int dI\,d\mu\,\frac{\pi^2I^2}{2mL^2} \times\left(-\frac{2}{L_0}\frac{\partial\delta L}{\partial\theta}\right)\delta\bar{f}^{(1)}\right], \tag{42}$$

where the collisionless terms have vanished in the integral over θ. Integrating by parts and substituting from Eq. (41) results in

$$\frac{\partial N(p_\theta)}{\partial t}=\frac{\partial}{\partial p_\theta}\left(\int\frac{d\theta}{2\pi}\int dI\,d\mu\,\frac{\pi^2I^2}{2mL^2}\frac{2\ \delta L}{L_0}\frac{1}{\bar{\omega}_R} \times C\left\{\bar{f}_0\left[1-\frac{1}{\bar{\omega}_R}\frac{\pi^2I^2}{2mL^2}\frac{2\ \delta L}{L_0} \times\left(\frac{\bar{\omega}_R}{T}+\frac{1}{\bar{f}_0}\frac{\partial\bar{f}_0}{\partial p_0}\right)\right]\right\}\right). \tag{43}$$

After changing variables of integration from (μ,I) to $(v_z,v_\perp)$, this equation may be written as

$$\frac{\partial N(p_\theta)}{\partial t}=\frac{\partial}{\partial p_\theta}\left(\int\frac{d\theta}{2\pi}\int d^3v\left(\frac{1}{2}mv_z^2\right)\frac{2\ \delta L}{L_0}\frac{1}{\bar{\omega}_R} \times C\left\{\bar{f}_0\left[1-\frac{1}{\bar{\omega}_R}\left(\frac{1}{2}mv_z^2\right)\frac{2\,\delta L}{L_0} \times\left(\frac{\bar{\omega}_R}{T}+\frac{1}{f_0}\frac{\partial\bar{f}_0}{\partial p_\theta}\right)\right]\right\}\right), \tag{44}$$

where

$$\bar{\omega}_R=\frac{\partial(e\Phi)}{\partial p_\theta}-\left(\frac{1}{2}mv_z^2\right)\frac{2}{L_0}\frac{\partial L_0}{\partial p_\theta}, \tag{45}$$

$$\frac{1}{\bar{f}_0}\frac{\partial\bar{f}_0}{\partial p_\theta}=\frac{1}{N}\frac{\partial N}{\partial p_\theta}-\frac{3}{2}\frac{1}{T}\frac{\partial T}{\partial p_\theta}+\left(\frac{1}{2}mv^2\right)\frac{1}{T^2}\frac{\partial T}{\partial p_\theta} +\left(\frac{1}{2}mv_z^2\right)\frac{1}{T}\frac{2}{L_0}\frac{\partial L_0}{\partial p_\theta}, \tag{46}$$

and

$$\bar{f}_0=N(p_\theta)f_{\rm M}, \tag{47}$$

with $f_{\rm M}$ a Maxwellian distribution. An expression very similar to the second term on the RHS of Eq. (45) was previously derived by Peurrung and Fajans.[18]

The velocity integral in this equation is simplified by assuming that the Debye length is small. This is typically the case in the experiments, and is consistent with our assumed form of the end potential. The ratio of the two terms in Eq. (45) scales as

$$\frac{\frac{1}{2}(mv_z^2)}{(\partial e\ \Phi/\partial p_\theta)}\frac{2}{L_0}\frac{\partial L_0}{\partial p_\theta}\sim\frac{T}{e\Phi}\sim\frac{\lambda_D^2}{r_p^2}, \tag{48}$$

where r_p is the radius of the plasma. We therefore neglect the velocity-dependent term in the rotation frequency, and take

$$\bar{\omega}_R\simeq\frac{\partial(e\Phi)}{\partial p_\theta}=\omega_R, \tag{49}$$

the usual **E×B** rotation frequency. Similarly, the ratio of the two terms inside the collision operator in Eq. (43) scales as

$$\frac{1}{(\bar{\omega}_R/T)}\frac{1}{\bar{f}_0}\frac{\partial\bar{f}_0}{\partial p_\theta}\sim\frac{T}{e\phi}\sim\frac{\lambda_D^2}{r_p^2}, \tag{50}$$

and therefore we will neglect the term, $1/\bar{f}_0\ \partial\bar{f}_0/\partial p_\theta$.

We take the collision operator in the general form

$$C[f]=\int d^3v_1\ d\sigma|v_{\rm rel}|[f(v_1')f(v')-f(v)f(v_1)], \tag{51}$$

where $d\sigma$ is the differential cross section and $v_{\rm rel}=v-v_1$. Using this form and the small Debye length approximation, we obtain

$$\frac{\partial N(p_\theta)}{\partial t}=-\frac{\partial}{\partial p_\theta}\left\{\int\frac{d\theta}{2\pi}\left(\frac{2\ \delta L}{L_0}\right)^2\frac{N}{\omega_R T}N\int d^3v\left(\frac{1}{2}mv_z^2\right)\times\int d^3v_1\ d\sigma|v_{\rm rel}|\left[\left(\frac{1}{2}mv_{z1}'^2+\frac{1}{2}mv_z'^2\right)f_{\rm M1}'f_{\rm M}'-\left(\frac{1}{2}mv_{z1}^2+\frac{1}{2}mv_z^2\right)f_{\rm M1}f_{\rm M}\right]\right\}. \tag{52}$$

To evaluate the velocity integral in this equation, it is instructive to consider the collisional relaxation of an anisotropic Maxwellian distribution,

$$f_{\rm A}(v)=\left(\frac{2\pi T_\parallel}{m}\right)^{-1/2}\left(\frac{2\pi T_\perp}{m}\right)^{-1}\times\exp\left(\frac{-\frac{1}{2}mv_z^2}{T_\parallel}-\frac{\frac{1}{2}mv_\perp^2}{T_\perp}\right). \tag{53}$$

The change in the parallel temperature due to collisions is given by

$$\frac{d}{dt}\left(\frac{T_\parallel}{2}\right)=N\int d^3v\left(\frac{1}{2}mv_z^2\right)\int d^3v_1\ d\sigma|v_{\rm rel}|[f_{\rm A}(v_1')f_{\rm A}(v')-f_{\rm A}(v_1)f_{\rm A}(v)]=\nu_{\perp,\parallel}(T_\perp-T_\parallel), \tag{54}$$

and may be used as a definition of the equipartition rate, $\nu_{\perp,\parallel}$. Consider the case $T_\perp=T$ and $T_\parallel=(1-\alpha)T$. Substituting this into Eq. (53) and taking the limit $\alpha\rightarrow 0$, one can easily show

$$N\int d^3v\left(\frac{1}{2}mv_z^2\right)\int d^3v_1\ d\sigma|v_{\rm rel}|\left[\left(\frac{1}{2}mv_{z1}'^2+\frac{1}{2}mv_z'^2\right)\times f_{\rm M1}'f_{\rm M}'-\left(\frac{1}{2}mv_{z1}^2+\frac{1}{2}mv_z^2\right)f_{\rm M1}f_{\rm M}\right]=-T^2\nu_{\perp,\parallel}. \tag{55}$$

This is precisely the integral that appears in the transport equation. Equation (52) can now be written in the simple form

$$\frac{\partial N(p_\theta)}{\partial t}=-\frac{\partial}{\partial p_\theta}\left[4\nu_{\perp,\parallel}\left\langle\frac{\delta L^2}{L_0^2}\right\rangle_\theta\left(\frac{T}{-\omega_R}\right)N\right]. \tag{56}$$

Changing variables from (p_θ,θ) to (r,θ) yields the result

$$\frac{\partial N(r)}{\partial t}=-\frac{1}{r}\frac{d}{dr}r\left(4\nu_{\perp,\parallel}\left\langle\frac{\delta L^2}{L_0^2}\right\rangle_\theta\frac{T}{-e\ \partial\Phi/\partial r}N\right). \tag{57}$$

The quantity in brackets is the radial particle flux, and is identical to the flux given by Eq. (12) in the introduction.

III. RESONANT PARTICLE TRANSPORT

In the previous section, we derived a transport equation in the adiabatic limit by assuming that $\omega_B\gg\omega_R$ for every particle in the system. This approach neglects the effect of particles that satisfy the resonance condition, $l\omega_R=2n\omega_B$, where n and l are small integers. When the bounce frequency for a thermal particle, $\bar{\omega}_B=(\pi/L_0)\sqrt{T/m}$, is comparable to the rotation frequency, and collisions are sufficiently weak, resonant particle transport dominates. In this regime, the damping rate of the $m=1$ diocotron mode is larger and scales differently than the damping rate given in Eq. (19). A similar effect occurs for transport in tandem mirrors in the "resonant-plateau" regime.[19,20]

The basic idea behind resonant particle transport is easy to understand. When a particle is reflected off the nonaxisymmetric end potential it experiences a force in the $\hat{\theta}$ direction, causing its angular momentum, $p_\theta=(eB/2c)r^2$, and radial position to change. The magnitude and direction of the radial step depends on the particle's azimuthal position at the point of reflection. In addition, fast particles take larger steps, because a larger force is required to reflect them. Consider a particle satisfying the lowest-order resonance condition $\omega_R=2\omega_B$. Such a particle reflects off each end of the plasma at the same θ position for many bounces and consequently takes many radial steps in the same direction. For nonresonant particles, the radial steps tend to cancel.

When collisions are sufficiently weak resonant particle transport is always present. The size of the transport is determined by the relative number of resonant particles and by the contribution from each resonant particle. In the adiabatic limit low-order bounce-rotation resonances are at low velocities. Although there are a large number of resonant particles, the contribution from each particle is small because the radial steps are small and infrequent. In this regime, resonant particle transport is negligible. In the opposite limit where $\omega_R\gg\bar{\omega}_B$, and the resonance is located at a large velocity, the contribution from each particle is large, but there are few particles on the tail of the Maxwellian, which interact resonantly. We will see from the formal treatment that resonant particle transport is important when $\bar{\omega}_B \approx \omega_R$.

Our starting point is again the drift-kinetic Boltzmann equation,

$$\frac{\partial f}{\partial t} + [f,H] = C(f), \tag{58}$$

where

$$H = \frac{p_z^2}{2m} + e\Phi(p_\theta) + B\mu + e\Phi_e(\theta,p_\theta,z). \tag{59}$$

The calculation is simplified by considering a system that consists of only one-half of the plasma, that is, we take

$$\Phi_e(\theta,p_\theta,z) = \begin{cases} 0, & 0<z<\frac{1}{2}L(\theta,p_\theta), \\ \infty, & \text{otherwise.} \end{cases} \tag{60}$$

Since the particles specularly reflect off a plane at $z=0$ without changing θ, p_θ, z, or $|v_z|$, the transport equation will be the same for this system as for the full length system.

Writing the poisson bracket in Eq. (58) as

$$[f,H] = \frac{\partial}{\partial z}\left(f\frac{\partial H}{\partial p_z}\right) - \frac{\partial}{\partial p_z}\left(f\frac{\partial H}{\partial z}\right) + \frac{\partial}{\partial \theta}\left(f\frac{\partial H}{\partial p_\theta}\right) - \frac{\partial}{\partial p_\theta}\left(f\frac{\partial H}{\partial \theta}\right), \tag{61}$$

and integrating over all variables except p_θ gives the result

$$\frac{\partial N(p_\theta)}{\partial t} = \frac{\partial}{\partial p_\theta}\left(\int \frac{d\theta}{2\pi}\int d^2v_\perp \int dv_z \int dz_f \frac{\partial H}{\partial \theta}\right), \tag{62}$$

where

$$N(p_\theta) = \int \frac{d\theta}{2\pi}\int d^2v_\perp \int dv_z \int dz\, f. \tag{63}$$

Solutions to Eq. (58) are assumed to take the form

$$f = f_0(H,p_\theta,t) + \delta f(p_z,z,p_\theta,\theta,t), \tag{64}$$

where $\delta f/f_0 \sim \delta L/L_0$ and

$$f_0 = \frac{N(p_\theta)}{(\frac{1}{2}L_0)(2\pi T/m)^{3/2}}\, e^{e\Phi/T} e^{-H/T}. \tag{65}$$

In velocity variables, this is a Maxwellian distribution. As in Sec. II, we neglect terms higher order in λ_D^2/r_p^2 by assuming L_0, T, and N are constant in p_θ.

To first order in $\delta L/L_0$, the drift-kinetic Boltzmann equation is

$$\frac{\partial(\delta f)}{\partial t} + [\delta f,H_0] - C(f_0+\delta f) = -[f_0,H], \tag{66}$$

where H_0 is the Hamiltonian with $\delta L=0$. The first two terms on the left-hand side can be thought of as a derivative along the unperturbed orbit,

$$\frac{\partial(\delta f)}{\partial t} + [\delta f,H_0] = \frac{d^{(0)}}{dt}(\delta f). \tag{67}$$

We approximate the collision operator by

$$C(f_0+\delta f) = -\nu\,\delta f, \tag{68}$$

where ν is an effective collision frequency. Clearly, this oversimplifies the transport problem in the adiabatic limit, because there is no reason to expect $\nu = \nu_{\perp,\parallel}$. For resonant particle transport, however, we will let $\nu \to 0$ at the end of the calculation, and the effective collision frequency drops out. We do not expect this transport to depend sensitively on the detailed nature of the collision operator.

Evaluating the RHS of Eq. (66), and using Eqs. (67) and (68) gives

$$\frac{d^{(0)}}{dt}\delta f + \nu\,\delta f = \frac{\omega_R}{T} f_0 \frac{\partial(e\Phi_e)}{\partial \theta}. \tag{69}$$

The solution to this equation is

$$\delta f(t) = \int_0^t dt'\, e^{-\nu(t-t')}\left(\frac{\omega_R}{T} f_0 \frac{\partial(e\Phi_e)}{\partial\theta}\right)', \tag{70}$$

where the prime indicates evaluation along the unperturbed orbit. To this order in δL, f_0 is constant along the unperturbed orbit and may be factored out of the integral, so that

$$\delta f(t) = \frac{\omega_R}{T} f_0 \int_0^t dt'\, e^{-\nu(t-t')} \frac{\partial(e\Phi_e)}{\partial\theta'}. \tag{71}$$

Consider Eq. (62), the transport equation. Since $\partial H/\partial\theta = 0$ everywhere except at the end of the plasma, we need to find δf only at the end of the plasma, where the particles are reflected. Therefore, it is convenient to write Eq. (71) as

$$\delta f(t) = \frac{\omega_R}{T} f_0 \int_0^{t_0} dt'\, e^{-\nu(t-t')} \frac{\partial(e\Phi_e)}{\partial\theta'} + \frac{\omega_R}{T} f_0 \int_{t_0}^t dt' \frac{\partial e\,\Phi_e}{\partial\theta'}, \tag{72}$$

where t_0 indicates the time just before the reflection. The first term consists of a sum of many reflections, and the second term is due to a single partial reflection.

We delay evaluation of the second term until δf is inserted into the transport equation. We evaluate the first term by calculating the effect of a single reflection and then summing over many reflections. The axial impulse exerted by the end potential on a particle during reflection is

$$-2|p_z^0| = -\int_{\text{turn}} dt' \frac{\partial(e\Phi_e)}{\partial z'}. \tag{73}$$

Since the end potential is only a function of the quantity $[z-\frac{1}{2}L(\theta,p_\theta)]$, this may be rewritten as

$$-2|p_z^0| = \frac{2}{\partial L/\partial\theta}\int_{\text{turn}} dt' \frac{\partial e\,\Phi_e}{\partial\theta'}. \tag{74}$$

Since $-\partial(e\Phi_e)/\partial\theta = \dot{p}_\theta$, the change in p_θ due to the reflection is

$$\Delta p_\theta = -\int_{\text{turn}} dt' \frac{\partial(e\Phi_e)}{\partial\theta'} = +|p_z^0| \frac{\partial(\delta L)}{\partial\theta}. \tag{75}$$

This equation has a simple physical interpretation. The force that reflects a particle is normal to the surface at the end of the plasma. Due to the asymmetry, there is a small component of this force in the $\hat{\theta}$ direction, which exerts a torque on the particle causing p_θ to change. A larger force is needed to reflect fast particles, and therefore these particles take larger steps in p_θ.

The first term in Eq. (72) can now be written as

$$\delta f_1 = \frac{-\omega_R}{T} f_0 \int_0^{t_0} dt' \, e^{-\nu(t-t')} \sum_{j=1}^{N} |p_z^0| \frac{\partial(\delta L)}{\partial \theta'} \times \delta(t'-t_j), \tag{76}$$

where the index j is summed over past reflections. The time at the jth reflection is given simply by

$$t_j = t - \frac{L_0}{|v_z^0|} j, \tag{77}$$

and along unperturbed orbits,

$$\theta' = \theta - \omega_R(t-t'). \tag{78}$$

After substituting in a Fourier series for δL, Eq. (76) becomes

$$\delta f_1 = \frac{-\omega_R}{T} f_0 |p_z^0| \sum_l il \, \delta L_l(p_\theta) e^{il\theta} \sum_{j=1}^{N} e^{-j(il\omega_R+\nu)L_0/|v_z^0|}. \tag{79}$$

This contribution to δf arises from a series of discrete "kicks" acting on f_0. Note that those kicks that occurred in the distant past (large j) are collisionally damped.

The sum over j can be evaluated exactly, so that

$$\delta f_1 = \frac{\omega_R}{T} f_0 |p_z^0| \sum_l il \, \delta L_l e^{il\theta} \left(\frac{e^{-(il\omega_R+\nu)L_0/v_z^0 N} - 1}{e^{i(l\omega_R - i\nu)L_0/|v_z^0|} - 1} \right). \tag{80}$$

The exponential in the numerator can be dropped because

$$\nu \frac{L_0}{v_z^0} N \sim \nu t \sim \frac{\nu}{\gamma} \gg 1. \tag{81}$$

Using the mathematical identity

$$\frac{1}{e^{iz}-1} = -\frac{1}{2} - i \sum_{n=-\infty}^{\infty} \frac{1}{z - 2\pi n}, \tag{82}$$

the full fluctuation distribution is written as

$$\delta f(t) = \frac{\omega_R}{T} f_0 |p_z^0| \sum_l il \, \delta L_l e^{il\theta} \times \left(\frac{1}{2} + i \sum_n \frac{1}{(l\omega_R - i\nu)L_0/|v_z^0| - 2\pi n} \right) + \frac{\omega_R}{T} \int_{t_0}^{t} dt' \, \frac{\partial(e\Phi_e)}{\partial \theta'}. \tag{83}$$

Inserting $f = f_0 + \delta f$ into Eq. (62) gives

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial p_\theta} \left(\int \frac{d\theta}{2\pi} \int d^2 v_\perp \int dv_z \int dz \, \delta f \frac{\partial e\,\Phi_e}{\partial \theta} \right), \tag{84}$$

where f_0 has vanished in the integral over θ. Since we have calculated δf as a function of time, it is useful to transform this integral into an integral over unperturbed orbits. First, note that the integrand is nonzero only at the end of the plasma, and so the transformation need only be valid for the short turning time. If a particle has phase space coordinates $(\theta^0, p_\theta^0, z^0, p_z^0)$ at time t_0 at some position just before the end of the plasma, the unperturbed orbits valid during the turn are

$$\theta = \theta^0, \tag{85a}$$

$$p_\theta = p_\theta^0, \tag{85b}$$

$$p_z = p_z(p_z^0, z^0, t), \tag{85c}$$

$$z = z(p_z^0, z^0, t). \tag{85d}$$

We change variables in the integral using

$$dz\,dp_z = \left| \frac{\partial(z,p_z)}{\partial(t,p_z^0)} \right| dt\,dp_z^0 = \left| \frac{\partial z}{\partial t} \frac{\partial p_z}{\partial p_z^0} - \frac{\partial p_z}{\partial t} \frac{\partial z}{\partial p_z^0} \right| dt\,dp_z^0. \tag{86}$$

The equations of motion are derived from the unperturbed Hamiltonian, H_0, and therefore this may be written as

$$dz\,dp_z = \left| \frac{\partial H_0}{\partial p_z} \frac{\partial p_z}{\partial p_z^0} + \frac{\partial H_0}{\partial z} \frac{\partial z}{\partial p_z^0} \right| dt\,dp_z^0. \tag{87}$$

The quantity in absolute value bars is just $\partial H_0/\partial p_z^0$, and therefore

$$dz\,dp_z = \left| \frac{\partial H_0}{\partial p_z^0} \right| dt\,dp_z^0 = |v_z^0| dt\,dp_z^0. \tag{88}$$

The transport equation can now be written in the form

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial p_\theta} \left(\int \frac{d\theta}{2\pi} \int d^2 v_\perp \int_0^\infty dv_z^0 |v_z^0| \int_{t_0}^{t_f} dt' \, \delta f' \frac{\partial H}{\partial \theta'} \right), \tag{89}$$

where the primes indicate evaluation along the unperturbed orbit, and t_f is the time just after the turn.

Using δf from Eq. (83), this becomes

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial p_\theta} \left\{ \int \frac{d\theta}{2\pi} \int d^2 v_\perp \int_0^\infty dv_z^0 \, v_z^0 \int_{t_0}^{t_f} dt' \, \frac{\partial(e\Phi_e)}{\partial \theta'} \times \left[\frac{\omega_R}{T} f_0' \int_{t_0}^{t'} dt'' \, \frac{\partial(e\Phi_e)}{\partial \theta''} + \frac{\omega_R}{T} f_0' p_z^0 \sum_l il \, \delta L_l e^{il\theta} \times \left(\frac{1}{2} + i \sum_n \frac{1}{(l\omega_R - i\nu)L_0/v_z^0 - 2\pi n} \right) \right] \right\}. \tag{90}$$

To this order, f_0 is constant along the orbit and may be pulled out of the integrals. In velocity variables, $f_0(t=t_0)$ is given by

$$f_0 = \frac{N(p_\theta)}{(\frac{1}{2}L_0)} f_{\mathrm{M}}, \tag{91}$$

where f_{M} is a Maxwellian.

The time integrals can be expressed in terms of Δp_θ as

$$\int_{t_0}^{t_f} dt' \frac{\partial(e\Phi_e)}{\partial\theta'} = -\Delta p_\theta \quad (92)$$

and

$$\int_{t_0}^{t_f} dt' \frac{\partial(e\Phi_e)}{\partial\theta'} \int_{t_0}^{t'} dt'' \frac{\partial(e\Phi_e)}{\partial\theta''} = \frac{1}{2}\Delta p_\theta^2. \quad (93)$$

Using Δp_θ from Eq. (75), Eq. (90) becomes

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial p_\theta}\left(\int \frac{d\theta}{2\pi}\int d^2 v_\perp \int_0^\infty dv_z^0\, m^2 (v_z^0)^3 f_0 \times \frac{\omega_R}{T}\sum_l il\, \delta L_l e^{il\theta} \sum_{l'} il'\, \delta L_{l'} e^{il'\theta} \times \sum_n \frac{i}{(l\omega_R - i\nu)L_0/v_z^0 - 2\pi n}\right). \quad (94)$$

After integrating over $v_\perp$ and θ and dropping the imaginary part of the integral, this becomes

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial p_\theta}\left(-\frac{2\omega_R}{T} N(p_\theta)\int_0^\infty dv_z^0\, m^2 (v_z^0)^4 f_M \times \sum_{l,n} l^2 \frac{|\delta L_l|^2}{L_0^2}\frac{\nu}{(l\omega_R - 2n\omega_B)^2 + \nu^2}\right), \quad (95)$$

where we have introduced $\omega_B = \pi|v_z|/L_0$.

First, consider the $n=0$ term in the sum. This may be thought of as the transport due to a resonance located at $v_z = \infty$, and corresponds physically to transport in the adiabatic limit. Evaluating the integral over v_z for this term yields

$$\frac{\partial N^{\text{adiabatic}}}{\partial t} = -\frac{\partial}{\partial p_\theta}\left[3\nu \frac{\langle \delta L^2\rangle_\theta}{L_0^2}\left(\frac{T}{-\omega_R}\right) N(p_\theta)\right]. \quad (96)$$

Identifying ν with $\frac{4}{3}\nu_{\perp,\parallel}$, we have the same result obtained in Sec. II. If we had kept ν as an operator, we would have recovered $\nu_{\perp,\parallel}$.

Now consider the terms in the sum for $n>0$. Since we are working in the small ν limit, we approximate

$$\frac{\nu}{(l\omega_R - 2n\omega_B)^2 + \nu^2} \simeq \pi\delta(l\omega_R - 2n\omega_B). \quad (97)$$

The factor of 2 appears because particles are reflected at both ends of the plasma. Particles with $\omega_B = \omega_R$, for example, may step radially outward at one end of the plasma, but will step inward at the other end.

Using the approximation of Eq. (97), the transport equation becomes

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial p_\theta}\left\{4\left[\frac{1}{64}\sqrt{\frac{\pi}{2}}\sum_{l,n}\frac{(l\omega_R)^6}{(n\bar{\omega}_B)^5}\frac{|\delta L_l|^2}{L_0^2} \times \exp\left(-\frac{l^2\omega_R^2}{8n^2\bar{\omega}_B^2}\right)\right]\frac{T}{-\omega_R}N(p_\theta)\right\}, \quad (98)$$

where $\bar{\omega}_B = \pi\sqrt{T/m}/L_0$ is the mean bounce frequency in the plasma. The size of the resonant particle transport is determined by two competing effects. Consider the case where the resonance is located at large v_z. The resonant particles take large radial steps as the end potential must exert a large force in order to reflect them. Furthermore, these fast moving particles are reflected very frequently. While these effects tend to increase the transport, the location of the resonance on the tail of the Maxwellian ensures that there are relatively few resonant particles. Similarly, when the resonance is located at small v_z, the contribution from each particle is small, but there are many particles that interact resonantly.

For moderate temperature plasmas, the $n=1$ term in Eq. (98) is largest. To determine which terms in the sum over l are largest we estimate the size of the Fourier components δL_l. The end shape of the plasma is axisymmetric about the central axis of the trap and by considering the shape of the vacuum equipotential contours we expect the radius of curvature of the end shape to be proportional to R_w, the radius of the conducting wall. If we model the end of the plasma as the intersection of a cylinder and a hemisphere of radius R_w, we find that the length of the plasma parallel to the magnetic field is

$$L(r,\theta) = L_0 - \sqrt{R_w^2 - (D^2 + 2\,Dr\cos\theta + r^2)}, \quad (99)$$

where D is the displacement of the center of charge off axis and (r,θ) is a cylindrical coordinate system centered on an axis through the center of charge. Taylor expanding this expression in the limit $Dr \ll R_w^2$, we find that

$$\delta L_l \propto R_w\left(\frac{Dr}{R_w^2}\right)^{|l|}. \quad (100)$$

Note that this agrees with the scaling given by Eq. (5) for a flat end ($l=\pm1$). In the experiments D/R_w and r/R_w are typically very small, and therefore the terms in Eq. (98) with $l \neq \pm1$ are negligible, despite the coefficient $\sim l^6$. Keeping only the $n=1$ and $l=\pm1$ terms and changing variables from (p_θ,θ) to (r,θ), the transport equation can be written as

$$\frac{\partial N(r)}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r} r\left\{4\left[\frac{1}{64}\sqrt{\frac{\pi}{2}}\frac{\omega_R^6}{\bar{\omega}_B^5}\exp\left(-\frac{\omega_R^2}{8\bar{\omega}_B^2}\right)\right] \times \left\langle\frac{\delta L^2}{L_0^2}\right\rangle_\theta \frac{T}{-e\,\partial\Phi/\partial r}N\right\}. \quad (101)$$

This result may be understood physically by considering the orbit of a single resonant particle. In one reflection, the particle takes a radial step as given by Eq. (75) [recall $p_\theta = (eB/2c)r^2$]. The particle takes approximately ω_B/ν of these steps before being converted to a nonresonant particle, so that the fundamental step size governing the transport is

$$\Delta r = \frac{\omega_B}{\nu}\left(\frac{c}{eBr} m V_z \frac{\partial \delta L}{\partial\theta}\right). \quad (102)$$

One can estimate the size of the diffusion coefficient as the average of the step size squared times the rate at which particles take steps, that is,

$$D = \nu\langle(\Delta r)^2\rangle_\theta. \quad (103)$$

The radial particle flux is given by

$$\Gamma = -D\left(\frac{\partial f_0}{\partial r}\right)_H \Delta V, \tag{104}$$

where the distribution function and the diffusion coefficient are to be evaluated at the resonant velocity; ΔV is the width of the resonance in velocity space and (along with the distribution function) indicates the relative number of particles that participate in the resonant interaction. For the case $n=1$, $l=\pm 1$, the resonance condition is

$$2\omega_B \pm \omega_R + i\nu = 0, \tag{105}$$

so the width of the resonance may be estimated to be

$$\Delta V = \frac{L_0}{\pi}\Delta\omega_B = \frac{L_0}{2\pi}\nu. \tag{106}$$

After some algebra, the estimate for the flux can be written as

$$\Gamma = \left[\frac{1}{64}\sqrt{2\pi^3}\frac{\omega_R^6}{\bar{\omega}_B^5}\exp\left(-\frac{\omega_R^2}{8\bar{\omega}_B^2}\right)\right] \times \left\langle\frac{\delta L^2}{L_0^2}\right\rangle_\theta \frac{T}{-e\,\partial\Phi/\partial r} N. \tag{107}$$

Except for a numerical coefficient, this expression agrees with the flux given in Eq. (101).

To find the damping rate of the $m=1$ diocotron mode, we would again use conservation of angular momentum [Eq. (14)]. The ratio of the resonant particle damping rate to the adiabatic damping rate is given by

$$\frac{\gamma^r}{\gamma^a} = \frac{1}{64}\sqrt{\frac{\pi}{2}}\frac{\omega_R^6}{\bar{\omega}_B^5 \nu_{\perp,\parallel}}\exp\left(-\frac{\omega_R^2}{8\bar{\omega}_B^2}\right). \tag{108}$$

When this ratio becomes larger than 1, we expect that resonant particle damping will dominate. In typical experiments, $\bar{\omega}_B \gg \omega_R$, and therefore resonance effects are negligible. Even when $\gamma^r/\gamma^a > 1$, we must be sure that the frequency ordering $\nu \ll \omega_B$ strictly holds. Even moderate collisionality will destroy the resonance effect. In the limit $\nu > \omega_B$, a fluid-like treatment is more appropriate.

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[1]J. H. Malmberg and J. S. deGrassie, Phys. Rev. Lett. **35**, 577 (1975).
[2]J. S. deGrassie and J. H. Malmberg, Phys. Fluids **23**, 63 (1980).
[3]C. F. Driscoll and K. S. Fine, Phys. Fluids B **2**, 1359 (1990).
[4]J. Tan and G. Gabrielse, Phys. Rev. Lett. **67**, 3090 (1991).
[5]D. R. Nicholson, *Introduction to Plasma Theory* (Wiley, New York, 1983).
[6]R. H. Levy, Phys. Fluids **8**, 1288 (1965); **11**, 920 (1968).
[7]W. D. White, J. H. Malmberg, and C. F. Driscoll, Phys. Rev. Lett. **49**, 1822 (1982).
[8]K. S. Fine, C. F. Driscoll, and J. H. Malmberg, Phys. Rev. Lett. **63**, 2232 (1989).
[9]B. Cluggish and C. F. Driscoll. Bull. Am. Phys. Soc. **37**, 1972 (1993).
[10]D. D. Ryutov and G. V. Stupakov, Fiz. Plazmy **4**, 521 (1978).
[11]R. J. Briggs, J. D. Dougherty, and R. H. Levy, Phys. Fluids **13**, 421 (1970).
[12]J. H. Malmberg and T. M. O'Neil, Phys. Rev. Lett. **39**, 1333 (1977).
[13]A. J. Peurrung and J. Fajans, Phys. Fluids B **5**, 11 (1993).
[14]J. M. Berger, W. A. Newcomb, J. M. Dawson, E. A. Frieman, R. M. Kulsrud, and A. Lenard, Phys. Fluids **1**, 301 (1958).
[15]M. E. Glinsky, T. M. O'Neil, M. N. Rosenbluth, K. Tsuruta, and S. Ichimaru, Phys. Fluids B **4**, 1156 (1992).
[16]B. R. Beck, J. Fajans, and J. H. Malmberg, Phys. Rev. Lett. **68**, 317 (1992).
[17]J. B. Taylor, Phys. Fluids **7**, 767 (1964).
[18]A. J. Peurrung and J. Fajans, Phys. Fluids B **5**, 4295 (1993).
[19]D. D. Ryutov and G. V. Stupakov, Dokl. Akad. Nauk. SSSR **240**, 1086 (1978) [Sov. Phys. Dokl. **23**, 412 (1978)].
[20]R. H. Cohen, Nucl. Fusion **19**, 1579 (1979).