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Criterion for the sign of wave energy ⓢ ⓕ

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Thomas M. O'Neil

AFFILIATIONS

Department of Physics, University of California at San Diego, La Jolla, California 92093-0319, USA

ABSTRACT

A criterion for the sign of wave energy is developed by using the symmetry properties of the plasma equilibrium and the fact that Vlasov dynamics is an incompressible flow in phase space, rather than the usual and more difficult procedure of calculating the value of the wave energy directly. Applications are made to the case of waves excited on a non-neutral plasma in a Malmberg–Penning trap and to waves excited on an infinitely long non-neutral beam.

I. INTRODUCTION

A plasma wave is said to have positive energy if energy must be added to the plasma when the wave is excited. Likewise, a wave is said to have negative energy if energy must be removed from the plasma when the wave is excited.[1,2] Since energy is reference frame dependent, the sign of wave energy is reference frame dependent. This paper considers weakly damped, electrostatic waves that propagate on a stable non-neutral plasma, and establishes criterion that the waves have negative energy as viewed in the laboratory reference frame.

The sign of wave energy is of interest because a negative energy wave can become unstable when it has access to an energy sink. The wave grows as it transfers energy to the sink. The resistive wall instability of the diocotron wave is a well-known example.[3] Also, negative energy waves and a positive energy wave can couple nonlinearly yielding a nonlinear instability, where the positive energy wave plays the role of the energy sink.[2,4] Linear instabilities, such as the beam plasma instability, are often described as resulting from the interaction of a negative and a positive energy wave. What is meant in this case is that there is a negative energy wave on the beam alone and a positive energy wave on a lab frame plasma alone, and when both the beam and plasma are present simultaneously, the negative energy wave and positive energy wave couple linearly forming a new unstable wave.

The usual way to calculate the sign of wave energy is to calculate the wave energy directly using second order perturbation theory. The calculation must be carried to second order because the wave energy is second order in wave amplitude. For analytic tractability, the calculation often assumes a particularly simple and idealized equilibrium configuration and some simplifying dynamical model.

Here, we use a different approach, which determines the sign of wave energy without calculating the value of the wave energy. This approach relies on the Vlasov–Poisson stability theorem of Davidson

and Krall[5] and on the symmetry properties of the equilibrium configuration. The stability theorem itself depends on the fact that Vlasov dynamics is an incompressible flow in phase space.

The closest previous work to this approach is that by Sturrock,[1] who focused on beam waves in a traveling wave tube. He showed that the sign of wave energy could be determined by general considerations rather than a direct calculation of wave energy. However, his work preceded much of modern plasma theory and did not mention a kinetic description of the waves. The present paper can be thought of as a modernization and extension of Sturrock's original ideas.

As a first example, we consider weakly damped, azimuthally propagating, electrostatic waves on a non-neutral plasma that is confined in a Malmberg–Penning trap.[6] The plasma in such a trap is radially confined by a uniform axial magnetic field and axially confined by electrostatic fields. The confinement geometry of a Malmberg–Penning trap is characterized by two symmetries: invariance under translations in time and invariance under translations in θ, where (r, θ, z) is a cylindrical coordinate system with the z-axis coincident with the axis of the trap. Using these symmetries and the fact that Vlasov dynamics is an incompressible flow in phase space, Davidson and Krall found that a class of rotating plasmas are stable under Vlasov–Poisson dynamics.[5] The particle distributions for these plasmas are monotonically decreasing functions of the Hamiltonian in the rotating frame. Important examples of such distributions are the thermal equilibrium distributions, where the distribution is an exponentially decreasing function of the Hamiltonian.[5,7,8]

We assume that the plasma is in a Davidson–Krall equilibrium characterized by rotation frequency ω_r, when a weakly damped electrostatic wave of the form $\psi(r, z)\cos(l\theta - \omega t)$ is excited on the plasma. The form of the wave, standing in r and z and propagating in θ, is dictated by the symmetry properties of the equilibrium. We will

see that the wave energy, as viewed in the laboratory reference frame, is negative if the ratio $(\omega - l\omega_r)/\omega$ is negative. This simple criterion applies to all of the weakly damped, electrostatic waves that can be excited on the plasma, including diocotron waves, Trivelpiece-Gould waves, cyclotron waves, etc.[5,9,10] Also, the criterion requires no knowledge of the eigenfunction $\psi(r, z)$.

A second example adds another symmetry: invariance under translation in z. We consider an infinitely long, non-neutral beam that is confined radially by a uniform axial magnetic field, a system with invariance under translations in t, θ, and z. Davidson and Krall showed that such a beam is stable under Vlasov–Poisson dynamics if the particle distribution is a monotonic decreasing function of the Hamiltonian in a frame that translates axially with velocity u and rotates with angular frequency ω_r. We will see that a weakly damped, electrostatic wave on the beam has negative energy, as seen in the laboratory frame, when the ratio $(\omega - l\omega_r - ku)/\omega$ is negative. Here, the wave propagates both azimuthally and axially, and l and k are the azimuthal and axial mode numbers. This result is a generalization of Sturrock's criterion for negative energy waves on a cold beam, $(\omega - ku)/\omega < 0$.

A third example returns to the finite length plasma in a Malmberg–Penning trap but simplifies the dynamics to 2D $E \times B$ drift dynamics. For the uniform magnetic field of the Malmberg–Penning trap, the density evolves incompressibly under 2D $E \times B$ drift dynamics. When the equilibrium plasma is a monotonically decreasing function of radius, all weakly damped, electrostatic waves described by 2D $E \times B$ drift dynamics have negative energy.

Also, there are non-neutral plasma equilibria that do not have cylindrical symmetry but are stable under 2D $E \times B$ drift dynamics, and we will see that all of the low frequency drift modes on these plasmas have negative energy.

II. WAVES ON A PLASMA CONFINED IN A MALMBERG–PENNING TRAP

The confinement region of a Malmberg–Penning trap is bounded by a conducting cylindrical shell that is divided axially into three sections, as shown in Fig. 1.[6–8] The figure illustrates the case of a non-neutral plasma composed of positive charges, where the central section is held at ground potential and the two end sections are held at a positive potential, with the resulting end electric fields providing axial confinement. Also, there is a uniform axial magnetic field $\boldsymbol{B} = \hat{z} B$, and the Lorentz force $\boldsymbol{J} \times \boldsymbol{B}/c$ provides the radial confinement. Here, the current $\boldsymbol{J}$ is due to plasma rotation. In equilibrium, the radially inward Lorentz force balances the radially outward space charge force, pressure gradient force, and centrifugal force. For the Lorentz force to be radially inward for a plasma of positive charges, the rotation frequency must be in the negative sense, relative to the magnetic field and the
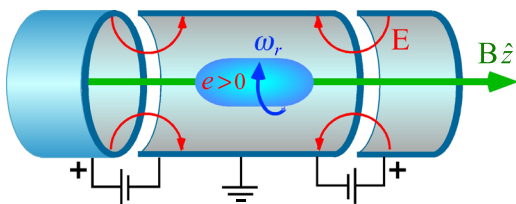

FIG. 1. Sketch of a Malmberg–Penning trap.

z-axis. For a plasma of negative charges, the rotation is in the positive sense.

During the wave launching process, the plasma dynamics is assumed to be well described by the Vlasov–Poisson equations

$$\frac{\partial f}{\partial t} + [f, H] = 0, \tag{1}$$

$$\nabla^2 \varphi_p = -4\pi e \int d^3 v f = -\frac{4\pi e}{m^3 r} \int dp_r dp_\theta dp_z f, \tag{2}$$

where

$$H = \left[\frac{p_r^2}{2m} + \frac{\left(p_\theta - \frac{eBr^2}{2c} \right)^2}{2mr^2} + \frac{p_z^2}{2m} \right] + e\phi_p(r, \theta, z, t) + e\phi_{ext}(r, \theta, z, t) \tag{3}$$

is the single particle Hamiltonian and the quantity $[f, H]$ is a Poisson bracket.[7,8,11] The quantities $[r, p_r = mr]$, $[\theta, p_\theta = mr^2\dot{\theta} + eBr^2/2c]$ and $[z, p_z = m\dot{z}]$ are canonically conjugate coordinate and momentum pairs. The quantities e and m are the charge and mass of the plasma particles. The plasma space charge potential, φ_p, satisfies Poisson's equation and vanishes on the cylindrical conducting wall. The quantity ϕ_{ext} is an external potential not produced by the plasma charge density. For the uniform axial magnetic field of the Malmberg–Penning trap, the vector potential is given by $A_\theta = Br^2/2$.

One may ask why we use the full Vlasov equation when analysis of low frequency waves on a magnetically confined non-neutral plasma often employs the drift-kinetic equation as an approximation to the Vlasov equation. For the typical operating regime, the cyclotron radius is small enough to justify the approximation. However, here there is no added complication in retaining the full Vlasov equation. Of course, the drift-kinetic equation is included as a limiting form of the Vlasov equation, and also the Vlasov equation can treat higher frequency modes such as cyclotron modes and is applicable near the Brillouin limit,[5] where the effective magnetic field in the rotating frame of the plasma goes to zero, and guiding center drift theory fails completely. In the rotating frame of the plasma, the Lorentz force and the Coriolis force compete, and they exactly cancel at the Brillouin limit.[5]

The total plasma energy is given by the integral

$$W = \int d\Gamma f(\Gamma, t) \left\{ \left[\frac{p_r^2}{2m} + \frac{\left(p_\theta - \frac{eBr^2}{2c} \right)^2}{2mr^2} + \frac{p_z^2}{2m} \right] \right.$$
$$\left. + \frac{1}{2} e\phi_p(r, \theta, z, t) + e\phi_{ext}(r, \theta, z, t) \right\}, \tag{4}$$

where the terms in the square brackets are the kinetic energy and the remaining two terms the electrostatic energy. The factor of $1/2$ appears in front of ϕ_p, but not in front of ϕ_{ext}, because ϕ_p is produced by the plasma charge itself and ϕ_{ext} is produced by external charge.[7,8] In the integral, Γ is a point in the six dimensional phase space. Of course, the energy is arbitrary up to an additive constant; for example, changing the value of $e\phi_{ext}$ by a constant would change the value of W by a constant. This arbitrariness will not be a problem because we will

need only changes in the value of W. The total canonical angular momentum and the total axial momentum are given by the integrals

$$L = \int d\Gamma f(\Gamma, t) p_\theta, \tag{5}$$

$$P = \int d\Gamma f(\Gamma, t) p_z. \tag{6}$$

For future reference, we note that the Hamiltonian in a frame rotating with angular frequency ω_r is $H_{\omega_r} = H - \omega_r p_\theta$.[7,8] This well-known result can be understood by noting that the square bracket in Hamiltonian (3) is the particle kinetic energy, $mv^2/2$, and by noting the algebraic identity

$$\frac{mv^2}{2} - \omega_r\left(mv_\theta r - \frac{eBr^2}{2c}\right) = \frac{m(\mathbf{v} - \hat{\boldsymbol{\theta}}\omega_r r)^2}{2} - \frac{m\omega_r^2 r^2}{2} + \frac{eB\omega_r r^2}{2}, \tag{7}$$

where the first term on the right hand side is the kinetic energy in the rotating frame, the second term is the centrifugal potential energy, and the third term is the electrostatic potential energy associated with the radial electric field induced by rotating through the magnetic field.

Alternately, one can construct a canonical transformation to the rotating frame using the generating function

$$F_2 = (\theta - \omega_r t)\bar{p}_\theta + r\bar{p}_r + z\bar{p}_z, \tag{8}$$

where the barred quantities are variables in the rotating frame and the notation of Goldstein[11] is used. Taking partial derivatives of F_2 yields the transformation

$$\bar{\theta} = \frac{\partial F_2}{\partial \bar{p}_\theta} = \theta - \omega_r t, \tag{9}$$

with all of the other variables remaining unchanged. Likewise, the Hamiltonian in the rotating frame is given by the expression

$$H_{\omega_r} \equiv \overline{H} = H + \frac{\partial F_2}{\partial t} = H - \omega_r p_\theta. \tag{10}$$

Subtracting $\omega_r p_\theta$ from the curly bracket in Eq. (4) yields the total energy in the rotating frame

$$W_{\omega_r} = \int d\Gamma f(\Gamma, t)\left\{\left[\frac{p_r^2}{2m} + \frac{\left(p_\theta - \frac{eBr^2}{2c}\right)^2}{2mr^2} + \frac{p_z^2}{2m}\right] + \frac{1}{2}e\phi_p(r,\theta,z,t) + e\phi_{ext}(r,\theta,z,t) - \omega_r p_\theta\right\} = W - \omega_r L. \tag{11}$$

By using Eq. (1) through (6) and the boundary condition that ϕ_p is zero on the cylindrical boundary wall, one obtains the time derivatives

$$\frac{dW}{dt} = \int d\Gamma f(\Gamma, t)\frac{\partial e\phi_{ext}}{\partial t}, \tag{12}$$

$$\frac{dL}{dt} = -\int d\Gamma f(\Gamma, t)\frac{\partial e\phi_{ext}}{\partial \theta}, \tag{13}$$

$$\frac{dP}{dt} = -\int d\Gamma f(\Gamma, t)\frac{\partial e\phi_{ext}}{\partial z}. \tag{14}$$

Physically, the rate of change of energy, angular momentum, and axial momentum depend only on the derivatives of $e\phi_{ext}$ because the interaction of the plasma particles with each other through $e\phi_p$ conserves the plasma energy, angular momentum, and axial momentum.

When no wave is being launched, $\phi_{ext} = \phi^{(0)}(r,z)$ is simply the confinement potential due to the time-independent voltage applied to the end cylinders. When the plasma has come to an equilibrium state in this external potential, we denote the plasma potential by $\phi_p^{(0)}(r,z)$ and the Hamiltonian by $H^{(0)}$. Any distribution function of the form $f^{(0)}(H^{(0)}, p_\theta)$ is an equilibrium of the Vlasov–Poisson equations. Furthermore, Davidson[5] found that the special class of equilibria $f^{(0)}(H^{(0)} - \omega_r p_\theta)$ is stable under Vlasov–Poisson dynamics, provided that the distribution is monotonically decreasing in its argument. The argument $H_{\omega_r}^{(0)} = H^{(0)} - \omega_r p_\theta$ is simply the equilibrium Hamiltonian in the rotating frame, so there is no shear in the rotational flow of the plasma. Consequently, these states are often referred to as rigid rotor states.

The most important examples are thermal equilibrium states,[5,7,8] where $f^{(0)}(H^{(0)} - \omega_r p_\theta)$ is exponentially decreasing in its argument. The thermal equilibrium states are routinely observed in experiments with non-neutral plasmas and are easily used as starting equilibria for wave launching experiments. We assume that the plasma is in a monotonically decreasing, rigid rotor state when the wave launching process begins.

Readers might rightly object that Vlasov dynamics cannot lead to a thermal equilibrium state. For a case where a thermal equilibrium state is to be the equilibrium on which waves are launched, we envision a long initial period during which weak collisions bring the plasma to thermal equilibrium, followed by a much shorter wave launching period during which Vlasov dynamics provides a good approximation.

When the plasma is in a monotonically decreasing rigid rotor state, we suppose a wave launching potential is switched on, changing the external potential to the form $\phi_{ext} = \phi^{(0)}(r,z) + \phi^{(1)}(r,\theta,z,t)$. In experiments, waves are launched by applying temporally varying potentials to electrically isolated sectors of the wall. However, the launching potential, $\phi^{(1)}$, then can have spatial overlap with the eigenmodes for several waves, and when two or more of these waves are nearly degenerate both can be launched. To avoid this complication, we envision a "theorists" launching potential, which exactly matches the eigenmode of the weakly damped wave to be launched. This theorists launching potential is produced by "imaginary charge density" that is distributed appropriately in the confinement region. Like the charge density on the electrodes that produce a wave launching potential in experiments, the imaginary charge density is not considered to be part of the plasma charge density, but rather to be an external charge density. We postulate the launching potential

$$\phi^{(1)} = \epsilon \Theta(t)\psi(r,z)\cos(l\theta - \omega t), \tag{15}$$

where ϵ is small compared to unity, $\Theta(t)$ is a step function affecting the turn on of the launching potential and $\psi(r,z)\cos(l\theta - \omega t)$ is the eigenmode of the wave to be launched. Throughout the discussion l is taken to be positive, but ω can be positive or negative, allowing waves to have positive or negative azimuthal phase velocity.

The wave launching potential is assumed to be small (i.e., $\epsilon \ll 1$) and to stay on for many wave cycles. The amplitude of a resonantly driven mode grows secularly (i.e., $A \sim \epsilon |\omega| t$), so the amplitude is finite if $|\omega| t \sim 1/_\epsilon$. Actually, we assume that the time t is such that the wave amplitude is finite, but still small enough that linear theory is a good approximation. Also, we have assumed that the Landau damping rate for the wave is small. The secular growth of wave amplitude continues only for a time $t \sim 1/_\gamma$, where γ is the damping rate, so we have assumed that $^\gamma/_{|\omega|} \sim \epsilon$ is small.

Substituting $\phi_{ext} = \phi^{(0)}(r, z) + \phi^{(1)}$ into Eq. (12) yields the time derivative

$$\frac{dW}{dt} = \delta(t) \int d\Gamma f^{(0)} e\epsilon \psi(r) \cos(l\theta)$$
$$+ \omega \int d\Gamma f(\Gamma, t) e\epsilon \Theta(t) \psi(r) \sin(l\theta - \omega t), \quad (16)$$

where the delta function coefficient of the first integral comes from the time derivative of the step function, and the terms in the integrand have been evaluated at $t = 0$. Since the wave launching potential has not yet had time to change the distribution at $t = 0$, the distribution in the first integral has been set equal to the equilibrium distribution $f^{(0)}$. Since this distribution is independent of θ, the integral vanishes, and Eq. (16) reduces to the result

$$\frac{dW}{dt} = \omega \int d\Gamma \ f(\Gamma, t) e\epsilon \Theta(t) \psi(r) \sin(l\theta - \omega t). \quad (17)$$

We limit consideration to modes with $l \neq 0$. Substituting $\phi_{ext} = \phi^{(0)}(r, z) + \phi^{(1)}$ into Eq. (13) yields the time derivative

$$\frac{dL}{dt} = l \int d\Gamma \ f(\Gamma, t) e\epsilon \Theta(t) \psi(r) \sin(l\theta - \omega t). \quad (18)$$

As expected, both the plasma energy W and angular momentum L change in value as the wave is driven up in amplitude.

However, by noting that the integrals in Eqs. (17) and (18) are identical, one finds a simple relationship between the time derivatives of W and L

$$\frac{1}{\omega}\frac{dW}{dt} = \frac{1}{l}\frac{dL}{dt}. \quad (19)$$

Integrating this equation from some time before the launching potential is turned on to some time t after the wave has reached finite amplitude yields the relation

$$\frac{\Delta W}{\omega} = \frac{\Delta L}{l}, \quad (20)$$

where ΔW and ΔL are the change in plasma energy and angular momentum during the launching process. This relation comes from the functional form of the wave, which in turn comes from the symmetry of the equilibrium, that is, the invariance under translation in t and θ. We will identify ΔW and ΔL as the wave energy and angular momentum in the laboratory frame, respectively.

Of course, one may ask how we know that all of the energy and angular momentum produced by the launching potential goes into the wave. In a Vlasov description, energy and angular momentum are transferred only by the mean electric field. The launching field can neither do work nor apply a torque on the wall since the wall is assumed

to have perfect conductivity and to be cylindrically symmetric. The launching potential can do work on and apply a torque only on the plasma charge density and there only on the perturbed charge density.

The launching potential creates a small perturbed charge density, the wave charge density, and then does work and exerts a torque on that charge density, gradually building up the wave amplitude and charge density. We assume that the buildup takes place over many cycles so that the perturbed charge density is always close to that for an undriven wave. The wave electric field does no net work and exerts no net torque on the wave charge density since the wave field is produced by the charge density. In contrast, the launching field is produced by an external charge density and does do net work and does exert net torque on the wave charge density.

By hypothesis, the wave experiences negligible damping during the launching process, that is, the wave electric field does negligible net work and exerts negligible net torque on any perturbed charge density other than that of the wave. Since the lunching potential is assumed to be of the same form as the wave potential, the launching potential transfers negligible energy and angular momentum to any perturbed charge density other than that of the wave.

Equation (20) has a simple interpretation in terms of wave action or quantum number. The energy is related to the wave action through the relation $\Delta W = \omega J$, and the angular momentum is related to the action through the relation $\Delta L = Jl$. Dividing these two equations yields Eq. (20). Sturrock[1] used this idea, postulating a quadratic Lagrangian for waves on a fluid medium. Here, we elected to derive Eq. (20) directly from symmetry considerations since that seems more fundamental.

Taking the same differences in Eq. (11) and using Eq. (20) yields the relation

$$\Delta W_{\omega_r} = \Delta W - \omega_r \Delta L = \frac{(\omega - l\omega_r)}{\omega}\Delta W, \quad (21)$$

where ΔW_{ω_r} is the change in energy as seen in the rotating frame of the plasma. If this quantity is positive, then the sign of ΔW is given by the sign of the ratio $(\omega - l\omega_r)/\omega$.

Before the wave is launched, the external potential is given by $\phi^{(0)}(r, z)$, the plasma potential by $\phi_p^{(0)}(r, z)$, and the distribution by $f^{(0)}(H^{(0)} - \omega_r p_\theta)$. At time t, the external potential includes the launching potential $\phi^{(1)}$, and the distribution and plasma potential have evolved to $f^{(0)} + \Delta f$ and $\phi_p^{(0)} + \Delta\phi_p$, respectively. By using the integral expression for W_{ω_r} in Eq. (11), one finds the change

$$\Delta W_{\omega_r} = \int d\Gamma \left(e\phi^{(1)} f^{(0)} + \Delta f H_{\omega_r}^{(0)} + \Delta f e\phi^{(1)} + \frac{\Delta f e\Delta\phi_p}{2}\right). \quad (22)$$

The integral over the first term vanishes because $f^{(0)}$ is independent of θ and $\phi^{(1)} = \epsilon\Theta(t)\psi(r)\cos(l\theta - \omega t)$ yields zero under θ-integration. Also, Poisson's equation, integration by parts, and the boundary condition that $\Delta\phi_p$ vanishes on the conducting boundary cylinder can been used to rewrite the last term, yielding the result

$$\Delta W_{\omega_r} = \int d\Gamma \Delta f H_{\omega_r}^{(0)} + \int d\Gamma \Delta f e\phi^{(1)} + \frac{\int d^3 x \left(\nabla\Delta\phi_p\right)^2}{8\pi}. \quad (23)$$

The third integral is positive. To show that the sum of the three integrals is positive, we show that the first integral is positive and that the second integral is small compared to the first.

The positive value of the first integral follows from the fact that the Vlasov equation for the evolution of $f(\Gamma, t)$ is a statement of incompressible flow in phase space and that $f^{(0)}(\Gamma)$ is a monotonically decreasing function of $H^{(0)}_{\omega_r}(\Gamma)$. Any change in $f(\Gamma, t)$ must increase the value of $\int d\Gamma f(\Gamma, t) H^{(0)}_{\omega_r}(\Gamma)$, that is, the first integral in Eq. (23) must be positive.

A more formal demonstration utilizes techniques from the Davidson–Krall stability theorem.[5] Because $f(\Gamma, t)$ evolves in phase space as an incompressible fluid, the integral $\int d\Gamma\, G(f(\Gamma, t))$ is constant in time for any function G. Taylor expanding the integrand in terms of the small quantity Δf yields the result

$$0 = \int d\Gamma \left[\frac{\partial G}{\partial f^{(0)}} \Delta f + \frac{1}{2}\frac{\partial^2 G}{\partial f^{(0)2}}(\Delta f)^2 \right], \tag{24}$$

where only second order terms in Δf have been retained. Subtracting this integral from the integral over the first term in Eq. (23) yields the equation

$$\int d\Gamma\, \Delta f(\Gamma, t) H^{(0)}_{\omega_r}(\Gamma) = \int d\Gamma\, \Delta f(\Gamma, t)\left[H^{(0)}_{\omega_r}(\overline{\Gamma}) - \frac{\partial G}{\partial f^{(0)}} \right]$$
$$- \int d\Gamma\, \frac{1}{2}\frac{\partial^2 G}{\partial f^{(0)2}}(\Delta f)^2. \tag{25}$$

Choosing the function $G(f^{(0)})$ to be such that $H^{(0)}_r(\Gamma) = \frac{\partial G}{\partial f^{(0)}}$ and recalling that $f^{(0)}$ is a monotonic decreasing function of $H^{(0)}_{\omega_r}$ (i.e., $\frac{\partial^2 G}{\partial f^{(0)2}} = \frac{\partial H^{(0)}_{\omega_r}(\overline{\Gamma})}{\partial f^{(0)}} < 0$) yields the result

$$\int d\Gamma\, \Delta f(\Gamma, t) H^{(0)}_{\omega_r}(\overline{\Gamma}) = -\int d\Gamma\, \frac{1}{2}\frac{\partial H^{(0)}_{\omega_r}(\overline{\Gamma})}{\partial f^{(0)}}(\Delta f)^2 > 0. \tag{26}$$

The argument that the second integral in Eq. (23) is small compared to the first integral may be understood in terms of the ordering argument in the paragraph following Eq. (15). The quantity $\Delta f(\Gamma, t)$ is the order of the wave amplitude $A \sim \epsilon|\omega|t$, so Eq. (26) shows that the first integral in Eq. (23) is of order $A^2 \sim (\epsilon\omega t)^2$. Since $\phi^{(1)}(\overline{\Gamma}, t)$ is of order ϵ, the second integral in Eq. (23) is of order $\epsilon^2|\omega|t$, which is smaller than the first term by order $1/(|\omega|t) \sim \epsilon$.

There is one subtle point in this ordering argument. Just looking at the first integral in Eq. (23), one might be tempted to say that the integral is first order in the wave amplitude $A \sim \epsilon|\omega|t$. However, $H^{(0)}_{\omega_r}$ is θ independent so the term in Δf which is first order in wave amplitude integrates to zero, leaving the contribution from a term that is second order in wave amplitude, $A^2 \sim (\epsilon\omega t)^2$. Eq. (26) simply makes the second order nature of the integral explicit.

Thus, the criterion that the wave energy, as viewed in the laboratory frame, be negative is the inequality

$$0 > \frac{(\omega - l\omega_r)}{\omega} = \left(1 - \frac{\omega_r}{\omega/l}\right). \tag{27}$$

In interpreting this criterion, recall that the azimuthal wave number l is assumed to be positive, but that ω_r is negative for positively charged species and positive for negatively charged species. For either sign, the criterion for negative energy is that the wave phase rotates in the same sense as the plasma, but more slowly.

Let us consider two simple applications of the criterion to waves that are excited on a long pure electron plasma column that has come to a state of thermal equilibrium. When the thermal equilibrium plasma is cold, the density is uniform out to some radius R_p and there drops to zero. We assume that the density is small compared to that at the Brillouin limit, which implies the frequency ordering $|\omega_r| \ll |\Omega_c|$. In this limit, the frequency of a diocotron wave is given by the relation[5,9,10] $\omega - l\omega_r = -\omega_r[1 - (R_p/R_W)^{2l}]$, where R_W is the radius of the conducting boundary wall. The diocotron waves have zero axial wave number. One can see that criterion (27) is satisfied for all l, so all of the diocotron waves have negative energy. For a realistic finite length column, there are corrections to the frequency, but experimentally the corrections are known to be relatively small for a sufficiently long column and a sufficiently low temperature. We will return to diocotron waves in Sec. III.

The frequency of a Trivelpiece-Gould wave is given by the relation[5,9,10]

$$(\omega - l\omega_r)^2 = \frac{\omega_p^2 k^2}{\left(k^2 + \dfrac{R_p^2}{\chi_{l,m}^2} \right)}, \tag{28}$$

where k is the axial wave number and $\chi_{l,m}/R_p$ is the m-th radial mode number for azimuthal mode l, and we have assumed that the Doppler shifted phase velocity $(\omega - l\omega_r)/k$ is large compared to the thermal velocity. Other than the Doppler shift and the quantization of the radial mode number, this is just the familiar dispersion relation for an electron plasma wave in a strong magnetic field, $\omega^2 = \frac{\omega_p^2 k^2}{k^2 + k_\perp^2}$, where $k_\perp$ is the transverse (here, radial) wave number. For a finite length column, the axial wave number must be quantized approximately as $k \approx n\pi/L'$, where L' is the length of the column. Choosing the lowest mode numbers (i.e., $l = 1$, $m = 1$ and $n = 1$), choosing the negative sign in the factor when taking the square root, and assuming that $k \ll \chi_{l,m}/R_p$ yields $\omega - \omega_r = -\omega_p R_p/2L'$, where we have set $\pi/\chi_{1,1} \approx 1/2$. Thus, the mode can be negative energy if $\omega_r > \omega_p R_p/2L'$.

For the frequency ordering mentioned above, the cyclotron waves have positive energy.[9]

Thus far, we have proceeded under the assumption that wave damping is negligible during the launching process, so that the quantity ΔW is all wave energy at the end of the launching process. However, if there is small but finite Landau damping, wave energy will gradually be transformed into resonant particle energy. Note that the wave energy and the resonant particle energy are both part of the energy associated with the overall wave perturbation. If the wave energy is positive, the resonant particles receive positive energy, and if the wave energy is negative the resonant particles receive negative energy. Thus, we write the relation $\Delta W = (\Delta W_{wave} + \Delta W_{res})$, where ΔW_{wave} and ΔW_{res} are of the same sign. Assuming that the wave launching process is over and that no energy sink has been introduced, ΔW is constant in time. The wave energy, ΔW_{wave}, is proportional to $|A|^2$, where A is the wave amplitude, and the rate of change of ΔW_{res} is proportional to $|A|^2$, with the sign of ΔW_{wave}. Thus, we find the relation

$$\frac{2}{|A|}\frac{d|A|}{dt}\Delta W_{wave} = \frac{d}{dt}\Delta W_{wave} = -\frac{d}{dt}\Delta W_{res} = -2\gamma\Delta W_{wave}. \tag{29}$$

By canceling the factor ΔW_{wave} from each side, one can recognize the proportionality constant γ as the Landau damping decrement. Of course, the wave can only damp since the plasma is stable.

Next, we suppose that the wave energy can access an energy sink. For sufficiently low wave amplitude, one expects the rate at which energy is deposited in the sink to be of the form $-S(\omega)|A|^2$, where $S(\omega)$ is a positive function dependent on the details of the energy sink. See Ref. 3 for a simple example of the function $S(\omega)$. The rate of the change of wave amplitude is given by the relation

$$\left(\frac{2}{|A|}\frac{d|A|}{dt} + 2\gamma\right)\Delta W_{wave} = -S(\omega)|A|^2 \qquad (30)$$

and the wave grows if $-\frac{-S(\omega)|A|^2}{\Delta W_{wave}} - 2\gamma > 0$.

Criterion (27) was developed for a single species plasma with an equilibrium particle distribution that satisfies the Davidson–Krall stability criterion. In general, criterion (27) does not apply for the case of a neutral plasma. The problem is that a neutral plasma cannot be confined and also have electron and ion distributions that satisfy Davidson–Krall criterion for the same rotation frequency. This statement is a generalization of the well-known fact that a neutral plasma cannot be confined by static electric and magnetic fields in a state of thermal equilibrium.[7,8] The confinement, that is, the spatial localization of the plasma, comes from the requirements that the equilibrium distribution function be monotonically decreasing in its argument, $H^{(0)} - \omega_r p_\theta$, and that the scalar and vector potential terms in this quantity become large for large r or large z. Since both the scalar and vector potentials are multiplied by a factor of e, the confinement of ions implies nonconfinement of electrons and *vice versa*.

However, there may be situations where one of the two species in a confined neutral plasma is characterized by a distribution that satisfies the Davidson–Krall stability criterion, and then a mode which involved only that species would be negative energy when criterion (27) is satisfied.

Also, a two species plasma of like signed charges, for example, electrons and negative ions, can be confined in a Malmberg–Penning trap with distributions that satisfy the Davidson–Krall criterion for the same rotation frequency, and then criterion (27) applies even if the mode involves both species.

III. WAVES ON AN INFINITELY LONG BEAM

A second example adds another symmetry: namely, invariance under translation in z. Consider a plasma beam that is confined radially by a uniform axial magnetic field in a region of space bounded by a conducting cylinder. Again, the boundary condition on the plasma potential, ϕ_p, is to be zero on the conducting cylinder. This configuration is essentially a Malmberg–Penning trap with the end confinement potentials moved off to $z = \pm\infty$.

We assume that a wave launching process on this beam is well described by Vlasov–Poisson equations (1) and (2), with the Hamiltonian (3). The energy, angular momentum, and axial momentum are given by Eqs. (4)–(6) and the time derivatives of these quantities by Eqs. (12)–(14).

When no wave is being launched, the external potential is zero and the plasma potential is of the form $\phi_p^{(0)}(r)$, so the corresponding Hamiltonian, $H^{(0)}$, is independent of t, θ, and z. Thus, $H^{(0)}$, p_θ, and p_z are constants of the motion, and any distribution of the form $f^{(0)}(H^{(0)}, p_\theta, p_z)$ is an equilibrium of the Vlasov–Poisson

equations. Furthermore, the Davidson–Krall theorem[5] proves Vlasov–Poisson stability for distributions of the form $f^{(0)}(H^{(0)} - \omega_r p_\theta - up_z)$, where $f^{(0)}$ is a monotonically decreasing function of its argument. Here the argument is simply the Hamiltonian in a frame that rotates with angular frequency ω_r and drifts axial with velocity u. Again, when $f^{(0)}$ is an exponentially decreasing function of its argument, the distribution describes a thermal equilibrium state.

Even when a wave is being launched, the Hamiltonian in this uniformly rotating and translating frame is given by $H_{\omega_r,u} = H - \omega_r p_\theta - up_z$, and likewise the energy in the frame is given by the relation

$$W_{\omega_r,u} = W - \omega_r L - uP \qquad (31)$$

again up to an additive constant.

Because the equilibrium is symmetric under translations in t, θ, and z, the wave launching potential is of the form

$$\phi^{(1)} = \epsilon\Theta(t)\psi(r)\cos(kz + l\theta - \omega t), \qquad (32)$$

where $\psi(r)\cos(kz + l\theta - \omega t)$ is the eigenfunction of the weakly damped mode to be launched, and the axial wave number k is taken to be positive. Substituting this external potential into Eqs. (12)–(14) yields the time derivative relations

$$\frac{1}{\omega}\frac{dW}{dt} = \frac{1}{l}\frac{dL}{dt} = \frac{1}{k}\frac{dP}{dt} \qquad (33)$$

and integrating from before the launching potential is turned on to some time t when the wave has reached finite amplitude yields the difference relations

$$\frac{\Delta W}{\omega} = \frac{\Delta L}{l} = \frac{\Delta P}{k}. \qquad (34)$$

Taking the same differences in Eq. (29) and using Eq. (32) then yields the relation

$$\Delta W_{\omega_r,u} = \Delta W - \omega_r\Delta L - u\Delta P = \left(\frac{\omega - l\omega_r - ku}{\omega}\right)\Delta W \qquad (35)$$

so the sign of ΔW is the same as the sign of the ratio $((\omega - l\omega_r - ku)/\omega)$ provided that $\Delta W_{\omega_r,u}$ is positive.

That $\Delta W_{\omega_r,u}$ is positive can be shown by the same proof used to show that ΔW_{ω_r} is positive. Taking differences in the integral expression for $\Delta W_{\omega_r,u}$ leads to the same expression as that given in Eq. (23), except that $\Delta W_{\omega_r,u}$ replaces ΔW_{ω_r} on the left hand side and $H_{\omega_r,u}^{(0)}$ replaces $H_{\omega_r}^{(0)}$ on the right hand side. Also, $H_{\omega_r,u}^{(0)}$ replaces $H_{\omega_r}^{(0)}$ as the argument of the equilibrium distribution $f^{(0)}$. With these replacements, the proofs are completely parallel. Thus, the criterion that a wave has negative energy in the laboratory frame is the inequality

$$0 > \left(\frac{\omega - l\omega_r - ku}{\omega}\right). \qquad (36)$$

IV. WAVES GOVERNED BY 2D $E \times B$ DRIFT DYNAMICS

A third example focuses on a plasma column that undergoes 2D $E \times B$ drift dynamics. Again, the confinement region is bounded by a conducting cylinder on which the plasma potential vanishes. The

continuity equation for the particle density evolving under this dynamics can be written in the form

$$\frac{\partial n}{\partial t} + [n, H] = 0, \tag{37}$$

where the Hamiltonian is simply

$$H = e\phi_p(\theta, p_\theta, t) + e\phi_{ext}(\theta, p_\theta, t). \tag{38}$$

Here, $\left(\theta, p_\theta = \frac{eBr^2}{2c}\right)$ are a canonically conjugate coordinate and momentum pair. Note that Eq. (37) is written in the same form as the Vlasov equation, which reflects the fact that 2D $E \times B$ drift flow in a uniform magnetic field is incompressible. As is expected, the 2D $E \times B$ drift Hamiltonian is obtained from Hamiltonian (3) simply by letting the particle mass approach zero and neglecting any z-dependence. These same procedures reduce expressions (4) and (5) to the form

$$W = \left(\frac{cL'}{eB}\right) \int d\theta dp_\theta\, n\left(\frac{e\phi_p}{2} + e\phi_{ext}\right), \tag{39}$$

$$L = \frac{cL'}{eB} \int d\theta dp_\theta n p_\theta, \tag{40}$$

where the coefficient $(cL')/eB$ is simply a normalization to account for the fact that n is the density. The factor L' is the column length. In 2D $E \times B$ drift dynamics, the total energy is simply the electrostatic energy, and the total angular momentum comes only from the vector potential term of the canonical angular momentum. Again, the Hamiltonian in a frame that rotates with frequency ω_r is $H_{\omega_r} = H - \omega_r p_\theta$ and the energy in the rotating frame is given by the expression

$$W_{\omega_r} = \int d\Gamma f(\Gamma, t)\left\{+\frac{1}{2}e\phi_p(r, \theta, z, t) + e\phi_{ext}(r, \theta, z, t) - \omega_r p_\theta\right\}$$
$$= W - \omega_r L. \tag{41}$$

By using continuity equation (37), Poisson's equation and the boundary condition that ϕ_p vanish on the conducting boundary wall, the time derivatives of Eqs. (39) and (40) can be written as the expressions

$$\frac{dW}{dt} = \left(\frac{cL'}{eB}\right) \int d\theta dp_\theta n \frac{\partial e\phi_{ext}}{\partial t}, \tag{42}$$

$$\frac{dL}{dt} = \frac{cL'}{eB} \int d\theta dp_\theta n \frac{\partial e\phi_{ext}}{\partial \theta}. \tag{43}$$

Consider any equilibrium density profile, $n^{(0)}(r)$, that is monotonically decreasing in radius. Let the equilibrium potential for this profile be $\phi_p^{(0)}(p_\theta)$ and the Hamiltonian be $H^{(0)} = e\phi_p^{(0)}(p_\theta)$. The external potential is zero when no launching potential is present. The equilibrium Hamiltonian in a frame that rotates with the angular frequency ω_r is given by the expression $H_{\omega_r}^{(0)}(p_\theta) = H^{(0)}(p_\theta) - \omega_r p_\theta$. We intend to write the density profile as a monotonically decreasing function of $H_{\omega_r}^{(0)}(p_\theta)$, but note at the outset that this functional form does not mean that ω_r is the $E \times B$ drift rotation frequency of the plasma.

Indeed, for a typical monotonically decreasing density profile, there is no single rotation frequency; there is shear in the $E \times B$ drift rotational flow.

Why is it that a Vlasov distribution of the form $f^{(0)}\left(H_{\omega_r}^{(0)}\right)$ implies that the plasma is in a shear-free flow, but a density distribution of the form $n^{(0)}\left(H_{\omega_r}^{(0)}\right)$ still allows shear in the $E \times B$ drift rotational flow? The answer is that the Hamiltonian in the Vlasov case includes velocity variables, and the functional dependence $f^{(0)}\left(H_{\omega_r}^{(0)}\right)$ imposes constraints on the velocity dependence, whereas the Hamiltonian for $E \times B$ drift dynamics does not contain any velocity variables. Even for the thermal equilibrium state $n^{(0)} = Cexp\left(-\frac{H_{\omega_r}^{(0)}}{T}\right)$, there is shear in the $E \times B$ drift flow. The fluid flow is shear-free for a thermal equilibrium plasma, but the fluid flow consists of the sum of an $E \times B$ drift flow and a diamagnetic flow. The $E \times B$ drift flow alone is not shear-free.

The density distribution $n^{(0)}\left(H_{\omega_r}^{(0)}\right)$ will be monotonically decreasing in $H_{\omega_r}^{(0)}$ provided that the following inequality is satisfied:

$$0 > \frac{\partial n^{(0)}}{\partial H_{\omega_r}^{(0)}} = \frac{c}{eB}\frac{\partial n^{(0)}}{\partial r}\frac{\partial p_\theta}{\partial H_{\omega_r}^{(0)}}. \tag{44}$$

The factor $\partial n^{(0)}/\partial r$ is negative by hypothesis, so the product of the other factors must be positive

$$0 < \frac{eB}{c}\frac{\partial H_{\omega_r}^{(0)}}{\partial p_\theta} = \frac{eB}{c}\left(\frac{\partial H^{(0)}}{\partial p_\theta} - \omega_r\right). \tag{45}$$

Here, the quantity $\partial H^{(0)}/\partial p_\theta$ is the local $E \times B$ drift rotation frequency and ω_r is a constant rotation frequency of our choice. For a plasma of positive charges, $\partial H^{(0)}/\partial p_\theta$ is negative, so the inequality is satisfied if ω_r is chosen to be negative and larger in magnitude than the largest value of $\partial H^{(0)}/\partial p_\theta$, typically the local rotation frequency at the center of the plasma. For a plasma of negative charges, all of the frequencies are positive, but the charge in front is negative, so the inequality is again satisfied if ω_r is chosen to be larger than the rotation frequency at the center of the trap.

When the column is in such an equilibrium state, we suppose that the wave launching potential

$$\phi_{ext} = \phi^{(1)} = \epsilon\Theta(t)\psi(p_\theta)\cos(l\theta - \omega t) \tag{46}$$

is switched on. Again, ϵ is small compared to unity, $\Theta(t)$ is a step function that effects the turn on of the potential, and $\psi(p_\theta)\cos(l\theta - \omega t)$ is the eigenfunction of the weakly damped wave to be launched.

Substituting this expression for ϕ_{ext} into Eqs. (42) and (43) yields the relation

$$\frac{1}{\omega}\frac{dW}{dt} = \frac{1}{l}\frac{dL}{dt}, \tag{47}$$

and integrating from $t < 0$ to some time t after the wave has reached finite amplitude yields the difference relation

$$\frac{\Delta W}{\omega} = \frac{\Delta L}{l}. \tag{48}$$

Finally, taking the same differences in Eq. (41) yields the result

$$\Delta W_{\omega_r} = \Delta W - \omega_r \Delta L = \frac{(\omega - l\omega_r)}{\omega} \Delta W. \tag{49}$$

Thus, if ΔW_{ω_r} is positive, the wave energy in the laboratory frame is negative if the ratio $(\omega - l\omega_r)/\omega$ is negative. That ΔW_{ω_r} is positive follows from the same argument as given in Sec. II [i.e., Eq. (22) through (26)], except that f is replaced everywhere by n.

Superficially, the result in this section looks to be the same as that in Sec. II; indeed, Eq. (49) appears to be the same as Eq. (21). However, in Sec. II, we required the equilibrium Vlasov distribution $f^{(0)}\left(H^{(0)}_{\omega_r}\right)$ to be monotonically decreasing in its argument, and here we allow any equilibrium density profile that is monotonically decreasing in radius. Also, in Sec. II, the rotation frequency ω_r is the rotation frequency of the plasma, not a free parameter, whereas here the only constraint on ω_r is that it be larger in magnitude than the $E \times B$ drift rotation frequency at the center of the column. Thus, for any ω and l, we can choose any value of ω_r such that the ratio $(\omega - l\omega_r)/\omega$ is negative. Thus, within the context of 2D $E \times B$ drift flow, all waves on a monotonically decreasing density profile have negative energy when viewed in the laboratory frame.

To help motivate the discussion in Sec. V, note that the energy in the laboratory frame is a local maximum in the space of states that are accessible under incompressible flow. This follows from the fact that ΔW is negative for any incompressible perturbation.

V. ASYMMETRICAL EQUILIBRIA

The Davidson–Krall stability theorem[5] is not applicable to plasma equilibria that are not cylindrically symmetric. The reason is that the theorem requires the laboratory frame equilibrium also to be an equilibrium in a rotating frame since the theorem proves stability by showing that the energy is minimum in the rotating frame.

However, within the context of 2D $E \times B$ drift flow, there is an alternate approach to stability that does not require the equilibrium to have cylindrical symmetry.[12] When no wave is being launched, the external potential ϕ_{ext} is constant and the electrostatic energy W is constant in time. In the space of states that are accessible under incompressible flow, the plasma evolves along a contour of constant W. When the initial state is a either a local minimum or a local maximum in the space of accessible states, the contour shrinks to a point and no evolution is possible. The plasma is in a stable equilibrium state.

The idea of proving stability at an energy maximum, rather than an energy minimum, was part of the pioneering work on stability by Thomson and Kelvin[13] and Arnold.[14] For general Vlasov dynamics, it is not profitable to look for an energy maximum because the kinetic energy can increase arbitrarily, but the kinetic energy does not appear in expression (39) for the energy in the limit of 2D $E \times B$ dynamics. More precisely, the kinetic energy is small and constant. The electric field does not change the kinetic energy because the $E \times B$ drift velocity is transverse to the electric field. More generally, when the characteristic frequencies of the drift motion are small compared to the cyclotron frequency and the axial bounce frequency of the particles, the cyclotron and axial bounce actions are good adiabatic invariants under the slow drift motion. These adiabatic invariants bind up the particle kinetic energy so that it cannot increase arbitrarily, and the total energy, including the kinetic energy, can be a local maximum against low frequency drift perturbations.[15]

Various asymmetric plasma equilibria are states of maximum electrostatic energy and, hence, stable against low frequency 2D $E \times B$ drift instabilities. For example, a cylindrical plasma on which an $l = 1$ diocotron mode has been excited can be thought of as an equilibrium state in the rotating frame of the mode. In this frame, the plasma is simply displaced off the axis of the trap. At least for modest displacements and modest plasma radii, the equilibrium is a state of maximum electrostatic energy in the rotating frame of the mode.[15] Asymmetric equilibria that are stationary in the lab frame can be produced by applying z-independent asymmetric fields on the plasma using wall electrodes; such equilibria have been observed to survive stably for seconds and are likely states of maximum electrostatic energy.[16] Also, toroidal plasmas have been showed to have equilibria that are states of maximum electrostatic energy[15] and are observed to survive stably for seconds.[17]

The relation of all of this work to the present paper is that low frequency drift modes excited on these plasmas must all be negative energy modes. If the plasma had access to an energy sink, the modes would have the potential to become unstable.

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