Vlasov theory of electrostatic modes in a finite length electron column

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A Vlasov theory of low-frequency electrostatic modes in a finite length electron column is presented. The column is assumed to have cylindrical symmetry and flat end surfaces at which the electrons undergo specular reflection. The eigenfrequencies and the eigenfunctions are obtained as a series expansion in the small parameter R/L (where R is a characteristic radius and L the half-length). In zeroth order, the modes are simply the modes for an infinitely long column with axial wavenumbers quantized as $k = n\pi/L$. In first order, these modes are weakly coupled. This means that a $k = 0$ diocotron mode can share in the Landau damping of higher k modes. For certain values of the plasma parameters, two zero-order modes are degenerate and the coupling is strong.

I. INTRODUCTION

Experiments have been performed recently on low-frequency electrostatic modes in a warm pure electron plasma column of finite length.[1] For these modes, we present a Vlasov theory which includes the bounce motion of the electrons and a self-consistent treatment of the mode structure on the finite length column. A fluid theory of these modes has been published previously.[2]

The theory is applicable to both plasma modes and diocotron modes,[3] but attention is focused on the $k = 0$ diocotron modes, especially the $l = 1$, $k = 0$ diocotron mode (l = azimuthal mode number, k = axial wavenumber). For an infinitely long column this mode is undamped.[4] The reason is that the resonant radius r_s [defined by $\omega - l\omega_r(r_s) = 0$, where $\omega_r(r)$ is the $\mathbf{E}\times\mathbf{B}$ rotation frequency of the column] is equal to the radius R of the conducting cylindrical wall which bounds the confinement region and the plasma density is zero at that radius. For the $l > 1$, $k = 0$ diocotron modes, the resonance radius r_s is less than R, but the mode is still undamped if the radial density profile falls to zero at a radius less than r_s. One of the main points of this paper is that these modes can be damped in a finite length column. They couple to $k \neq 0$ plasma modes and share in their Landau damping.

The confinement geometry used in the experiments is shown in Fig. 1. The plasma resides inside a conducting cylindrical tube adjoined on either side by a coaxial cylindrical section. The outer sections are given a strong negative bias $-\Delta V$ which serves to confine the plasma axially. A strong magnetic field B directed along the common axis of the three cylindrical sections provides the radial confining force on the plasma.

We model the plasma column as a right circular cylinder of radius a and of length $2L$ (see Fig. 2). In other words, the radial density profile is assumed to be rectangular and the column ends to be flat. The main features of our results are insensitive to slight modification of this geometry (e.g., slight rounding of the rectangular radial density profile), provided the resonant radius remains outside the plasma.

We choose the ordering of the relevant parameters to be in accord with the experiments.[1] The plasma radius, the radius of the conducting cylindrical wall and the column length are ordered as $a \lesssim R \ll 2L$. The plasma frequency is much smaller than the cyclotron frequency (i.e., $\omega_p \ll \omega_c$), which in turn implies that the $\mathbf{E}\times\mathbf{B}$ rotation frequency,

$$\omega_r = cE_r/Br = \omega_p^2/2\omega_c, \tag{1}$$

is small compared to the plasma frequency. The modes under discussion have frequencies in the range of the rotation frequency. The inequality $\omega_p \ll \omega_c$ implies that the Larmor radius is much smaller than the Debye length, and the Debye length is smaller than the plasma radius (i.e., $r_L \ll \lambda_D < a$).

Under these conditions, the electron dynamics may be described by the drift-kinetic equation. The electron motion consists of $\mathbf{E}\times\mathbf{B}$ drift across the magnetic field and streaming along the field with specular reflection at the ends of the column. The modes are simultaneous solutions of the linearized drift-kinetic equation and Poisson's equation, subject to the boundary condition that the mode potential vanish at $r = R$ and at $z = \pm\infty$. Note that as regards the mode potential, we are treating the three cylindrical sections as one continuous grounded conductor. To justify this procedure, we note that potential differences between the sections are asso-

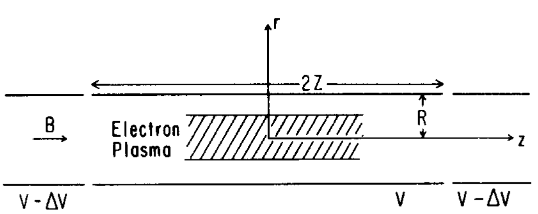

FIG. 1. Plasma confinement geometry.

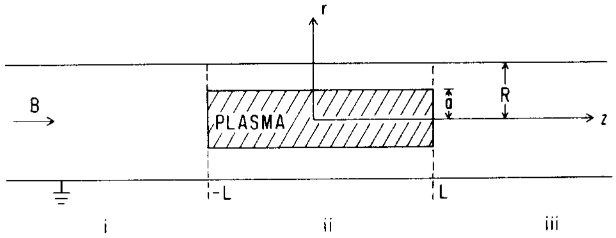

FIG. 2. Model for the wave theory.

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ciated with the inability of the currents in the leads to keep pace with the oscillating potential (finite capacitance effects, etc.). However, for the $l>0$ modes we consider in this paper, the $e^{il\theta}$ angular dependence assures that the total charge induced on the inside of any one cylindrical section is zero and hence there are no currents flowing in the leads. Thus the potential perturbation at the walls of all three sections due to the modes is zero.

The solution for the modes is obtained as a perturbation expansion in the small parameter R/L. In the lowest order in this parameter, the modes are just the modes of an infinitely long column, but with axial wavenumbers quantized in multiples of π/L. In the next order, these modes are coupled to each other (i.e., the modes do not have a single axial wavenumber $k=n\pi/L$, but have other Fourier components mixed in). As mentioned above, this has the consequence of enabling modes to share their damping properties. For example, the $k=0$ diocotron mode which normally exhibits no damping can be damped because of its coupling with the Landau damped $k\neq 0$ plasma modes. For certain values of the plasma parameters (ω_p, ω_c, a, R, and L), this normally weak coupling between the $k=0$ diocotron mode and a $k\neq 0$ plasma mode increases rapidly to order unity. The modes can then be regarded as degenerate and the damping of both modes can be significant.

The paper is organized in the following way. In Sec. II, we use the basic equations to obtain a matrix equation for the eigenfrequencies and the eigenfunctions. In Sec. III we solve the matrix equation for the $k=0$ diocotron mode and discuss the corrections to the eigenfrequency and the eigenfunction due to the finite length of the column. We find that the correction to the real part of the frequency occurs in order R/L, while the correction to the imaginary part occurs in order $(R/L)^2$. In Sec. IV, the degeneracy of the modes is discussed. Modes other than the $k=0$ diocotron mode are discussed in Sec. V.

II. THE DISPERSION EQUATION

The drift-kinetic equation is given by

$$\frac{\partial f}{\partial t}-c\,\frac{\nabla\varphi\times\mathbf{B}}{B^2}\cdot\frac{\partial f}{\partial \mathbf{x}_\perp}+v\,\frac{\partial f}{\partial z}+\frac{e}{m}\,\frac{\partial\varphi}{\partial z}\,\frac{\partial f}{\partial v}=0, \tag{2}$$

where $f(\mathbf{x},v,t)$ is the electron distribution function and $\varphi(\mathbf{x},t)$ is the electric potential; $\mathbf{x}$ refers to the spatial variables (r,θ,z) and v to the z component of the electron velocity. This equation must be solved simultaneously with Poisson's equation

$$\nabla^2\varphi=4\pi e\int_{-\infty}^{\infty} f(\mathbf{x},v)dv, \tag{3}$$

subject to the boundary conditions on the potential.

To discuss the eigenmodes of the system, we linearize Eqs. (2) and (3). Since f_0 and φ_0 have no azimuthal dependence, modes of different azimuthal numbers l can be considered separately. Assuming the azimuthal dependence $e^{il\theta}$ for the perturbed quantities, we obtain

$$\frac{\partial f_1}{\partial t}+il\omega_r f_1+v\,\frac{\partial f_1}{\partial z}=-\frac{e}{m}\,\frac{\partial\varphi_1}{\partial z}\,\frac{\partial f_0}{\partial v}+\frac{e}{m\omega_c}\,\frac{il\varphi_1}{r}\,\frac{\partial f_0}{\partial r}, \tag{4}$$

and

$$\frac{1}{r}\,\frac{\partial}{\partial r}\,r\,\frac{\partial\varphi_1}{\partial r}-\frac{l^2}{r^2}\,\varphi_1+\frac{\partial^2\varphi_1}{\partial z^2}=4\pi e n_1, \tag{5}$$

where

$$n_1=\int_{-\infty}^{\infty} f_1(\mathbf{x},v)dv. \tag{6}$$

We assume an asymptotic time dependence $e^{-i\omega t}$ for φ_1, f_1, and n_1 and solve Eqs. (4)–(6) for the eigenfrequencies ω and the eigenfunctions φ_1. We shall use the usual method of integration along unperturbed orbits for the solutions. The unperturbed electron drift motion perpendicular to the magnetic field consists of uniform rotation (with angular velocity ω_r) at a constant distance from the axis of the cylinder. The unperturbed axial motion is modeled as a uniform motion with specular reflections at the ends $z=\pm L$.

The unpertubed system is mirror symmetric about the plane $z=0$, and this fact together with the assumption that $f_0(v)$ is an even function of v enables us to consider modes odd and even in z separately. Confining ourselves to even modes [i.e., modes such that $\varphi_1(-z)=\varphi_1(z)$], we expand the mode potential in region ii in terms of a complete set of basis functions involving Bessel functions and cosine functions

$$\varphi_{\rm ii}(r,z)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}A_{mn}J_l(K_{lm}r)\cos\frac{n\pi z}{L}, \tag{7}$$

where K_{lm} is the solution of $J_l(K_{lm}R)=0$. In region iii, φ_1 is expanded in terms of the solutions of Laplace equation which vanish at $r=R$ and at $z=\infty$:

$$\varphi_{\rm iii}(r,z)=\sum_m B_m J_l(K_{lm}r)e^{-K_{lm}z}. \tag{8}$$

We shall not consider $\varphi_{\rm i}(r,z)$ since $\varphi_{\rm i}(r,z)=\varphi_{\rm iii}(r,-z)$.

The continuity of φ_1 across $z=L$ along with the orthogonality relation for Bessel functions yields

$$B_m e^{-K_{lm}L}=\sum_n(-1)^n A_{mn}. \tag{9}$$

Thus the determination of the coefficients A_{mn} implies the determination of B_m and hence of the complete wave potential. The second matching condition at $z=L$ is the continuity of $\partial\varphi_1/\partial z$. Using Eq. (9), this can be expressed as

$$\left.\frac{\partial\varphi_{\rm ii}}{\partial z}\right|_{z=L}=-\sum_m K_{lm}J_l(K_{lm}r)\sum_n(-1)^n A_{mn}. \tag{10}$$

We now consider the problem of an infinitely long column[5] in which $\varphi_{\rm ii}$, which is an even function of z, has been periodically continued to $\pm\infty$ [$\varphi_1(2nL+z)=\varphi_1(z)$, $n=\pm 1,\pm 2,\ldots$] and in which the electrons can stream without restriction along the z axis. If we now impose the restriction that $\partial\varphi_1/\partial z|_{L_-}$ ($=\partial\varphi_1/\partial z|_{(2n+1)L_-}$) in this problem should have the value (10), it is easy to see that our new problem is exactly equivalent to the original problem in which the electrons were specularly reflected at $z=\pm L$. The restriction on $\partial\varphi_1/\partial z$ can be most conveniently effected by introducing the surface charge densities

$$\frac{1}{2\pi e}\left(\sum_{m=1}^{\infty} K_{lm} J_l(K_{lm} r) \sum_{n=0}^{\infty} (-1)^n A_{mn}\right)$$
$$\times \sum_{n=-\infty}^{\infty} \delta[z-(2n+1)L], \qquad (11)$$

so that

$$\left.\frac{\partial \varphi_1}{\partial z}\right|_{L_+} - \left.\frac{\partial \varphi_1}{\partial z}\right|_{L_-} = -2\left.\frac{\partial \varphi_1}{\partial z}\right|_{L_-}$$
$$= 2\sum_m K_{lm} J_l(K_{lm} r) \sum_n (-1)^n A_{mn}.$$

Having noted the equivalency of the two problems, we shall proceed to solve the simpler infinite length problem. Since the unperturbed electron motion now consists just of uniform rotation ($\mathbf{E}\times\mathbf{B}$ drift) with angular velocity ω_r and of uniform axial motion with velocity v and since φ_1 is a superposition of plane waves propagating in z, integration of Eq. (4) is straightforward. For an unperturbed distribution of the form

$$f_0(r,v) = \tilde{n}(r) g(v)$$
$$= \tilde{n}[1-U(r-a)](2\pi\bar{v}^2)^{-1/2}\exp(-v^2/2\bar{v}^2) \qquad (12)$$

(U is the unit step function), we obtain

$$f_1(r,z,v) = \frac{e}{2m}\sum_{m,n} A_{mn} J_l(K_{lm} r)$$
$$\times\left(\frac{(n\pi/L)\tilde{n}(dg/dv) - (l/r\omega_c)(d\tilde{n}/dr) g}{\Omega - n\pi v/L}\right.$$
$$\left.\times e^{in\pi z/L} + (n \to -n)\right), \qquad (13)$$

where $\Omega = \omega - l\omega_r$. Note that in the present infinite length problem f_0 is not a function of z.

The perturbed charge density produced by φ_1 is given by $\int f_1\, dv$, where the velocity integration is performed over the appropriate Landau contours. Since g is an even function of v, the two terms in the large parentheses of Eq. (13) contribute equally to the velocity integral. This perturbed charge density along with the surface charge density (11) is inserted into Poisson's equation (5). Using the orthogonality properties of the Bessel functions and the cosine functions, we obtain

$$0 = \left[K_{lm}^2 + \left(\frac{n\pi}{L}\right)^2\right]\frac{R^2}{2} J_{l+1}^2(K_{lm} R) A_{mn}$$
$$+ \sum_{m'} A_{m'n}\left(\omega_p^2 \int_0^a r\, dr J_l(K_{lm} r) J_l(K_{lm'} r)\right.$$
$$\times \int_{-\infty}^{\infty} dv \frac{(n\pi/L)(dg/dv)}{\Omega - (n\pi/L)v}$$
$$\left. + 2l\omega_r J_l(K_{lm} a) J_l(K_{lm'} a)\int_{-\infty}^{\infty} dv \frac{g}{\Omega - (n\pi/L)v}\right)$$
$$+ 2\frac{(1-\frac{1}{2}\delta_{n0})}{L}(-1)^n K_{lm}\frac{R^2}{2} J_{l+1}^2(K_{lm} R)$$
$$\times \sum_p (-1)^p A_{mp}. \qquad (14)$$

We can rewrite Eq. (14) in matrix notation as

$$\mathscr{M} a = \begin{pmatrix} M_0 + B & -B & B & \cdots \\ -2B & M_1 + 2B & -2B & \cdots \\ 2B & -2B & M_2 + 2B & \cdots \\ \vdots & & & \end{pmatrix}\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \end{pmatrix} = 0, \qquad (15)$$

where A_n's are column vectors whose components are $(A_n)_j = A_{jn}$; M_n's and B are infinite matrices whose components are

$$(M_n)_{ij} = \left[K_{li}^2 N_i + \left(\frac{n\pi}{L}\right)^2 N_i\right]\delta_{ij} + \frac{2l\omega_r}{\Omega}\beta_{ij}$$
$$+\left(\frac{l\omega_r}{\Omega}\beta_{ij} - \frac{1}{2}\alpha_{ij}\frac{\omega_p^2}{\bar{v}^2}\right) Z'\left(\frac{\Omega}{\sqrt{2}(n\pi/L)\bar{v}}\right), \qquad (16)$$

and

$$B_{ij} = (K_{li}/L) N_i \delta_{ij}, \qquad (17)$$

where

$$N_i = (R^2/2) J_{l+1}^2(K_{li} R), \qquad (18)$$

$$\alpha_{ij} = \int_0^a r\, dr J_l(K_{li} r) J_l(K_{lj} r),$$
$$\beta_{ij} = J_l(K_{li} a) J_l(K_{lj} a), \qquad (19)$$

and Z is the plasma dispersion function.[6] In writing the expression (16) for $(M_n)_{ij}$ we have made use of the Maxwellian nature of g [see Eq. (12)].

The solution of Eq. (15) provides the eigenfrequencies ω and the corresponding eigenvectors a. Since these a's determine the potential wavefunctions completely via Eqs. (7)–(9), we shall often refer to the a's themselves as the wavefunctions.

III. SOLUTION OF THE MATRIX EQUATION

The matrices B are of order R/L smaller than the matrices M_n. With this in mind we develop a series solution to Eq. (15) in which R/L is the smallness parameter. In the order $(R/L)^0$, the matrix $\mathscr{M}$ is block diagonal and Eq. (15) can be factored as

$$M_n(\omega_n^{(0)}) A_n^{(0)} = 0 \quad (n = 0,1,\ldots). \qquad (20)$$

For each n, there are an infinite number of eigenfrequencies $\omega_{nd}^{(0)}, \omega_{n1}^{(0)}, \omega_{n2}^{(0)}, \ldots$ and eigenvectors $A_{nd}^{(0)}, A_{n1}^{(0)}, A_{n2}^{(0)}, \ldots$. These are just the frequencies and the wavefunctions corresponding to the various radial modes (with an axial wavenumber $n\pi/L$) in an infinitely long column. We note that in practice, taking a z dependence $\sim e^{in\pi z/L}$ and solving the radial eigenvalue problem embodied in Eqs. (4)–(6) is a more direct and more accurate way of obtaining these solutions than solving Eq. (20). The various values of ω plotted versus $n\pi/L$ form the various branches of the dispersion curve.[2] The frequencies $\omega_{nd}^{(0)}$ lie on the "diocotron branch," while the frequencies $\omega_{nm}^{(0)}$ lie on the "mth plasma branch" ($m = \pm 1, \pm 2, \ldots$). For $n = 0$, all the plasma branches converge to $\omega = l\omega_r$, and so $\omega = l\omega_r$ is an eigenfrequency with infinite degeneracy. The diocotron branch, on the other hand, stops at

$$\omega_{0d}^{(0)} = \omega_D \equiv \omega_r[l - 1 + (a/R)^{2l}]. \qquad (21)$$

Figure 3 shows the qualitative behavior of the first few

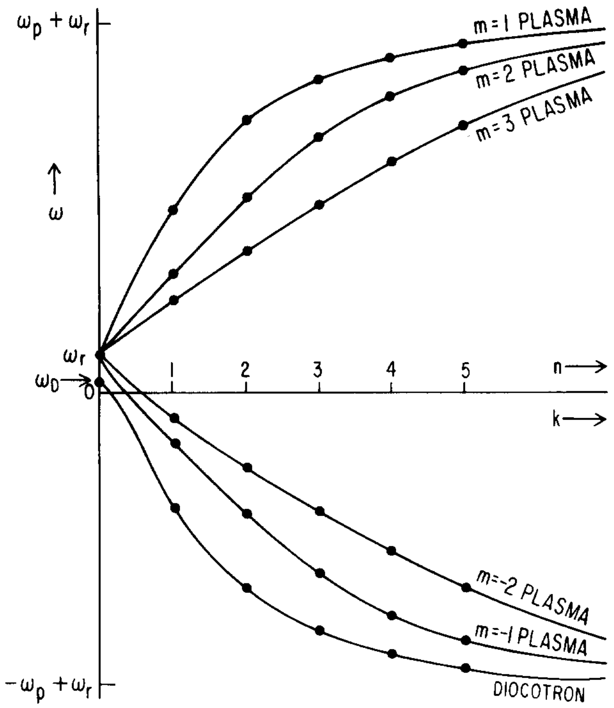

FIG. 3. Qualitative behavior of the first few branches of the $l = 1$ dispersion curve for a cold infinitely long plasma column. The dispersion curve for $l > 1$ is qualitatively similar if the frequencies are shifted down by $(l-1)\omega_r$. The dots are the solutions of Eq. (20) and represent D_n modes and P_{nm} modes corresponding to axial wavenumbers $k = n\pi/L$.

branches in the limit $\bar{v} = 0$. For $\bar{v} \neq 0$, the frequencies on all the branches develop imaginary components (increasing monotonically from zero as $n\pi/L$ is increased) which represent Landau damping.

The various modes on a finite length electron plasma column reduce, in order $(R/L)^0$, to the solutions of Eq. (20). This provides us a basis for nomenclature. We shall refer to the mode corresponding to the solution $\omega_{nd}^{(0)}$ as the D_n mode. The mode corresponding to the solution $\omega_{nm}^{(0)}$ $(m = \pm 1, \pm 2, \ldots)$ will be referred to as the P_{nm} mode. In the rest of this section, we shall concentrate on the D_0 mode, i.e., the mode whose lowest-order frequency is $\omega_{0d}^{(0)} = \omega_D$ and calculate the corrections introduced for the frequency and the wavefunction due to the finite length of the plasma column.

For the D_0 mode, the expansions (in powers of R/L) for the frequency ω and the wavefunction $\mathcal{a} = (A_0, A_1, \ldots)$ take the form

$$\omega = \omega^{(0)}(=\omega_{0d}^{(0)}) + \omega^{(1)} + \omega^{(2)} + \ldots, \tag{22a}$$

$$A_0 = A_0^{(0)}(=A_d^{(0)}) + A_0^{(1)} + A_0^{(2)} + \ldots, \tag{22b}$$

$$A_{n>0} = A_n^{(1)} + A_n^{(2)} + \ldots . \tag{22c}$$

In writing Eq. (22c), we have assumed that $M_{n>0}(\omega_{0d}^{(0)})$ is not close to being singular so that we can set $A_{n>0}^{(0)} = 0$. We consider the special case of $M_{n>0}(\omega_{0d}^{(0)})$ being singular in Sec. IV.

In orders $(R/L)^1$ and $(R/L)^2$, Eq. (15) yields

$$M_0(\omega^{(0)})A_0^{(1)} + \omega^{(1)}\frac{dM_0}{d\omega^{(0)}}A_0^{(0)} + BA_0^{(0)} = 0, \tag{23}$$

and

$$M_0(\omega^{(0)})A_0^{(2)} + \omega^{(1)}\frac{dM_0}{d\omega^{(0)}}A_0^{(1)} + \left(\omega^{(2)}\frac{dM_0}{d\omega^{(0)}} + \frac{1}{2}(\omega^{(1)})^2\frac{d^2M_0}{(d\omega^{(0)})^2}\right)A_0 + \sum_{n=0}^{\infty}(-1)^n BA_n^{(1)} = 0, \tag{24}$$

respectively, where $A_{n>0}^{(1)}$ is determined by

$$(-1)^n 2BA_0^{(0)} + 2B\sum_{p=1}^{\infty}(-1)^{n+p}A_p^{(1)} + M_n(\omega^{(0)})A_n^{(1)} = 0. \tag{25}$$

Since $M_0(\omega^{(0)})$ is symmetric, $(A_0^{(0)})^T M_0 = (M_0 A_0^{(0)})^T = 0$. Using this result, Eq. (23) can at once be solved for $\omega^{(1)}$:

$$\omega^{(1)} = \omega_{0d}^{(1)} = -\frac{(A_0^{(0)})^T BA_0^{(0)}}{(A_0^{(0)})^T(dM_0/d\omega^{(0)})A_0^{(0)}}. \tag{26}$$

From Eq. (16) and Eq. (20), we have

$$(A_0^{(0)})^T\frac{dM_0}{d\omega^{(0)}}A_0^{(0)} = \sum_{i,j}(A_0^{(0)})_i\frac{d(M_0)_{ij}}{d\omega^{(0)}}(A_0^{(0)})_j = \sum_{i,j}(A_0^{(0)})_i\frac{-[M_0(\omega_0^{(0)})]_{ij} + K_{li}^2N_i\delta_{ij}}{\omega^{(0)} - l\omega_r}(A_0^{(0)})_j = \frac{1}{\omega^{(0)} - l\omega_r}\sum_i K_{li}^2N_i[(A_0^{(0)})_i]^2. \tag{27}$$

$(A_0^{(0)})_i$ can be obtained by solving Eq. (20). However, we shall evaluate it using the well-known radial dependence of the $n = 0$ diocotron wave function:

$$\varphi_d(r) = \begin{cases} r^l, & r < a, \\ (a^{2l}/r^l)[(r^{2l} - R^{2l})/(a^{2l} - R^{2l})], & a < r < R. \end{cases} \tag{28}$$

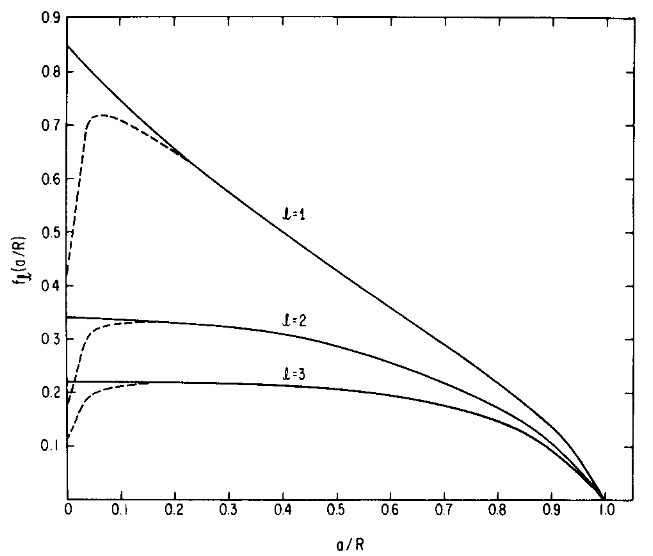

FIG. 4. Plot of $f_l(a/R)$ vs a/R for $l = 1, 2, 3$. The solid curves are from Eq. (30). The dotted lines indicate qualitatively the corrections due to second-order terms (see Appendix). The corrections are significant only for $a/R \lesssim |\Omega_0/\omega_p|$.

Substituting this function in Eq. (7) yields

$$(A_0^{(0)})_i = \frac{\int_0^R r\,dr\,\varphi_d J_l(K_{li}r)}{\int_0^R r\,dr\,J_l^2(K_{li}r)} = \frac{2la^l}{1-(a/R)^{2l}}\frac{2}{R^2}\frac{J_l(K_{li}a)}{K_{li}^2 J_{l+1}^2(K_{li}R)}. \qquad (29)$$

From Eqs. (26), (27), (29), and (17), we obtain an explicit expression for $\omega^{(1)}$:

$$\omega^{(1)} = \omega_{0d}^{(1)} = (l\omega_r - \omega_D)\frac{a}{L} \times \left(\frac{\Sigma_i\,[J_l^2(K_{li}a)/(K_{li}a)^3 J_{l+1}^2(K_{li}R)]}{\Sigma_i\,[J_l^2(K_{li}a)/(K_{li}a)^2 J_{l+1}^2(K_{li}R)]}\right). \qquad (30)$$

We note that $\omega^{(1)}$ is real and is independent of the temperature. Also, it depends linearly on the ratio a/L. The dependence of $\omega^{(1)}$ on a/R is via ω_D $\{=\omega_r[l-1+(a/R)^{2l}]\}$ and via the dimensionless expression within the large parentheses which we shall designate $f_l(a/R)$. A plot of f_l vs a/R is presented in Fig. 4 for the cases $l=1$, 2, and 3 (solid lines).

The expression for $\omega^{(1)}$ given by Eq. (30) is strictly true only if a/R is greater than ω_r/ω_p. For smaller values of a/R, we shall show in the Appendix that the last term in Eq. (24) which is an infinite sum of terms each of order $(R/L)^2$, adds up to a sum of order R/L and hence has to be included in Eq. (23). However, since we are already working in the limit $\omega_r/\omega_p \ll 1$, this correction to $\omega^{(1)}$ can generally be ignored.

The correction $A_0^{(1)}$ to the eigenvector can also be obtained from Eq. (23). Since $M_0(\omega^{(0)})$ is a real symmetric matrix, its eigenvalues λ_i are all real and the corresponding real eigenvectors X_i form a complete set. Since the eigenvector corresponding to the nondegenerate eigenvalue $\lambda_1 = 0$ is $X_1 = A_0^{(0)}$, we find from Eq. (23) that

$$A_0^{(1)} = -\sum_{i=2}^{\infty}\left(X_i^T\frac{(dM_0/d\omega^{(0)})\omega^{(1)} + B}{\lambda_i}X_1\right)X_i. \qquad (31)$$

We note that $A_0^{(1)}$ is real since all the quantities on the right-hand side of Eq. (31) are real. We also note that the component of $A_0^{(1)}$ in the direction of $A_0^{(0)}$ is not determined by Eq. (23). Imposition of a normalization condition of A_0, however, allows us to set this component equal to zero.

We shall now evaluate the imaginary part of the frequency ω_{0d} caused by the finite length of the column. This imaginary part arises from the coupling of the $n=0$ diocotron mode to the various $n\neq 0$ components which are Landau-damped. The diocotron mode can be envisioned as sharing in this Landau-damping. To obtain the damping rate of the D_0 mode, we inspect the solution of Eq. (24) for $\omega^{(2)}$:

$$\omega^{(2)} = \frac{-(A_0^{(0)})^T\{\omega^{(1)}(dM_0/d\omega^{(0)})A_0^{(1)} + \frac{1}{2}(\omega^{(1)})^2[d^2M_0/d(\omega^{(0)})^2]A_0^{(0)} + B\,\Sigma_{n=0}^{\infty}(-1)^n A_n^{(1)}\}}{(A_0^{(0)})^T(dM_0/d\omega^{(0)})A_0^{(0)}}. \qquad (32)$$

Since $A_0^{(0)}$, $A_0^{(1)}$, $M_0(\omega^{(0)})$, $\omega^{(1)}$, and B are all real, we immediately obtain

$$\mathrm{Im}\,\omega^{(2)} = -\frac{(A_0^{(0)})^T B\,\mathrm{Im}\,\Sigma_{n=1}^{\infty}(-1)^n A_n^{(1)}}{(A_0^{(0)})^T(dM_0/d\omega^{(0)})A_0^{(0)}}. \qquad (33)$$

Using Eqs. (A4), (A7), (27), (29), and (33), we finally obtain, in the limit $1 \gg (l\omega_r - \omega_D)/\omega_p \gg \lambda_D/L$ (where $\lambda_D = \bar{v}/\omega_p$ is the Debye length),

$$\mathrm{Im}\,\omega^{(2)} = -(l\omega_r - \omega_D)\left(\frac{l\omega_r - \omega_D}{\omega_p}\ln\frac{\omega_p}{l\omega_r - \omega_D}\right)\frac{\lambda_D}{L}\left[\left(\frac{32}{\pi}\right)^{1/2}\frac{\Sigma_i\Sigma_j\,[\alpha_{ij}J_l(K_{li}a)J_l(K_{lj}a)/N_iN_jK_{li}K_{lj}]}{(1/2l)[1-(a/R)^{2l}]}\right]. \qquad (34)$$

The term in large brackets is just a function of a/R and l. We denote it by $F_l(a/R)$ and display its dependence on a/R in Fig. 5 for the cases $l=1$, 2, and 3.

Finally, we investigate the behavior of the D_0 wavefunction in region ii for $\lambda_D/R \ll 1$. In the order $(R/L)^0$, the wavefunction $\varphi_{ii}^{(0)}(r,z) = \Sigma_m A_{m0}^{(0)} J_l(K_{lm}r)$ is independent of z and has a radial dependence shown in Eq. (28). The first-order corrections $A_n^{(1)}$ to the wavefunction are determined by Eq. (25). We shall consider $a/R \simeq 1$ in which case the middle term in Eq. (25) is small by order R/L and thus can be ignored. Keeping just the leading term in the expansion of M_n^{-1} (see Appendix), we obtain from Eqs. (7) and (25) the first-order correction part of the D_0 wavefunction

$$\varphi_{ii}^{(1)}(r,z) \simeq \sum_m \frac{2K_{lm}}{L}N_m A_{m0}^{(0)} J_l(K_{lm}r)\sum_n \frac{(-1)^{n+1}\cos(n\pi z/L)}{K_{lm}^2 N_m + (n\pi/L)^2 N_m - \frac{1}{2}N_m(\omega_p^2/\bar{v}^2)Z'[\Omega_0/\sqrt{2}(n\pi/L)\bar{v}]}, \qquad (35)$$

where $\Omega_0 = \omega_D - l\omega_r$. In the limit of small λ_D/R, this expression can be approximated by

$$\varphi_{ii}^{(1)}(r,z) \simeq \sum_m A_{m0}^{(0)} J_l(K_{lm}r)\left(-\frac{\Omega_0}{\omega_p}\frac{\cos(\Omega_0/\omega_p)K_{lm}z}{\sin(\Omega_0/\omega_p)K_{lm}L} - K_{lm}\lambda_D\frac{\cosh(z/\lambda_D)}{\sinh(L/\lambda_D)}\right). \qquad (36)$$

The first term on the right-hand side represents the temperature-independent correction to the wave function introduced by the finite length of the column. The second term (which is significant only for points within a few Debye lengths from the ends of the column) depends on temperature through λ_D.

Ignoring the temperature-dependent second term of Eq. (36), we find for the wavefunction $\varphi_{ii}\,(=\varphi_{ii}^{(0)} + \varphi_{ii}^{(1)})$,

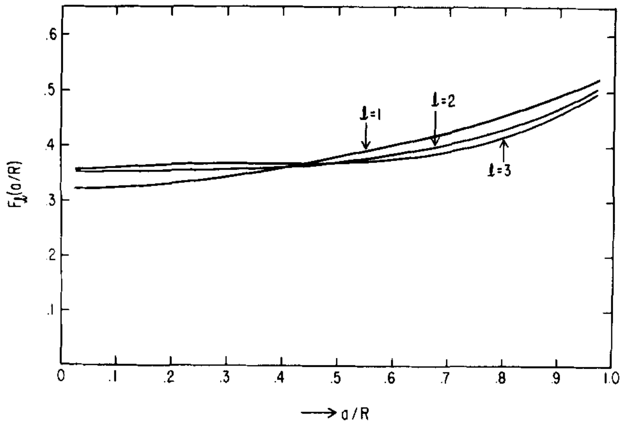

FIG. 5. Plot of $F_l(a/R)$ vs a/R for $l = 1, 2, 3$.

$$\epsilon_{zz} \frac{\partial \varphi_{ii}}{\partial z}\Big|_L \simeq -\sum_m A^{(0)}_{m0} K_{lm} J_l(K_{lm} r) \simeq \frac{\partial \varphi_{iii}}{\partial z}\Big|_L, \quad (37)$$

which is just the cold plasma boundary condition at $z = L$ for the z component of the wave electric field. In obtaining Eq. (37) we have used Eqs. (8) and (9) and the relation

$$\epsilon_{zz} = 1 - \omega_p^2/\Omega_0^2 \simeq -\omega_p^2/\Omega_0^2. \quad (38)$$

Since $|\epsilon_{zz}| \gg 1$, $\partial \varphi_{ii}/\partial z \simeq 0$ at $z = L$ for the cold plasma wavefunction.

Including the temperature-dependent correction in φ_{ii}, we find that it provides the dominant contribution to $\partial \varphi_{ii}/\partial z$ at $z = L$. We then obtain

$$\frac{\partial \varphi_{ii}}{\partial z}\Big|_L \simeq -\sum_m K_{lm} A^{(0)}_{m0} J_l(K_{lm} r) \simeq \frac{\partial \varphi_{iii}}{\partial z}\Big|_L, \quad (39)$$

i.e., the z component of the wave electric field is continuous across $z = L$.

Thus the overall picture of the wavefunction that emerges is the following. Over most of the column, the wavefunction is well approximated by the cold plasma wavefunction. Its radial structure is very close to that of the $k = 0$ diocotron wavefunction, and as a function of z it is slightly concave enabling it to satisfy the cold plasma boundary condition (37) at $z = L$. Equation (37) implies a discontinuity in $\partial \varphi/\partial z$ and hence a surface charge layer at $z = L$. Since the plasma is warm this charge layer is actually spread over a width $\sim \lambda_D$ and the wavefunction bends over in the region $L - \lambda_D \lesssim z < L$ to join smoothly with φ_{iii}.

In our model, the electrons are all reflected at the plane $z = L$. A more realistic model, one in which the region of reflection has a finite width $\sim \lambda_D$, will not qualitatively change the behavior of the wavefunction described above. It should be pointed out that a real plasma presents the added complexity that the end surface is not flat but is rounded with a shape that depends on the confining potential, the plasma density and the geometry of the system.[7]

IV. DEGENERACY

In writing Eqs. (22), we assumed that $M_{n>0}(\omega^{(0)}_{0d})$ was not close to being singular; this enabled us to set $A^{(0)}_{n>0} = 0$.

We now consider the case in which a solution $\omega^{(0)}_{sm}$ of the equation $|M_s(\omega)| = 0$ is very near $\omega^{(0)}_{0d}$ (m is negative as is clear from Fig. 3). Then the D_0 mode will be degenerate with the P_{sm} mode. In the zeroth order, the vector α is now of the form $\alpha = (A_0 = A^{(0)}_{0d}, A_1 = 0, A_2 = 0, \ldots, A_s = \alpha A^{(0)}_{sm}, A_{s+1} = 0, \ldots)$, where α is a number of order unity.

Assuming $|\omega^{(0)}_{0d} - \omega^{(0)}_{sm}| \ll \omega^{(0)}_{0d}$, we set $\omega = \omega^{(0)}_{sm} + \omega^{(1)}$ and Taylor-expand $M_0(\omega)$ and $M_s(\omega)$. Ignoring quadratic and higher-order terms in the expansion, we obtain the equations for A_0 and A_s [see Eq. (15)].

$$\begin{aligned}
&[M_0(\omega^{(0)}_{0d}) + (\omega^{(0)}_{sm} - \omega^{(0)}_{0d} + \omega^{(1)})(dM_0/d\omega^{(0)}_{0d}) + B\,] \\
&\quad \times (A^{(0)}_{0d} + A^{(1)}_{0d}) + (-1)^s B(\alpha A^{(0)}_{sm} + A^{(1)}_{sm}) = 0, \\
&(-1)^s 2B(A^{(0)}_{0d} + A^{(1)}_{0d}) \\
&\quad + [M_s(\omega^{(0)}_{sm}) + \omega^{(1)}(dM_s/d\omega^{(0)}_{sm}) + 2B\,] \\
&\quad \times [\alpha A^{(0)}_{sm} + A^{(1)}_{sm}] = 0.
\end{aligned} \quad (40)$$

Proceeding as before, we obtain

$$\begin{aligned}
\omega^{(1)} &= -\frac{(A^{(0)}_{0d})^T B\,[A^{(0)}_{0d} + (-1)^s \alpha A^{(0)}_{sm}]}{(A^{(0)}_{0d})(dM_0/\omega^{(0)}_{0d}) A^{(0)}_{0d}} - (\omega^{(0)}_{sm} - \omega^{(0)}_{0d}) \\
&= -\frac{2(A^{(0)}_{sm})^T B\,[(-1)^s A^{(0)}_{0d} + \alpha A^{(0)}_{sm}]}{\alpha (A^{(0)}_{sm})^T (dM_s/d\omega^{(0)}_{sm}) A^{(0)}_{sm}}.
\end{aligned} \quad (41)$$

Equation (41) can be regarded as a quadratic in α and can be readily solved for the two values of α and the corresponding pair of frequencies $\omega^{(1)}$. The degeneracy is strong when $|\alpha| \sim O(1)$ and is weak when $|\alpha| \to 0$ or $\to \infty$. Using the fact that $(A^{(0)}_{0d})^T (dM_0/d\omega^{(0)}_{0d}) A^{(0)}_{0d} \sim (A^{(0)}_{sm})^T (dM_s/d\omega^{(0)}_{sm}) A^{(0)}_{sm} \sim 1/\omega_r$, it can be easily shown that $|\alpha| \sim O(1)$ only if $|\omega^{(0)}_{sm} - \omega^{(0)}_{0d}| < \omega_r (R/L)$. Defining the width of degeneracy $\Delta\omega$ as $\Delta\omega = \omega_r (R/L) \sim |\omega^{(1)}|$, we find that in order to have strong degeneracy between the P_{sm} mode and the D_0 mode, $\omega^{(0)}_{sm} - \omega^{(0)}_{0d}$ must satisfy

$$|\omega^{(0)}_{sm} - \omega^{(0)}_{0d}| < \Delta\omega \sim |\omega^{(1)}|. \quad (42)$$

The frequencies $\omega^{(0)}_{sm}$ are complex because of Landau damping. For values close to $\omega^{(0)}_{0d}$, $\omega^{(0)}_{sm}$ is given by

$$\mathrm{Re}\,\omega^{(0)}_{sm} \simeq l\omega_r - \frac{\omega_p}{j_{l+1,|m|}} \frac{s\pi a}{L}, \quad (43)$$

$$\mathrm{Im}\,\omega^{(0)}_{sm} \simeq -\sqrt{2\pi} \frac{s\pi}{L} \bar{v} \left(\frac{\omega_p a}{\sqrt{2} j_{l+1,|m|} \bar{v}} \right)^4 \times \exp[-(\omega_p a/\sqrt{2} j_{l+1,|m|} \bar{v})^2]. \quad (44)$$

Since $\mathrm{Im}\,\omega^{(0)}_{sm}$ increases monotonically with s, there is a threshold value of s above which the inequality (42) cannot be satisfied and hence there can be no degeneracy between the D_0 mode and the P_{sm} mode. Also, for a given P_{sm} mode, there is a threshold temperature above which it cannot be degenerate with the D_0 mode. Both these thresholds are encompassed in the following necessary condition for strong degeneracy between the P_{sm} mode and the D_0 mode:

$$s\left(\frac{a}{j_{l+1,|m|} \lambda_D} \right)^3 \exp\left[-\frac{1}{2} \left(\frac{a}{j_{l+1,|m|} \lambda_D} \right)^2 \right] \lesssim \frac{\omega_r}{\omega_p} \frac{R}{a} j_{l+1,|m|}. \quad (45)$$

For example, for $\omega_r/\omega_p = 10^{-2}$, $l = 1$, and $a/R = 1/2$, the $P_{1,-1}$ mode can be degenerate with the D_0 mode only if $\lambda_D \lesssim 15a$. It should be noted that even if the inequality (45) is satisfied, one must still have the real part of $\omega_{sm}^{(0)}$ close enough to $\omega_{0d}^{(0)}$ to satisfy (42). This can be achieved by tuning one of the plasma parameters, most conveniently ω_c. From the expressions (43) for $\omega_{sm}^{(0)}$ and (21) for $\omega_{0d}^{(0)}$, one finds that Re $\omega_{sm}^{(0)} = \omega_{0d}^{(0)}$ when

$$\frac{\omega_c}{\omega_p} = \frac{1}{2}\frac{j_{l+1,|m|}}{s\pi}\frac{L}{a}\left[1-\left(\frac{a}{R}\right)^{2l}\right]. \tag{46}$$

When there is no degeneracy, the damping rate of the $n = 0$ diocotron mode is extremely weak, being of order $(R/L)(\lambda_D/L)\omega_{0d}^{(0)}$ [see Eq. (34)]. In the presence of degeneracy, however, the damping rate can be much higher. In fact, Eqs. (41) and (42) suggest that Im ω^1 can become as large as Re $\omega^1 \sim (R/L)\omega^{(0)}$. This effect provides a clear signature for degeneracy in an experiment. If one locked a receiver to the diocotron frequency $\omega_{0d}^{(0)}$ and continuously varied ω_c, one would see an undamped wave showing sudden damping as ω_c crossed the degeneracy value given by Eq. (46).

V. OTHER MODES

The present theory can be applied to the $n \neq 0$ modes in a straightforward manner. As mentioned in Sec. III, the zero-order solutions $\omega_n^{(0)}$ and $A_n^{(0)}$ are complex. The first-order correction $\omega_n^{(1)}$ is also, in general, complex. Whether this correction produced by the finite length of the column tends to stabilize or destabilize the zero-order mode depends on the particular situation considered.

The only modes we have not considered are the $n = 0$ plasma modes. For these egregious modes, $\omega_{0m}^{(0)} = l\omega_r$ $(m = \pm 1, \pm 2, ...)$ and all the particles are resonant with these modes; the modes seem to be intrinsically nonlinear. Furthermore, they are infinitely degenerate. These difficulties cannot be handled by the present theory.

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APPENDIX: EVALUATION OF Im $\Sigma_{n=1}^{\infty}(-1)^n A_n^{(1)}$

1. Evaluation of $\Sigma_{n=1}^{\infty}(-1)^n A_n^{(1)}$

We shall evaluate the sum $S = \Sigma_{n=1}^{\infty}(-1)^n A_n^{(1)}$ in terms M_n, B, and $A_0^{(0)}$. This result is used in obtaining the expression (34) for Im $\omega^{(2)}$. We shall also show that S, which is normally of order (R/L), turns out to be of order $(R/L)^0$ when $a/R \to 0$. In this limit, the term $B\,\Sigma(-1)^n A_n^{(1)}$ should be transferred from the second-order equation (24) to the first-order equation (23). As a result, the expression (26) for $\omega^{(1)}$ is modified for very small values of a/R.

From Eq. (25), it follows that

$$2\sum_{n=1}^{\infty} M_n^{-1} B A_0^{(0)} + 2\sum_{n=1}^{\infty} M_n^{-1} B \sum_{p=1}^{\infty}(-1)^p A_p^{(1)} + \sum_{n=1}^{\infty}(-1)^n A_n^{(1)} = 0,$$

or

$$\sum_{n=1}^{\infty}(-1)^n A_n^{(1)} = -\left(I + 2\sum_{n=1}^{\infty} M_n^{-1}(\omega^{(0)})B\right)^{-1} \times 2\sum_{n=1}^{\infty} M_n^{-1}(\omega^{(0)}) B A_0^{(0)}. \tag{A1}$$

We shall first evaluate the second term inside the large parentheses to see under what conditions it can be ignored in comparison with the first. In doing so we shall ignore thermal effects as being small. Then, the ijth element of the matrix M_n is given by

$$(M_n)_{ij} = \left[K_{li}^2 N_i + \left(\frac{n\pi}{L}\right)^2 N_i\right]\delta_{ij} - \left(\frac{n\pi}{L}\right)^2 \alpha_{ij}\frac{\omega_p^2}{\Omega_0^2} + \frac{2l\omega_r}{\Omega_0}\beta_{ij}, \tag{A2}$$

where α_{ij}, β_{ij}, and N_i are defined by Eqs. (19) and (18), respectively; also, $\Omega_0 = \omega^{(0)} - l\omega_r$. As $a \to 0$, the off-diagonal elements of M_n go to zero. With this in mind, we shall expand M_n^{-1} in powers of the nondiagonal elements. In the lowest order, M_n^{-1} is given by

$$(M_n^{-1})_{ij} = \delta_{ij}/(M_n)_{ii}.$$

Substituting $(M_n)_{ii}$ from Eq. (A2), we sum over n to obtain[8]

$$\sum_{n=1}^{\infty}(M_n^{-1})_{ij} = \delta_{ij}\left\{\frac{L}{2}\coth\left[L\left(\frac{K_{li}^2 N_i + (2l\omega_r/\Omega_0)N_i}{N_i - (\omega_p^2/\Omega_0^2)\alpha_{ii}}\right)^{1/2}\right] \times\left[\left(K_{li}^2 N_i + \frac{2l\omega_r}{\Omega_0}\beta_{ii}\right)^{1/2}\left(N_i - \frac{\omega_p^2}{\Omega_0^2}\alpha_{ii}\right)^{1/2}\right]^{-1} - \frac{1}{2[K_{li}^2 N_i + (2l\omega_r/\Omega_0)\beta_{ii}]}\right\}. \tag{A3}$$

In the limit $a \to 0$, $\Sigma(M_n^{-1})_{ij} \to \delta_{ij}[L/2K_{li}N_i + O(1)]$. For $\omega_p^2/\Omega_0^2 \gg N_i/\alpha_{ii} \simeq R^2/a^2$, $\Sigma(M_n^{-1})_{ij} \to \delta_{ij}\{O[(L/2K_{li}N_i) \cdot [(\Omega_0/\omega_p)(R/a)]]\}$. Similar conclusions hold in higher orders of the expansion. Thus we see that for $a \to 0$, $2\,\Sigma\, M_n^{-1}B \to I$, but for $a/R > |\Omega_0/\omega_p|$, $2\,\Sigma\, M_n^{-1}B$ is small in comparison with I and can be ignored. In the limit $a \to 0$, $\Sigma(-1)^n A_n^{(1)} \to -\frac{1}{2}A_0^{(0)}$, and the normally second-order (in R/L) quantity $B\,\Sigma(-1)^n A_n^{(1)}$ happens to be a first-order quantity. This necessitates the shifting of the term from the second-order equation (24) to the first-order equation (23). As a result, the value of $\omega^{(1)}$ in the limit $a \to 0$ is half the value given by Eq. (26). As a/R increases beyond $|\Omega_0/\omega_p|$, the magnitude of $B\,\Sigma(-1)^n A_n^{(1)}$ rapidly drops off, and ω approaches the value given by Eq. (26). The modified plots of $f_l(a/R)$ [see Eq. (30)] represented qualitatively by the dashed lines in Fig. 4 agree with results previously obtained.[2]

Excluding very small values of a/R, we can replace the large parentheses in Eq. (A1) with the identity matrix I. Thus we have

$$\sum_{n=1}^{\infty}(-1)^n A_n^{(1)} = -2\sum_{n=1}^{\infty} M_n^{-1}(\omega^{(0)}) B A_0^{(0)}. \tag{A4}$$

Since B and $A_0^{(0)}$ are known real quantities, the imaginary part of the left-hand side [which determines Im $\omega^{(2)}$ through Eq. (33)] is known if Im $\Sigma M_n^{-1}(\omega^{(0)})$ is known.

2. Evaluation of Im $\Sigma_{n=1}^{\infty} M_n^{-1}(\omega^{(0)})$

The value of Im M_n^{-1} is determined by the value of Im Z', and Im Z' vanishes rapidly when its argument increases beyond unity. Thus, in evaluating Im $\Sigma_n M_n^{-1}$ we can restrict ourselves to $n \gtrsim |\Omega_0| L/\sqrt{2}\pi\bar{v}$. For $\bar{v}\to 0$, the following approximation is valid in this regime:

$$(M_n)_{ij} = (n\pi/L)^2 N_i \delta_{ij} - \tfrac{1}{2}\alpha_{ij}(\omega_p^2/v^2) Z'. \tag{A5}$$

Using the identity

$$\delta(r-r') = r'\sum_i \frac{J_l(K_{li}r)J_l(K_{li}r')}{N_i},$$

it can be easily shown that

$$(M_n^{-1})_{ij} = \left(\frac{L}{n\pi}\right)^2 \frac{\delta_{ij}}{N_i} + \frac{(L/n\pi)^2 Z'}{2(n\pi\bar{v}/L\omega_p)^2 - Z'}\,\frac{\alpha_{ij}}{N_i N_j}. \tag{A6}$$

Thus we have

$$\mathrm{Im}\sum_{n=1}^{\infty}(M_n^{-1})_{ij} = \frac{\alpha_{ij}}{N_i N_j}\,\mathrm{Im}\sum_{n=1}^{\infty}\left(\frac{L}{n\pi}\right)^2 Z'\left(\frac{\Omega_0}{\sqrt{2}(n\pi/L)\bar{v}}\right) \times\left[\left(\sqrt{2}\frac{n\pi}{L}\frac{\bar{v}}{\omega_p}\right)^2 - Z'\left(\frac{\Omega_0}{\sqrt{2}(n\pi/L)\bar{v}}\right)\right]^{-1}.$$

In the limit $|\Omega_0/\omega_p|\ (\gg \bar{v}/L\omega_p)\to 0$,

$$\mathrm{Im}\sum_{n=1}^{\infty}(M_n^{-1})_{ij} \simeq \frac{\alpha_{ij}}{N_i N_j}\frac{L^2}{\pi^2}\frac{(2\pi)^{3/2}}{4}\frac{\lambda_D}{L}\left|\frac{\Omega_0}{\omega_p}\right| \ln\left(\left|\frac{\Omega_0}{\omega_p}\right|\right), \tag{A7}$$

where $\lambda_D = \bar{v}/\omega_p$ is the Debye length.

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