A nonlinear diocotron mode

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A perturbation procedure for the construction of a nonlinear diocotron mode of an infinite length cylindrical pure electron plasma column is presented. The plasma is modeled as a cold fluid executing $\mathbf{E}\times\mathbf{B}$ drift in a strong axial magnetic field. The mode has no axial variation and depends on θ (azimuthal angle) and t (time) through the combination $\theta - \Omega t$. Thus the potential and the density are stationary in the frame rotating with angular frequency Ω. In this frame, the potential satisfies the relation $\nabla^2\phi = F(\phi)$ where F is a function of ϕ only. This equation is solved perturbatively to determine a nonlinear mode supported by a cylindrically symmetric equilibrium density profile. The properties of the mode depend only on the offset of the plasma column from the axis of the bounding conducting cylinder and the frequency of the mode is larger than the value predicted by linear theory by an amount roughly proportional to the square of the offset.

I. INTRODUCTION

The diocotron mode is a low-frequency drift mode supported by an electron plasma (more generally, by any nonneutral plasma) confined by a magnetic field. A simple physical picture for the mode can be given for the typical electron plasma confinement geometry[1]—a cylindrical electron plasma confined by a strong axial magnetic field inside a concentric grounded conducting tube. A state of dynamical equilibrium for such an electron column is one of $\mathbf{E}\times\mathbf{B}$ rotation (right-handed circular motion relative to the magnetic field) caused by the radial self electric field. A displacement of the column away from the equilibrium position induces extra positive charge on the nearest portion of the wall and the resulting extra electric field causes the displaced column to $\mathbf{E}\times\mathbf{B}$ drift about the axis in the same direction as the equilibrium drift but with a lower frequency (Fig. 1). For a small displacement, this second drift is seen by a stationary observer as the linear diocotron mode with azimuthal mode number $l = 1$ (having a density perturbation $\sim\cos\theta$) moving counterclockwise along the surface of the plasma column.

An interesting property of the diocotron mode is that in a frame rotating counterclockwise along with the equilibrium $\mathbf{E}\times\mathbf{B}$ drift, the mode is moving clockwise and as is the rule in such a situation,[2] the mode has negative energy. As a consequence, any process that takes energy out of the mode causes it to grow.[3] This characteristic has been used in recent experiments[4] to drive the diocotron mode quasistatically to large amplitudes. The conducting tube surrounding the plasma is split into sections (Fig. 1), which are connected through a resistance. The diocotron mode loses energy and thus grows, because the image charges on the wall induced by the mode generate heat as they traverse the resistor. Typically, only the diocotron mode with azimuthal mode number $l = 1$ is observed to grow in these experiments. The growth of the mode can be stopped at any level by shorting the resistance, and the wave persists with a very small damping rate. A quasistatic reduction in the wave amplitude can also be achieved[5] by inserting a suitable negative effective resistance between the sections of the tube, i.e., by negative feedback. The equilibrium density profile seems to remain unchanged to good accuracy after a cycle of growth and reduction and the frequency of the mode (i.e., the frequency with which the perturbed density profile rotates around the axis of the tube) increases with amplitude.[6]

In this article, we develop a systematic perturbation theory for the construction of such a nonlinear diocotron mode, which can be continuously grown from zero amplitude on an infinite length cylindrical pure electron plasma column having a smooth radial equilibrium density profile. The plasma

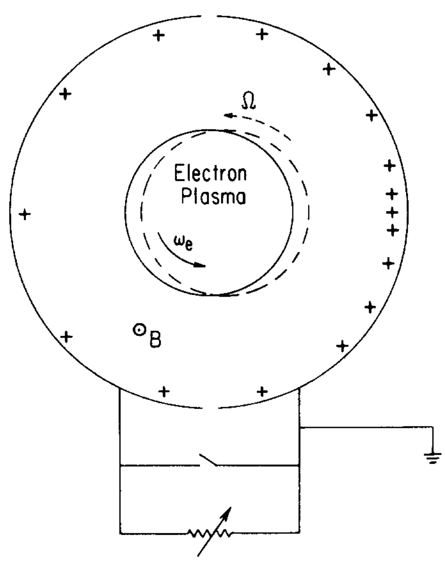

FIG. 1. Schematic diagram for the quasistatic growth of the diocotron mode. Here ω_e is the angular frequency of the equilibrium $\mathbf{E}\times\mathbf{B}$ rotation and Ω is the frequency of the diocotron mode with azimuthal mode number 1.

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density profile is predicted not only to move away from the axis of the tube, but to also deform with increasing amplitude. Using conservation of the total number of particles, we also obtain the variation of the mode frequency with amplitude; it is shifted upwards from the value predicted by linear theory by an amount proportional to the square of the mode amplitude.

We restrict our theory to diocotron modes that have no axial variation ($k_z = 0$). The electron plasma is modeled as a cold dissipationless fluid executing $\mathbf{E}\times\mathbf{B}$ drift (resulting from $\mathbf{E}$ produced by the plasma itself) in a strong axial magnetic field. Thus, our basic equations [(1)–(3)] can also describe two-dimensional incompressible flows in fluid dynamics,[3] in convective cells (in the zero temperature limit)[7] and in ideal magnetohydrodynamics (MHD).[8]

The article is organized as follows. In Sec. II, we review the conventional linear theory as a benchmark. In Sec. III, we develop our nonlinear perturbational approach, which we carry out explicitly up to the third order. Our theory applies to a "steady-state" diocotron mode for which the density and the potential depend on time t and azimuthal angle θ only through the combination $\theta - \Omega t$, i.e., the density and the potential are stationary in a frame rotating with angular velocity Ω. In this frame, the potential is found to satisfy the relation $\nabla^2\phi = F(\phi)$, where F is a function of ϕ only. Our theory determines the function $F(\phi)$ for a given equilibrium density profile and solves this relation in a perturbative manner. The first-order solution [$\propto \cos(\theta - \Omega t)$] agrees with the conventional linear analysis for the diocotron mode having azimuthal mode number $l = 1$. Higher-order terms represent corrections with more complicated angular dependence, introduced by noninfinitesimal wave amplitude. In Sec. IV, we obtain the nonlinear frequency shift by demanding that the total number of particles be the same with or without the diocotron mode. The frequency shift is positive and is proportional to the square of the wave amplitude. In Sec. V, we present numerical solutions calculated for a typical experimental profile.

II. LINEAR THEORY

We assume that the electrons can be treated as a cold dissipationless fluid and that the motion perpendicular to the magnetic field $B\hat{z}$ is just the $\mathbf{E}\times\mathbf{B}$ drift motion resulting from the self-electric field. The equations governing the two-dimensional motion of the electrons (with charge $= -e$) are

$$\frac{\partial n}{\partial t} + \mathbf{v}\cdot\nabla n = 0, \tag{1}$$

$$\mathbf{v} = -(c/B)\nabla\Phi\times\hat{z}, \quad \nabla\cdot\mathbf{v} = 0, \tag{2}$$

$$\nabla^2\Phi = 4\pi e n, \tag{3}$$

where, as in the rest of the article, $\nabla \equiv \nabla_\perp$. Linearizing the above equations and assuming a dependence $e^{-i\omega t + il\theta}$ for the linearized quantities, we obtain, for the continuity equation

$$[-i\omega + il\omega_e(r)]n_1 = -\frac{dn^0_{\rm lab}}{dr}v_{1r} = \frac{dn^0_{\rm lab}}{dr}\frac{ilc}{Br}\Phi_1, \tag{4}$$

where $n^0_{\rm lab}(r)$ is the equilibrium radial density profile and $\omega_e(r)$ is the equilibrium $\mathbf{E}\times\mathbf{B}$ drift frequency of the plasma column;

$$\omega_e(r) = -\frac{c}{r}\frac{E_r}{B} = \frac{4\pi ec}{Br^2}\int_0^r r'\,dr'\,n^0_{\rm lab}(r'). \tag{5}$$

Combining (4) and the linearized Poisson's equation yields the eigenvalue equation

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\Phi_1 - \frac{l^2}{r^2}\Phi_1 = -\frac{4\pi ec}{Br}\frac{dn^0_{\rm lab}}{dr}\frac{l\Phi_1}{\omega - l\omega_e(r)}, \tag{6}$$

with Φ_1 satisfying the boundary conditions $\Phi_1 = 0$ at $r = 0, R$ (ignoring the trivial $l = 0$ case), where R is the radius of the surrounding conducting tube.

For the special case[9] of $l = 1$, the only solution of Eq. (6) for an arbitrary choice of density profile $n^0_{\rm lab}(r)$ is $\Phi_1 \propto r[\omega_e(r) - \omega_e(R)]$ corresponding to the eigenvalue $\omega = \omega_e(R)$; from Eq. (5) it is clear that the eigenvalue depends only on the total charge in the system and is independent of the structure of the density profile. For $l \geqslant 2$, there are no solutions to Eq. (6) unless $dn^0_{\rm lab}/dr = 0$ at points where the resonant denominator vanishes. This is related to the fact that particles that are at the resonant radius (where $\omega - l\omega_e = 0$) have an $\mathbf{E}\times\mathbf{B}$ drift velocity equal to the azimuthal phase velocity of the wave and cause the wave to damp or grow depending on whether $dn^0_{\rm lab}/dr$ is negative or positive, respectively, at the resonant radius. In analogy with the conventional Landau damping, this process is sometimes referred to as "spatial Landau damping."[3] As with conventional Landau damping, 'normal modes' (valid for all times) of the system do not exist if there is damping. Therefore, if $n^0_{\rm lab}$ is a monotonically decreasing function of r, vanishing only at $r = R$, then $dn^0_{\rm lab}/dr \neq 0$ for all $r < R$ and hence the $l \geqslant 2$ normal mode solutions to Eq. (6) do not exist [except, of course, the trivial kind, $\Phi_1(r) = 0$].

III. NONLINEAR THEORY

We shall now construct a theory to describe steady-state nonlinear diocotron waves in an electron column. We assume a realistic model of a monotonically decreasing radial profile for the equilibrium density $n^0_{\rm lab}(r)$. The theory is a perturbation expansion in the wave amplitude and reduces in the limit to the linear theory described above.

We begin by noting that for a "steady-state" diocotron wave, there exists a frame (rotating at an angular frequency, say Ω) in which the flow pattern is independent of time, i.e., in which $\partial/\partial t = 0$. We can combine Eqs. (1)–(3) in this frame to obtain

$$\nabla\phi\times\hat{z}\cdot\nabla\nabla^2\phi = 0$$

or

$$\nabla\nabla^2\phi\times\nabla\phi = \mathbf{0}, \tag{7}$$

where ϕ is the potential measured in the moving frame. Equation (7) implies that $\nabla^2\phi$ is a function of ϕ:

$$\nabla^2\phi = F(\phi). \tag{8}$$

To determine the functional dependence F and to then solve Eq. (8), we expand ϕ as

$$\phi = \phi^0(r) + \sum_{i=1}^{\infty}\sum_{l=0}^{\infty} \epsilon_l^i(r)\cos l\theta, \tag{9}$$

where the superscripts denote the order of perturbation (see Secs. III A and III B) and $\theta = 0$ is the axis of symmetry. Similarly,

$$\Omega = \sum_{i=0}^{\infty} \Omega^i. \tag{10}$$

Since the laboratory frame electric field is given in terms of the moving frame electric field $\mathbf{E}$ by

$$\mathbf{E}_{\rm lab} = \mathbf{E} - (\mathbf{v}_{\rm rot}\times\mathbf{B})/c = \mathbf{E} - (r\Omega B/c)\hat{r},$$

we have the following relationship between quantities measured in the laboratory frame and the rotating frame:

$$\phi_{\rm lab} = \Phi = \phi + (\Omega B/2c)r^2, \tag{11a}$$

$$4\pi e n_{\rm lab} = \nabla^2\phi_{\rm lab} = F(\phi) + (2\Omega B/c). \tag{11b}$$

In the absence of the wave, $n^0_{\rm lab}$ is a smooth decreasing function of r and thus F is a smooth function of the equilibrium potential ϕ^0. Since the mode we have in mind involves a displacement of the equilibrium profile (with some possible deformation), we expect F to be a smooth function of ϕ even when the wave is present. Substituting (9) into (8), and Taylor expanding $F(\phi)$ about $\phi = \phi^0$, we obtain

$$\nabla^2\left(\phi^0(r) + \sum_{i=1}^{\infty}\sum_{l=0}^{\infty}\epsilon_l^i(r)\cos l\theta\right) = F[\phi^0(r)] + \sum_{m=1}^{\infty}\frac{1}{m!}F^{(m)}\left(\sum_{i=1}^{\infty}\sum_{l=0}^{\infty}\epsilon_l^i(r)\cos l\theta\right)^m, \tag{12}$$

with the abbreviation

$$F^{(m)} = \left.\frac{d^mF(\phi)}{d\phi^m}\right|_{\phi=\phi^0}. \tag{13}$$

Since $\phi(r=R,\theta) = \text{const}$, we must have

$$\epsilon^i_{l>1}(r=R) = 0, \quad \text{for all } i. \tag{14a}$$

Furthermore, since the angular part of ∇^2 in Eq. (12) introduces terms like $-l^2\epsilon_l^i(r)/r^2$, we must also have

$$\epsilon^i_{l>1}(r=0) = 0, \quad \text{for all } i. \tag{14b}$$

The $l = 0$ components ϵ_0^i are not constrained by the conditions (14a) and (14b). Instead, they obey the single condition

$$\left.\frac{d\epsilon_0^i(r)}{dr}\right|_{r=0} = 0, \quad \text{for all } i, \tag{15}$$

which follows immediately from the structure of ∇^2.

To solve Eq. (12) subject to the boundary conditions (14) and (15), we need to know $F(\phi^0)$. The determination of this functional dependence follows from the observation that in the limit of the wave amplitude going to zero ($\epsilon_l^i\to 0$), Eqs. (11a) and (11b) imply

$$\phi^0 = \phi^0_{\rm lab} - \frac{\Omega^0B}{2c}r^2, \quad F(\phi^0) = 4\pi e n^0_{\rm lab} - \frac{2\Omega^0B}{c}, \tag{16}$$

where $n^0_{\rm lab}(r)$ and $\phi^0_{\rm lab}(r)$ are the laboratory frame density and the electric potential of the equilibrium cylindrically symmetric electron column. Since $n^0_{\rm lab}(r)$ is assumed to be a monotonically decreasing function of r vanishing only at $r = R$, F is also a monotonically decreasing function of r. Also,

$$\frac{d\phi^0}{dr} = \frac{1}{r}\int_0^r dr'\, r'F[\phi^0(r')] = \frac{1}{r}\left(4\pi e\int_0^r dr'\, r' n^0_{\rm lab}(r') - \frac{\Omega^0 B}{c}r^2\right). \tag{17}$$

When Ω^0 is determined in the first-order analysis below, we shall find that the quantity in the large parentheses is positive. So ϕ^0 is a monotonically increasing function of r and thus can be inverted to obtain a smooth single-valued function $F(\phi^0)\equiv F[r(\phi^0)]$.

A. First-order analysis

In the first order, Eq. (12) reads

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{l^2}{r^2}\right)\epsilon_l^1(r) = F^{(1)}\epsilon_l^1(r)\,(l = 0,1,2,\ldots). \tag{18}$$

Since

$$F^{(1)} = \frac{dF}{d\phi^0} = \frac{dF/dr}{d\phi^0/dr} = \frac{4\pi e\, dn^0_{\rm lab}/dr}{d\phi^0_{\rm lab}/dr - (\Omega^0B/c)r},$$

we see that Eq. (18) is identical to Eq. (6) of the linear theory with the identification $l\Omega^0\to\omega$. As discussed in connection with Eq. (6), Eqs. (18) for $l\geqslant 2$ have only trivial solutions $\epsilon^1_{l>2} = 0$ for any choice of Ω^0. On the other hand, for one particular value (the linear $l = 1$ diocotron frequency) of Ω^0, we can find a nontrivial solution to the $l = 1$ equation. To determine this Ω^0, we note that $\epsilon_1^1(r) = \epsilon R\, d\phi^0/dr$ is a solution to the $l = 1$ component of Eqs. (18):

$$\text{lhs} = \epsilon R\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\frac{d\phi^0}{dr} - \frac{1}{r^2}\frac{d\phi^0}{dr}\right) = \epsilon R\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}r\frac{d\phi^0}{dr}\right) = \epsilon R\frac{dF(\phi^0)}{dr} = \epsilon R\frac{dF}{d\phi^0}\frac{d\phi^0}{dr} = \text{rhs}.$$

The dimensionless parameter ϵ is the small parameter of our perturbation theory. Constraints on its value will be obtained in Sec. III B.

The value of Ω^0 for which $\epsilon_1^1 = \epsilon R\, d\phi^0/dr$ is nontrivial is obtained by imposing the boundary condition $\epsilon_1^1(r=R) = 0$ together with Eq. (17). This yields

$$\Omega^0 = \frac{4\pi ec}{BR^2}\int_0^R dr'\, r'n^0_{\rm lab}(r') = \omega_e(R). \tag{19}$$

For a monotonically decreasing $n^0_{\rm lab}(r)$, the integral $4\pi e\int_0^r dr'\, r'n^0_{\rm lab}(r')$ in the expression (17) for $d\phi^0/dr$ increases like r^2 near the origin but less rapidly near $r = R$. Since from Eq. (19) the quadratic Ω^0Br^2/c is equal to the value of the integral at $r = R$, it is clearly less than the value of the integral for $r<R$. Therefore, $d\phi_0/dr\geqslant 0$ (the equality sign holding at $r = 0,R$) and hence, as claimed before, $\phi^0(r)$ is a monotonically increasing function of r. Thus, one can indeed invert $\phi^0(r)$ to obtain a smooth single-valued function $r(\phi^0)$ and hence a smooth single-valued function $F(\phi^0)\equiv F[r(\phi^0)]$.

With the choice of Ω^0 given by Eq. (19), we go back to the $l=0$ component of Eqs. (18) and solve for ϵ_0^1. Since ϵ_0^1 has only one boundary condition [Eq. (15)], it is determined only to within an arbitrary multiplicative constant. This interdeterminacy will be removed in the second-order analysis.

B. Second-order analysis

The second-order equations are

$$\frac{1}{r}\frac{d}{dr}r\frac{d\epsilon_0^2}{dr} = F^{(1)}\epsilon_0^2 + F^{(2)}\left(\frac{\epsilon_0^1\epsilon_0^1}{2} + \frac{\epsilon_1^1\epsilon_1^1}{4}\right), \qquad (20a)$$

$$\frac{1}{r}\frac{d}{dr}r\frac{d\epsilon_1^2}{dr} - \frac{1}{r^2}\epsilon_1^2 = F^{(1)}\epsilon_1^2 + F^{(2)}\epsilon_0^1\epsilon_1^1, \qquad (20b)$$

$$\frac{1}{r}\frac{d}{dr}r\frac{d\epsilon_2^2}{dr} - \frac{4}{r^2}\epsilon_2^2 = F^{(1)}\epsilon_2^2 + F^{(2)}\frac{\epsilon_1^1\epsilon_1^1}{4}, \qquad (20c)$$

$$\frac{1}{r}\frac{d}{dr}r\frac{d\epsilon_l^2}{dr} - \frac{l^2}{r^2}\epsilon_l^2 = F^{(1)}\epsilon_l^2 \quad (l=3,4,...)\,. \qquad (20d)$$

Using the same argument given in the first-order analysis, we set $\epsilon_{l>3}^2(r)=0$. We then turn our attention to the equation for ϵ_1^2, which can be rewritten as

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{1}{r^2} - F^{(1)}\right)\epsilon_1^2 = F^{(2)}\epsilon_0^1\epsilon_1^1. \qquad (21)$$

The operator in the parentheses is a second-order Sturm—Liouville operator whose eigenfunctions [satisfying the boundary conditions (14a) and (14b)] form a complete set. From the first-order analysis, we know that $\epsilon_1^1(r) = \epsilon R\, d\phi^0/dr$ is the eigenfunction of this operator corresponding to the eigenvalue 0. It follows that in order to have any allowed solution to Eq. (21), the right-hand side of Eq. (21) must be orthogonal to ϵ_1^1, i.e.,

$$\int_0^R r\,dr\,\epsilon_1^1(F^{(2)}\epsilon_0^1\epsilon_1^1) = 0. \qquad (22)$$

We use this condition to find the multiplicative constant in ϵ_0^1 left undetermined in the first-order analysis. In general, it is zero and thus we obtain $\epsilon_0^1(r)=0$. The equation for ϵ_1^2 is then the same as that for ϵ_1^1 and without loss of generality we can set $\epsilon_1^2(r)=0$.

The inhomogeneous terms in the Eqs. (20a) and (20c) are now completely determined and we can solve for ϵ_0^2 and ϵ_2^2 subject to the boundary conditions (14) and (15). Again, since ϵ_0^2 is subject to the single boundary condition (15), it can be determined only to within an arbitrary multiple of the homogeneous equation solution. This indeterminacy is removed via an integral constraint in the third-order analysis.

The validity of our perturbation theory requires that the second-order quantities ϵ^2 be smaller than the first-order quantities ϵ^1. From Eqs. (18) and (20),

$$\frac{\epsilon^2}{\epsilon^1} \sim \frac{F^{(2)}\epsilon^1}{F^{(1)}} \sim \epsilon R\frac{dF/dr}{F} \sim \epsilon\frac{R}{L},$$

where L is the characteristic length of variation of the equilibrium density profile. Thus, it is really $\tilde{\epsilon} \equiv \epsilon(R/L)$ that is the expansion parameter and we must have

$$\tilde{\epsilon} = \epsilon(R/L) < 1, \qquad (23)$$

for our theory to be valid.

C. Third-order analysis

Using previous arguments, we find that $\epsilon_2^3(r) = \epsilon_{l>4}^3(r) = 0$. Also $\epsilon_0^3(r)$ is in general restricted to be zero by a condition analogous to (22) in the fourth-order analysis. The function ϵ_1^3 satisfies the equation

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\epsilon_1^3 - \frac{1}{r^2}\epsilon_1^3 - F^{(1)}\epsilon_1^3 = F^{(2)}\left(\epsilon_1^1\epsilon_0^2 + \frac{\epsilon_1^1\epsilon_2^2}{2}\right) + F^{(3)}\left(\frac{\epsilon_1^1}{2}\right)^3. \qquad (24)$$

As before, the right-hand side has to satisfy the orthogonality condition

$$\int_0^R r\,dr\,\epsilon_1^1\left[F^{(2)}\left(\epsilon_1^1\epsilon_0^2 + \frac{\epsilon_1^1\epsilon_2^2}{2}\right) + F^{(3)}\left(\frac{\epsilon_1^1}{2}\right)^3\right] = 0, \qquad (25)$$

which fixes the multiplicative factor in ϵ_0^2. Not surprisingly, it is $\sim O(\epsilon^2)$. With the value of ϵ_0^2 determined by Eq. (25), we can solve Eq. (24) for ϵ_1^3. Note that we can restrict ϵ_1^3 to be orthogonal to ϵ_1^1 without a loss of generality.

The last remaining third-order quantity is $\epsilon_3^3(r)$, which satisfies the equation

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\epsilon_3^3 - \frac{9}{r^2}\epsilon_3^3 - F^{(1)}\epsilon_3^3 = F^{(2)}\frac{\epsilon_1^1\epsilon_2^2}{2} + \frac{F^{(3)}}{3}\left(\frac{\epsilon_1^1}{2}\right)^3. \qquad (26)$$

The perturbation analysis can be continued along these lines to any order desired. The general trend should be clear by now. In the ith order, $\epsilon_{l>1}^i(r)$ is uniquely determined, but $\epsilon_0^i(r)$ is determined only to within a multiple of the homogeneous equation solution. This indeterminacy is removed by demanding that the inhomogeneous term in the equation for ϵ_1^{i+1} be orthogonal to ϵ_1^1. Furthermore, without loss of generality, we restrict $\epsilon_1^{i>1}$ to be orthogonal to ϵ_1^1.

IV. FREQUENCY SHIFT

The frequency shift resulting from the nonlinearity is determined in a perturbation series using the physical requirement that the total number of particles be the same with and without the wave, i.e.,

$$\int_0^{2\pi} d\theta \int_0^R r\,dr\,n_{\rm lab}(r,\theta,t) = 2\pi\int_0^R r\,dr\,n_{\rm lab}^0(r), \qquad (27)$$

where, from Eq. (12),

$$4\pi e n_{\rm lab}(r,\theta,t) = F[\phi(r,\theta-\Omega t)] + (2B/c)\Omega.$$

We use the expansion (10) for Ω and the Taylor expansion [right-hand side of Eq. (12)] for F in Eq. (27) and equate terms of the same order to obtain Ω^i in terms of the integrals of the known functions $\epsilon_l^i(r)$. For example, keeping terms up to the third order, Eq. (27) implies

$$\int_0^R r\,dr\left(F^{(1)}\epsilon_0^2 + F^{(2)}\frac{\epsilon_1^1\epsilon_1^1}{4} + \frac{2B}{c}(\Omega^1+\Omega^2+\Omega^3)\right) = 0.$$

Therefore, $\Omega^1 = \Omega^3 = 0$ and

$$\frac{\Omega^2}{\Omega^0} = \frac{-\int_0^R r\,dr(F^{(1)}\epsilon_0^2 + F^{(2)}\epsilon_1^1\epsilon_1^1/4)}{4\pi e\int_0^R r\,dr\,n_{\rm lab}^0(r)}. \qquad (28)$$

There has been some speculation[9] that the linear diocotron frequency might remain unchanged in the nonlinear regime.

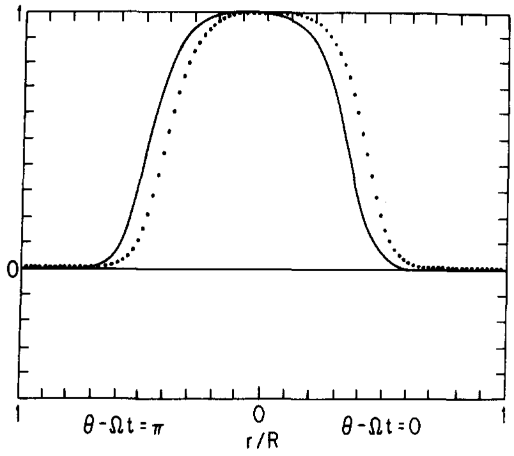

FIG. 2. Typical experimental equilibrium density profile modeled by $n^0_{\rm lab}(r) = [1 + \exp(-a_2)]/[1 + \exp(a_1 r^2/R^2 - a_2)]$ with $a_1 = 20$ and $a_2 = 3.2$ (dotted line). The calculated perturbed density profile for $\epsilon = 0.07$ is plotted along the $\theta - \Omega t = 0, \pi$ axis (solid line).

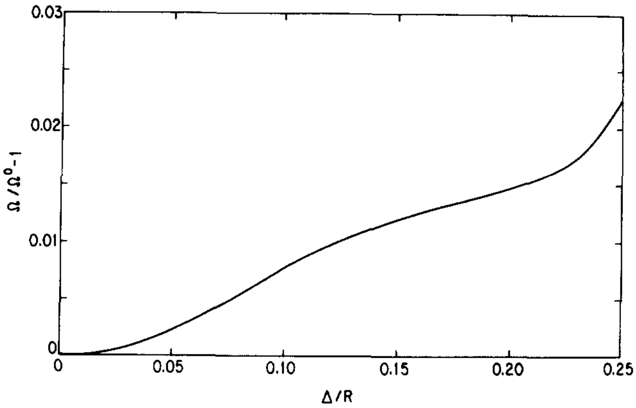

FIG. 4. The frequency shift $\Omega/\Omega^0 - 1$ versus Δ/R, where Ω^0 is the linear diocotron frequency and Δ is the displacement of the density peak from axis axis of the conducting tube.

From (28), it is clear that, in general, there is an amplitude dependent frequency shift. However, if $F^{(2)} = F^{(3)} = \cdots = 0$ [i.e., if $F(\phi^0) = \alpha\varphi^0 + \beta$], it follows from Eq. (12) that $\epsilon_l^{i>1}$ obey the same differential equations (and boundary conditions) as ϵ_l^1 and thus are just multiples of ϵ_l^1. Without loss of generality, we can set the multiplicative constants to zero. We also note that the functions ϵ_0^i $(i = 1,2,...)$ satisfy the same equation as $(\phi^0 + \beta/\alpha)$ and without loss of generality can be set equal to zero. From the structure of the above equations for Ω^i, it is clear that $\Omega^1 = \Omega^2 = \Omega^3 = \cdots = 0$, and the frequency shift is zero independent of the amplitude ϵ of the wave. In fact, the speculation in Ref. 9 was based on an analysis of this particular choice for $F(\phi^0)$. A linear dependence on ϕ^0 does not, however, describe a realistic situation because the corresponding density $n_{\rm lab}(r,\theta,t)$ cannot satisfy the condition $n_{\rm lab}(r = R,\theta,t) = 0$ without becoming negative somewhere within the boundary.

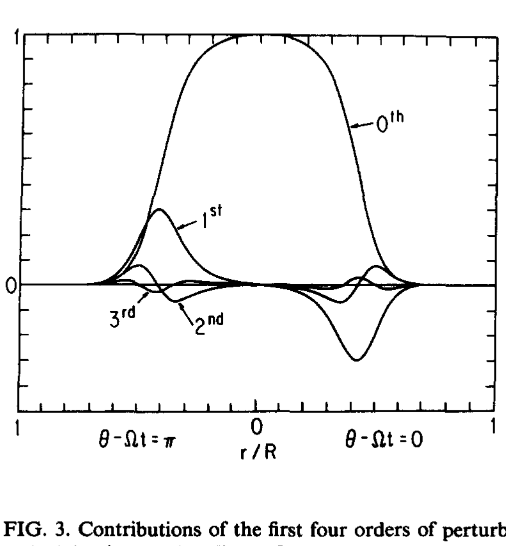

FIG. 3. Contributions of the first four orders of perturbation to the perturbed density profile (Fig. 2) for $\epsilon = 0.07$.

V. NUMERICAL RESULTS

We use a two-parameter smooth curve representation

$$[1 + \exp(-a_2)]/[1 + \exp(a_1 r^2/R^2 - a_2)]$$

for $n^0_{\rm lab}(r)$. Its profile (with $a_1 = 20$, $a_2 = 3.2$) displayed in Fig. 2 as the dotted line, models a typical experimental profile. The calculated perturbed density profile (including terms up to the third order) along the $\theta - \Omega t = 0,\pi$ axis is shown in Fig. 2 as the solid line for the value $\epsilon = 0.07$ (or $\tilde{\epsilon} \simeq 0.3$). For small values of ϵ, the perturbed profile looks like the equilibrium profile displaced to one side with no change in shape. With increasing ϵ, the profile tends to get deformed with a tail appearing on the inner edge. However, for these values of ϵ, higher-order corrections may have to be included.

The contributions of the various orders of perturbation to the density for the case $\epsilon = 0.07$ are displayed in Fig. 3. The first-order term is sizable compared to the zeroth order term and the mode is far from being linear. Since $\tilde{\epsilon} \simeq 0.3$, the contribution of each order of perturbation is approximately one third that of the preceding order.

There is no specific condition in our theory that forces $n^0_{\rm lab}(r,\theta - \Omega t)$ to satisfy the physical requirement of being nonnegative for $0 \leqslant r \leqslant R$. However, our numerical solutions seem to satisfy this requirement very well. Regions of negative density in a given order of perturbation tend to fill up in the next higher order (see Fig. 3). We do not, however, have a mathematical proof that if we sum all orders, the resultant density is always non-negative.

The nonlinear frequency shift given by Eq. (28) is illustrated in Fig. 4 where we have plotted $\Omega/\Omega^0 - 1$ versus Δ/R, where Ω^0 is the linear $l = 1$ diocotron frequency and Δ is the experimentally measurable displacement of the density maximum from the axis of the conducting tube. Since $\Omega^2 \propto (\epsilon)^2$, the frequency shift is a quadratic effect in terms of the wave amplitude. The value of ϵ ranges from 0 to 0.15 in Fig. 4 and thus the next nonvanishing correction $\Omega^4 \propto (\epsilon)^4$ is small.

VI. SUMMARY

We have developed a perturbative method for treating the $l = 1$ nonlinear diocotron mode. The theory is used to find the nonlinear steady-state dynamic equilibrium that can be continuously approached from a linear diocotron mode having azimuthal mode number $l = 1$. The theory predicts the density profile as well as the frequency shift for this mode as a function of amplitude.

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