Thermal equilibrium of a cryogenic magnetized pure electron plasma

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The thermal equilibrium correlation properties of a magnetically confined pure electron plasma (McPEP) are related to those of a one-component plasma (OCP). The N-particle spatial distribution ρ_s and the Helmholtz free energy F are evaluated for the McPEP to $O(\lambda_d^2/a^2)$, where λ_d is the thermal de Broglie wavelength and a is an interparticle spacing. The electron gyromotion is allowed to be fully quantized while the guiding center motion is quasiclassical. The distribution ρ_s is shown to be identical to that of a classical OCP with a slightly modified potential. To $O(\lambda_d^2/a^2)$ this modification does not affect that part of F that is caused by correlations, as long as certain requirements concerning the size of the plasma are met. This theory is motivated by a current series of experiments that involve the cooling of a magnetically confined pure electron plasma to the cryogenic temperature range.

Recent experiments have involved the confinement of an unneutralized collection of electrons of sufficient density to be called a plasma, that is, a pure electron plasma.[1] In contrast to the case of a neutral plasma, a pure electron plasma can be confined by static electric and magnetic fields and also be in a state of thermal equilibrium.[2,3] The experimentally achieved confinement times (10^5 sec) are such that these thermal equilibrium states are very likely realized in practice.

Since there are negligibly few ions in the confinement region, recombination cannot occur for the plasma even if it is cooled to very low temperatures, and temperatures in the cryogenic range have likely been achieved.[4] As the plasma cools, one expects the electrons to become strongly correlated, to develop the short-range order characteristic of a liquid, and ultimately to experience a phase transition to a solid state.[5–8] Conditions achieved in recent experiments are such that the physics of strong correlation should be important.[4]

This letter presents a theoretical discussion of the electron–electron correlation properties for such a plasma. In particular we show that the thermal equilibrium correlation properties of the magnetically confined pure electron plasma (McPEP) are nearly identical to those of a one-component plasma (OCP), that is, a plasma consisting of point charges embedded in a uniform neutralizing charge. This relation is very useful since the OCP has been the subject of extensive theoretical and computational research, especially in the correlated regime $\Gamma = e^2/akT \gg 1$. Here, T is the temperature and a is a measure of the distance between electrons, that is, $\frac{4}{3}\pi a^3 n_e = 1$, where n_e is the bulk electron density.

For the case where the electron dynamics can be described classically, it has been shown previously[5] that the thermal equilibrium correlation properties of a McPEP are identical to those of an OCP. However, the temperature and magnetic field strength of recent experiments are such that the electron cyclotron motion is quantized (i.e., $kT \sim \hbar\Omega_e$, where Ω_e is the electron cyclotron frequency), and the conditions for validity of the classical theorem are not satisfied.

In this letter we use the results of Alastuey and Jancovici[9] to extend the theorem to this quantum regime. Under the assumptions that $\lambda_d/a \ll 1$ (where $\lambda_d^2 \equiv \hbar^2/mkT$ is the thermal de Broglie wavelength) and $\hbar\omega_{pe} \ll kT$ (where $\omega_{pe}^2 = 4\pi e^2 n_e/m$ is the electron-plasma frequency), Alastuey and Jancovici derived the Wigner function to $O(\lambda_d^2/a^2)$ for an OCP. By writing the equilibrium density matrix for the McPEP in a suitable form, we are able to use the results of their calculation and show that the thermal equilibrium correlation properties of a quantum mechanical McPEP are the same as those of a classical OCP with a slightly modified potential energy. The modification can be understood as an averaging of the potential over the quantum uncertainty in electron position. The modification does not affect the correlation contribution to the free energy to $O(\lambda_d^2/a^2)$, so this contribution is identical to that of a classical OCP. The free energy of a classical OCP has been evaluated using Monte Carlo simulations,[6–8] and the other thermodynamic functions follow from the free energy.

Before presenting the results of the quantum calculation, we first review the classical theorem. The confinement geometry for the McPEP is cylindrical with radial confinement of electrons provided by a uniform axial magnetic field and axial confinement by negatively biased end cylinders.

For a classical system of electrons confined in such a cylindrically symmetric geometry, the distribution for a canonical ensemble is of the form

$$\rho = Z^{-1} \exp[-\beta(H - \omega L)] , \tag{1}$$

where

$$H = \sum_{j=1}^{N} \frac{m}{2} v_j^2 + V_e(\mathbf{x}_1, ..., \mathbf{x}_N) \tag{2}$$

is the N-particle Hamiltonian, v_j is the jth electron's velocity, and

$$L = \sum_{j=1}^{N} \left(m v_{\theta_j} r_j - \frac{m}{2e} \Omega_e r_j^2 \right) \tag{3}$$

is the canonical angular momentum in cylindrical coordinates (r, θ, z). The function V_e is the electrostatic potential energy, and the vector potential

$$\mathbf{A} = \hat{\theta}(m/e)\Omega_e(r^2/2)$$

has been used. Relativistic and diamagnetic effects are negligible for the low electron velocities under discussion. The partition function $Z(\beta, N, \omega)$ is determined by normalization of ρ to unity taking into account classically identical

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states, $\beta = (kT)^{-1}$, and ω is the frequency of rotation of the plasma about the z axis. (The fact that ρ describes rigid rotation about the z axis will become apparent shortly.)

A straightforward rearrangement of terms shows that we may write the function $H_R = H - \omega L$, which is the Hamiltonian in the rotating frame, as

$$H_R = \frac{m}{2} \sum_j [v_{r_j}^2 + v_{z_j}^2 + (v_{\theta_j} - \omega r_j)^2] + V, \qquad (4)$$

where

$$V = V_e + \sum_j \frac{m\omega}{2} (\Omega_e - \omega) r_j^2. \qquad (5)$$

Thus the velocity dependence of ρ is given by a product of Maxwellians in a frame rotating with angular velocity ω, and the spatial dependence is determined exclusively by V. The term in V that is quadratic in r_j may be interpreted as being caused by a hypothetical cylinder of uniform positive charge, the density of which is

$$n_+ = (m\omega/2\pi e^2)(\Omega_e - \omega)$$

($\Omega_e > \omega$ in the experiment). The canonical ensemble for the McPEP therefore differs only by rotation from that for electrons confined by a cylinder of positive charge, and the bulk properties of such a system are those of a classical OCP.

The derivation relies on the fact that neither the magnetic field nor the Coriolis force affect the energy of the system in the classical limit. This is no longer true in the quantum regime $\hbar\Omega_e \sim kT$, since Landau levels appear with energy eigenvalues that depend on the magnetic field strength; for an isolated electron they are given by $\hbar\Omega_e(n + 1/2)$. Electron spin also plays an important role in the quantum regime, since the spin eigenstates for an isolated electron are also separated by $\hbar\Omega_e$. Quantum uncertainty in position affects the interaction energy of the system. For the case of zero magnetic field, this effect can be visualized as a smearing of an average electron's charge over a sphere of radius λ_d.[10] The confining effect of a strong magnetic field reduces the size of this spatial uncertainty in the plane perpendicular to the magnetic field. There is a corresponding loss of certainty in the electron's perpendicular momentum that may be thought of as an increase in the effective perpendicular temperature. The final quantum effect is the exchange force caused by the antisymmetry of the N-electron wavefunction. This force is important only if a significant fraction of electrons are in nearly identical states. Alastuey and Jancovici show that mutual repulsion of electrons makes such configurations highly improbable as long as $\hbar\omega_{pe} \ll kT$; the exchange force is therefore neglected.

In order to construct the quantum thermal equilibrium distribution for the N-particle system, we follow Alastuey and Jancovici by defining the Wigner distribution

$$f_\sigma(\mathbf{x}, \mathbf{p}) = \int ds^{3N} e^{i\mathbf{p}\cdot\mathbf{s}} \times \langle \mathbf{x} - \mathbf{s}/2, \sigma_z | e^{-\beta(\hat{H} - \omega\hat{J})} | \mathbf{x} + \mathbf{s}/2, \sigma_z \rangle, \qquad (6)$$

where

$$\hat{H} = \sum_{j=1}^{N} \left\{ \frac{1}{2m} \left[\hat{p}_{r_j}^2 + \left(\frac{\hat{p}_{\theta j} + (m/2)\Omega_e r_j^2}{r_j^2} \right)^2 + \hat{p}_{z_j}^2 \right] + \frac{\hbar\Omega_e}{2} \hat{\sigma}_j \right\} + V_e \qquad (7)$$

is the Hamiltonian operator for the system and

$$\hat{J} = \sum_{j=1}^{N} \left(\hat{p}_{\theta_j} + \frac{\hbar}{2} \hat{\sigma}_{z_j} \right) \qquad (8)$$

is the canonical angular momentum operator. Both operators include spin; $\hat{\sigma}_{z_j}$ is the Pauli matrix for the jth particle. The momentum operators are given in cylindrical coordinates, and the ket $|\mathbf{x}, \sigma_z\rangle$ means $|\mathbf{x}_1, ..., \mathbf{x}_N, \sigma_{z1}, ..., \sigma_N\rangle$. This Wigner function can be written in the form used by Alastuey and Jancovici by rewriting the operator $\hat{H} - \omega\hat{J}$ as

$$\hat{H} - \omega\hat{J} = \sum_j \left\{ \frac{1}{2m} \left[\hat{p}_{r_j}^2 + \left(\frac{\hat{p}_\theta + (m/2)\Omega r_j^2}{r_j^2} \right)^2 + \hat{p}_{z_j}^2 \right] + \hbar(\Omega_e - \omega)\hat{\sigma}_{z_j} \right\} + V, \qquad (9)$$

where V is defined as in Eq. (5) and $\Omega = \Omega_e - 2\omega$ is the vortex frequency. Although Alastuey and Jancovici dealt with spinless particles, the addition of spin is trivial since the kets used in the definition of f commute with $\hat{\sigma}_{z_j}$.

We may thus simply quote the results of their analysis: the spatial distribution is given by

$$\rho_s(\mathbf{x}_1, ..., \mathbf{x}_N) = \int d\mathbf{p}^{3N} f(\mathbf{x}, \mathbf{p}) = Ce^{-\beta V} \left\{ 1 - \frac{\beta\lambda_d^2}{12} \sum_j \left[\frac{\partial^2 V}{\partial z_j^2} - \frac{\beta}{2} \left(\frac{\partial V}{\partial z_j} \right)^2 + \alpha \left(\nabla_{\perp_j}^2 V - \frac{\beta}{2} (\nabla_{\perp_j} V)^2 \right) \right] \right\}, \qquad (10)$$

where

$$\alpha = \frac{6}{\hbar\Omega\beta} \left(\frac{1}{\tanh(\hbar\Omega\beta/2)} - \frac{2}{\hbar\Omega\beta} \right),$$

C is a constant independent of position, and $\nabla_\perp$ is the gradient operator perpendicular to $\mathbf{B}$. This formula is good to $O(\lambda_d^2/a^2)$ and is useful for all magnetic fields. The function α is always less than or equal to unity and represents the confining effect of the magnetic field on quantum fluctuations perpendicular to $\mathbf{B}$.

Equation (10) shows that the N-particle spatial distribution for the McPEP is only slightly altered by quantization of the gyromotion and to lowest order in λ_d/a is given by the distribution for a classical OCP. To $O(\lambda_d^2/a^2)$, ρ_s can be written as $\exp(-\beta\bar{V})$, where

$$\bar{V} = V + \frac{\lambda_d^2}{12} \sum_{j=1}^{N} \left[\frac{\partial^2 V}{\partial z_j^2} - \frac{\beta}{2} \left(\frac{\partial V}{\partial z_j} \right)^2 + \alpha \left(\nabla_{\perp_j}^2 V - \frac{\beta}{2} (\nabla_{\perp_j} V)^2 \right) \right] \qquad (11)$$

is the modified potential energy, the modification being caused by averaging over the quantum uncertainty.

The Helmholtz free energy in the frame rotating with the plasma is given by

$$F = -kT \ln \sum_{\sigma = \pm 1} \int \frac{d\mathbf{p}^{3N} d\mathbf{x}^{3N} f}{(2\pi\hbar)^{3N} N!}$$

$$= F_c + F_K + \Delta F , \tag{12a}$$

where

$$F_c = -kT \ln \int \frac{d\mathbf{x}^{3N}}{\Lambda} \exp(-\beta V)$$

is the classical configurational free energy, Λ is a normalization constant,

$$F_K = NkT \ln \left[\frac{(2\pi\hbar)^2 N}{m\Omega\Lambda} \frac{\sinh(\hbar\Omega\beta/2)}{\cosh[\hbar(\Omega_e - \omega)\beta/2]} \left(\frac{\beta}{2\pi m}\right)^{1/2} \right] \tag{12b}$$

is the kinetic contribution, and

$$\Delta F = NkT \frac{\lambda_d^2}{24} \left\langle \sum_j \frac{\partial^2 V}{\partial z_j^2} + \alpha \nabla_{\perp_j}^2 V \right\rangle \tag{12c}$$

is the small contribution resulting from quantum uncertainty. Here, the brackets denote an average weighted by $\exp(-\beta V)$. The kinetic contribution contains the effects of parallel motion and quantized gyromotion as well as quantized spin. By reference to Eq. (5), one can see that there are two contributions to V and, therefore, to ΔF. The first is caused by the positive background charge and may be evaluated explicitly to give

$$\Delta F_1 = (NkT/8)\Gamma(\lambda_d^2/a^2)\alpha . \tag{13}$$

The second is caused by interelectron interactions; it may also be evaluated explicitly as long as the dimensions of the plasma are much larger than both the Debye length and the interparticle spacing, a condition well satisfied in the experiments. In this case the two-particle probability distribution $P(\mathbf{x}_1, \mathbf{x}_2)$ and the interparticle potential $\phi(\mathbf{x}_1, \mathbf{x}_2)$ can be written as functions of $|\mathbf{x}_1 - \mathbf{x}_2|$ only and this contribution to ΔF vanishes. Then $\Delta F = \Delta F_1$ and does not depend on correlations, so the correlation contribution to the free energy is given by that of a classical OCP to $O(\lambda_d^2/a^2)$. Furthermore, ΔF can be combined with F_k to $O(\omega/\Omega_e)$ to give

$$F_k + \Delta F = NkT \ln \left[\frac{(2\pi\hbar)^2}{m(\Omega_e - \omega)} \frac{N}{\Lambda} \left(\frac{\beta}{2\pi m}\right)^{1/2} \times \tanh\left(\frac{\hbar(\Omega_e - \omega)\beta}{2}\right) \right], \tag{14}$$

so that quantum effects are parametrized by a single variable $\hbar(\Omega_e - \omega)\beta$. Note that the ordering used in the quantum regime (i.e., $\hbar\Omega_e \sim kT$ but $\hbar\omega_{pe} \ll kT$) taken together with the relation[2] $2\omega_{pe}^2 \simeq \omega(\Omega_e - \omega)$ implies that $\omega \ll \Omega_e$.

In summary, by applying the theory of Alastuey and Jancovici to a magnetized pure electron plasma, we have derived the N-particle spatial distribution and the free energy of a McPEP to $O(\lambda_d^2/a^2)$, including the effects of quantized gyromotion. The N-particle distribution is given by that of a classical OCP with a slightly modified potential energy; this modification does not affect the correlation free energy.

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