Wall losses for a single-species plasma near thermal equilibrium

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The effect of a radially bounding wall on a magnetically confined single-species plasma near thermal equilibrium is considered. Solutions to the like-particle collisional transport equation are obtained; the boundary conditions at the wall follow from simple physical considerations. Integral constraints on the plasma evolution imply that only a fraction of the plasma can ever be lost to the wall. Analytic estimates and numerical solutions give the scaled wall flux in terms of the unperturbed equilibrium density at the radius of the wall.

I. INTRODUCTION

Recent theoretical and experimental work has treated the containment of a nonneutral, single-species plasma column.[1-7] The containment geometry is cylindrical, with an axial magnetic field providing radial confinement, and electrostatic potentials providing axial confinement. The plasma column rotates, due to the strong radial electric field resulting from the unneutralized space charge.

Radial transport of particles is strongly constrained by conservation of the total angular momentum of the plasma and fields.[3,8-10] In a cylindrically symmetric system with no external torques acting on the particles, there can be no bulk expansion of the plasma. H-theorems have been formulated,[3,11] demonstrating that like-particle interactions drive the plasma monotonically toward a confined thermal equilibrium state. In any real confinement device, there are always small effects such as collisions with neutrals or wall resistance which apply torques to the plasma and allow it to expand radially. However, electron plasma experiments[2] are now entering regimes in which like-particle interactions dominate, and it is this regime which will be treated in this paper.

For a nonneutral plasma free of all external torques, the equilibrium density profiles decrease exponentially with radius for large radii, but are never zero.[1,3] Any bounding wall at a finite radius will interact with the plasma and prevent it from being completely in equilibrium with itself. There will be a transfer of particles and angular momentum to the wall, and the question arises as to whether this could lead to rapid radial loss of the plasma. Confinement theorems have been developed[4] which place a bound on the total possible particle loss, but these give no information as to loss rates.

In this paper, we consider the effect of an absorbing wall on a single-species plasma which is free from any other external torques. The like-particle collisional transport equation[3] is solved for the radial particle and angular momentum losses; this transport equation includes only lowest-order terms in a $1/B$ expansion. The plasma is assumed to be quiescent and symmetric in the axial and azimuthal directions. The wall is assumed to be perfectly absorbing, and to exert no long-range torques on the plasma. Further, the plasma–wall interaction is taken to be weak, in the sense that the resulting plasma evolution occurs on a time scale long compared with the time scale for relaxation toward equilibrium; this will be true if the equilibrium plasma would have a low density at the radius of the wall.

The boundary conditions at the wall are that the particle density be zero, and that the angular momentum and energy lost is just that carried by the particles at the radius of the wall. The requirement that the radial particle flux be essentially constant with radius in the low-density region near the wall determines the density profile near the wall. Matching this constant flux profile to the equilibrium density profile gives a simple analytic estimate of the loss rate for any given equilibrium. These loss-rate estimates are verified by numerical integration of the radial transport equation with the appropriate boundary conditions.

By considering the radial integrals of particle density, angular momentum, and energy, constraints are formulated on the time evolution of the plasma: These indicate that the wall losses become exponentially small with time, and that in general only a small fraction of the plasma will be lost. The essential results are that the plasma will tend to move radially inward, away from the wall, since a disproportionately large angular momentum per particle is being lost to the wall; and that the plasma will not heat and thereby extend further toward the wall, since the average thermal energy per particle is being lost to the wall. While any given experiment might have additional effects which alter the loss rate for angular momentum and energy these results indicate that like-particle collisional transport does not by itself lead to rapid particle losses.

II. TRANSPORT AND EQUILIBRIA

Cross-field particle transport due to like-particle collisions has been derived from the fluid force equations,[8] from a random walk equation for the guiding centers,[9] and from an explicit perturbation expansion of the Boltzmann equation.[3] All three derivations give the same form for the particle flux, which is strongly constrained by conservation of canonical angular momentum.

The macroscopic force equation for a fluid of particles of charge $-e$ and density $n(r, t)$ may be written

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$$\nabla \cdot \mathsf{P} = -en[-\nabla\phi + (1/c)\mathbf{v}\times\mathbf{B}], \tag{1}$$

where P is the stress tensor as derived by Chapman and Cowling.[12] In cylindrical coordinates (r, θ, z) with the assumed symmetries and $\mathbf{B}=B\hat{z}$, the relevant terms are

$$P_{rr}=P_{\theta\theta}=nT,$$
$$P_{r\theta}=P_{\theta r}=-\frac{3}{8}\frac{\nu}{n}r_L^2 n^2 mr\frac{\partial}{\partial r}\frac{1}{r}v_\theta . \tag{2}$$

The temperature $T(t)$ is independent of radius, since heat transport is not constrained by conservation of angular momentum, and therefore occurs on a much faster time scale than does particle transport.[3] The off-diagonal terms in P represent the force from one fluid element to another due to any shear in the fluid velocity $v_\theta(r)$. Here, we have defined the Larmor radius $r_L \equiv \bar{v}/\Omega \equiv (T/m)^{1/2}(eB/mc)^{-1}$ and the like-particle collision frequency $\nu = 16\sqrt{\pi}\, e^4 n \ln\Lambda/15m^{1/2}T^{3/2}$, where $\ln\Lambda$ is the Coulomb logarithm, which is assumed to be essentially independent of r.

The r component of Eq. (1) gives

$$v_\theta = -\frac{T}{m\Omega}\left(\frac{n'}{n}-\frac{e\phi'}{T}\right), \tag{3}$$

where the prime represents $\partial/\partial r$. The θ component then gives the radial flux

$$J(r,t) \equiv nv_r = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{1}{m\Omega}P_{r\theta}$$
$$= \frac{3}{8}r_L^4\frac{\nu}{n}\frac{1}{r^2}\frac{\partial}{\partial r}r^2 n^2 r\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{n'}{n}-\frac{e\phi'}{T}\right)$$
$$\equiv \left(\frac{3}{8}\epsilon^3\delta n_0\bar{v}\right)\lambda_D^3\frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(\frac{n}{n_0}\right)^2 r\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{n'}{n}-\frac{e\phi'}{T}\right). \tag{4}$$

The derivations of this flux are all expansions in $1/B$; this is most explicitly seen in Ref. 3, in which the expansion parameters are $\epsilon \equiv r_L/\lambda_D$ and $\delta \equiv \nu/\Omega$, where $\lambda_D^2 = T/4\pi e^2 n_0$.

The time evolution of the plasma density is given by the continuity equation

$$\frac{\partial n}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}rJ=0. \tag{5}$$

Two integrals of Eq. (5) are of particular interest,

$$\dot{N}(R,t) \equiv \int_0^R 2\pi r\,dr\,\dot{n}(r,t) = -2\pi RJ(R,t), \tag{6}$$

and

$$\dot{L}(R,t) \equiv \int_0^R 2\pi r\,dr\,r^2\dot{n}(r,t)$$
$$= -2\pi R[R^2 J(R,t) - 2RP_{r\theta}(R,t)/m\Omega], \tag{7}$$

where the dot represents $\partial/\partial t$. The integral $N(R,t)$ is the total number of particles within radius R (per unit length in $\hat{z}$); and $L(R,t)$ is $(-2/m\Omega)$ times the total canonical angular momentum within radius R (per unit length in $\hat{z}$), to lowest order in ϵ. The latter follows from the canonical angular momentum for a single particle, $p_\theta = mrv_\theta - (e/c)A_\theta r \approx (-m\Omega/2)r^2$, since the second term dominates for large B.

The time evolution of the plasma temperature $T(t)$ is determined by the time derivative of the total energy (per unit length),

$$\dot{W} = \frac{d}{dt}\left(\frac{3}{2}N(R_W,t)T(t)\right)+\frac{d}{dt}\int_0^{R_W} 2\pi r\,dr\frac{\phi'^2}{8\pi}, \tag{8}$$

where R_W is the radius of the bounding wall. The contribution to the energy from the rotation of the plasma has been ignored, since this term is higher order in ϵ.

For the case where R_W is arbitrarily large and $\dot{W}=0$, H-theorems have been formulated to show that like-particle interactions drive the plasma monotonically toward a confined thermal equilibrium distribution.[3,11] The equilibrium is given by

$$f_e(r,\mathbf{v}) = n_0(m/2\pi T)^{3/2}\exp[-(1/T)(H-\omega p_\theta)], \tag{9}$$

where $H=\frac{1}{2}mv^2 - e\phi$ is the energy of a particle. The equilibrium velocity distribution is a Maxwellian with temperature T when viewed from a frame rotating with angular velocity ω; the particle density is given by

$$n_e(r) = n_0\exp[e\phi/T - (m\omega/2T)(\Omega-\omega)r^2] \equiv n_0\exp[\psi(r)]. \tag{10}$$

The equilibrium is completely specified by the three parameters n_0, ω, T. Alternately, the equilibrium can be specified by the total number of electrons, canonical angular momentum, and energy per unit length, i.e., by the complete radial integrals N, L, W.

With the functional form (10) for the density, Poisson's equation for $\phi(r)$ determines $\psi(r)$ as

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\psi = \frac{1}{\lambda_D^2}[e^\psi - (1+\gamma)]. \tag{11}$$

with boundary conditions $\psi(0)=\psi'(0)=0$. The Debye length is relative to the central density n_0, and the parameter γ is given by

$$\gamma \equiv (2m\omega/4\pi e^2 n_0)(\Omega-\omega)-1.$$

As seen from Eqs. (11) and (10), the equilibrium density profiles form a one-parameter family, when the radius is scaled in terms of the central Debye length, and the density is scaled to the central density. The parameter γ determines the radius of the plasma in Debye lengths: for $\gamma \ll 1$, $\gamma \approx \ln(2)[2\pi R_p/\lambda_D]^{1/2}\exp[-R_p/\lambda_D]$, where $n_e(R_p) = n_0/2$.

III. INTERACTION WITH A WALL

We now consider the case where the wall is at a finite radius. As an initial value problem, the particle transport equations (4) and (5) are determined by the initial conditions $n(r,0)$ and the boundary conditions at $r=0$ and $r=R_w$. Since the transport is fourth order in $\partial/\partial r$, with ϕ' completely specified by Poisson's equation, four boundary conditions are required. The energy equation (8) for $T(t)$ is determined by $T(0)$ and the energy loss rate $\dot{W}(t)$.

The two boundary conditions at $r=0$ are given by symmetry as

$$n'(0,t)=n'''(0,t)=0. \tag{12}$$

These are implicit in cylindrical coordinates, and are required for all terms in Eqs. (4) and (5) to be finite at $r=0$. The two boundary conditions at $r=R_w$ and the energy loss rate $\dot{W}$ are determined by the physical characteristics of the wall. We assume that the wall is perfectly absorbing and that it exerts no long-range torques on the plasma. When a particle is absorbed, it loses only the angular momentum and energy that it carried: The angular momentum loss is $(-2/m\Omega)R_w^2$, to lowest order in ϵ, and the energy loss is $\frac{3}{2}T(t)$, since the plasma is isothermal. The two remaining boundary conditions and the energy loss rate are thus

$$\begin{aligned} &n(R_w, t)=0, \\ &\dot{L}(R_w, t)=R_w^2\dot{N}(R_w, t), \qquad (13) \\ &\dot{W}(t)=\tfrac{3}{2}T(t)\,\dot{N}(R_w, t). \end{aligned}$$

We note that this boundary condition is equivalent to specifying that the plasma stress tensor component $P_{r\theta}(R_w, t)$ is zero. Using Eqs. (7) and (2), the second wall condition may also be written

$$n^2 r\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{n'}{n}-\frac{e\phi'}{T}\right)=0. \qquad (14)$$

One might worry about applying the boundary conditions at the wall, since the transport equation breaks down in a thin shell near the wall.[3] This shell is a few Larmor radii thick. However, by employing continuity of particle, angular momentum, and energy flow across the shell, one can check that the boundary conditions as given are accurate to lowest order in ϵ.

An interesting aspect of the boundary conditions is that they preclude a solution by separation of variables, i.e., as $n(r, t)=n(r)g(t)$. For example, when the electric field is negligible, the particle transport is proportional to n^2, and one might attempt a solution[6] with $g(t)=(1+\alpha t)^{-1}$. However, the condition $\dot{L}=R_w^2\dot{N}$ cannot be satisfied by any solution of this form. Specifically, one obtains $\dot{N}=\dot{g}N_0$ and $\dot{L}=\dot{g}\langle r^2\rangle N_0$, or $\dot{L}=\langle r^2\rangle\dot{N}$. This indicates that the plasma profile $n(r)$ must evolve with time, since a disproportionate amount of angular momentum is lost relative to the number of particles lost.

With the boundary conditions in hand, one could attempt a direct initial value solution of Eqs. (4), (5), and (8) by numerical methods. However, the transport is fourth order in $\partial/\partial r$ and cubically nonlinear in $n(r, t)$, making accurate numerical solutions quite difficult. Further, we are not interested in the details of the initial evolution, but rather in the rate of particle loss to the wall, and in the resulting equilibrium evolution. These questions can be partially answered by simple analytic considerations.

Near the wall, where the particle density is low, the density profile will quickly evolve so as to give a flux which is almost constant with radius; that is, the particles being lost from the higher density regions merely pass through the low-density tail. The constant flux profile is easily seen to be

$$n_f(r)=An_0[(R_w-r)/\lambda_D]^{3/2}, \qquad (15)$$

which has flux

$$J=(\tfrac{3}{8}\epsilon^3\delta n_0\bar{v})\tfrac{3}{2}A^2,$$

where we have ignored terms higher order in (R_w-r). This solution satisfies both wall boundary conditions (13).

An estimate of the loss rate to be expected from a plasma near thermal equilibrium can be obtained by requiring the low-density tail to join smoothly with an equilibrium profile. The minimum requirements which uniquely determine the flux are that n and n' be continuous; that is, we require

$$n_f(r_m)=n_e(r_m), \quad n_f'(r_m)=n_e'(r_m),$$

varying A and r_m so as to satisfy both conditions. This match does not, of course, make the stress tensor or flux continuous, which would be the case if n'' and n''' were continuous. Using $n_e(r)=n_0\exp[\psi(r)]$, the matching conditions give $A^2=(R_w-r_m)^{-3}\lambda_D^3\exp[2\psi(r_m)]$, and $(R_w-r_m)^{-1}=-(\frac{2}{3})\psi'(r_m)$. For equilibria with $\gamma\gg 1$, the function $\psi(r)$ is known analytically, as $\psi(r)=-(1+\gamma)r^2/4\lambda_D^2$. The flux is then

$$\begin{aligned} J&=\left(\frac{3}{8}\epsilon^3\delta n_0\bar{v}\right)\frac{3}{2}\left(\frac{n_e(r_m)}{n_0}\right)^2\left(\frac{R_w-r_m}{\lambda_D}\right)^{-3} \\ &\approx\left(\frac{3}{8}\epsilon^3\delta n_0\bar{v}\right)(1+\gamma)^{3/2}\frac{4}{9}\exp(3)\left(\frac{n_e(R_w)}{n_0}\right)^2 \\ &\quad\times[-\ln(n_e(R_w)/n_0)]^{3/2}, \end{aligned} \qquad (16)$$

where the last expression used $\psi'(r_m)\approx\psi'(R_w)$. The flux is seen to be parametrized by the fractional density which the unperturbed equilibrium would have at the radius of the wall. The factor $(1+\gamma)^{3/2}$ appears because the scale length for the equilibrium is $\lambda_D/(1+\gamma)^{1/2}$, when $\gamma\gg 1$.

We define a scaled flux

$$\hat{J}\equiv J[\tfrac{3}{8}\epsilon^3\delta n_0\bar{v}(1+\gamma)^{3/2}]^{-1}, \qquad (17)$$

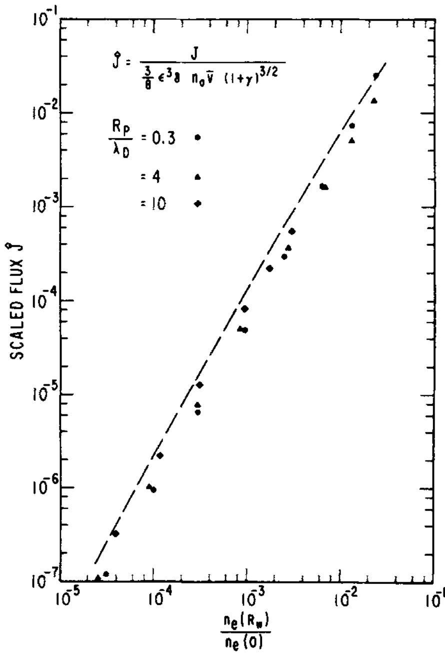

FIG. 1. Scaled flux $\hat{J}$ vs equilibrium density at the radius of the wall. The dotted line is an analytic estimate, whereas the points are numerical computations for three separate equilibria.

so as to remove the dependence on the magnetic field, central density, temperature, and equilibrium scale length. The dashed line of Fig. 1 shows $\hat{J}$ vs $n_e(R_w)/n_0$ from Eq. (16). Although this estimate was derived for equilibria with $\gamma \gg 1$, a similar estimate can be obtained for arbitrary γ using numerical integration for $\psi(r)$. The resulting estimates of $\hat{J}$ vs $n_e(R_w)/n_0$ are within 50% of that shown in Fig. 1.

Figure 1 also displays the scaled wall flux results obtained by numerical integration of the transport equation, as described in Sec. IV. It is apparent that the analytic estimates give the correct dependence on the density at the wall over five decades in flux magnitude. The estimates are approximately a factor of two larger than the computed fluxes; this discrepancy arises from the matching of a solution with constant flux to an equilibrium profile with zero flux.

If the interaction with the wall is weak, then the body of the plasma will be near an equilibrium at any given time, and one may consider the evolution of this "equilibrium" as particles, angular momentum, and energy are slowly lost to the wall. A simple estimate of the equilibrium evolution can be obtained from the square profile ($\gamma \ll 1$) integrals $N=\pi R_p^2 n_0$, $L=\pi R_p^4 n_0/2$, $W=\frac{3}{2}NT + e^2N^2[\frac{1}{4}+\ln(R_w/R_p)]$. The first two are easily inverted to give $n_0=N^2/2\pi L$ and $R_p^2=2L/N$.

The particle loss rate may be written

$$\frac{\dot{N}}{N}=\frac{-2\pi R_w J}{\pi R_p^2 n_0}=-\left(\frac{3}{8}\epsilon^3\delta\frac{\bar{v}}{\lambda_D}\right)2\frac{R_w}{R_p}\frac{\lambda_D}{R_p}(1+\gamma)^{3/2}\hat{J}, \tag{18}$$

where the quantity in large parentheses is the rate of relaxation toward an equilibrium with gradient lengths λ_D. The evolution of the equilibrium may then be given in terms of this particle loss rate. Using the boundary conditions $\dot{L}=R_w^2\dot{N}$ and $\dot{W}=\frac{3}{2}T\dot{N}$, we obtain

$$\begin{aligned}
&\frac{\dot{n}_0}{n_0}=2\frac{\dot{N}}{N}-\frac{\dot{L}}{L}=-2\frac{\dot{N}}{N}\left(\frac{R_w^2}{R_p^2}-1\right)>0,\\
&\frac{\dot{R}_p}{R_p}=\frac{\dot{N}}{N}\left(\frac{R_w^2}{R_p^2}-\frac{1}{2}\right)<0,\\
&\frac{3}{2}N\dot{T}=e^2N\dot{N}\left[\frac{R_w^2}{R_p^2}-1-2\ln\left(\frac{R_w}{R_p}\right)\right]\leq 0.
\end{aligned} \tag{19}$$

The central density increases, since conservation of angular momentum requires that some particles flow inward as others flow out to the wall. The radius of the plasma is decreasing both due to the decreasing number of particles, and due to the central density increase. The work that the inward-flowing particles do on the electric field is marginally greater than the work done by the field on the outward flowing particles, causing a slight decrease in the thermal energy. Although, these variations were derived for equilibria with $\gamma \ll 1$, the same effects would be observed for arbitrary equilibria, except that the relevant integrals would be less tractable.

For all equilibria, the combined effects of the variations in n_0, R_p, and T are to move the plasma edge away from the wall, decreasing the particle loss rate. The loss rate never becomes zero, but could well become negligible on the time scale of any experiment. Containment theorems have been rigorously formulated[4] showing that the total number of particles lost cannot exceed an upper bound, given approximately by $\Delta N/N \leq R_p^2/2R_w^2$. When this number of particles are lost, the remaining particles must be localized at $r=0$. This limiting state is probably less relevant to a laboratory experiment than is a consideration of the losses on the time scale of the experiment.

IV. NUMERICAL SOLUTIONS

We have obtained numerical solutions to the transport equations (4) and (5) for comparison with the loss rate estimates made in the previous section. Again, we assume the plasma to have reached a quasistatic state near thermal equilibrium, perturbed only by the losses to the wall. We solve the transport equation for this quasistatic state at a particular time t by assuming an appropriate functional form for the term $\partial n/\partial t$. Equation (5) is then solved as an ordinary differential equation having a "forcing function" $\partial n/\partial t$, with spatial boundary conditions at $r=0$ and $r=R_w$.

The functional form for $\partial n/\partial t$ is based on the assumption that the plasma is near an equilibrium profile, with time variations arising solely from the wall interaction. If we write

$$n(r,t)=n_e(r;n_0,\omega,T)+\delta n(r,t), \tag{20}$$

then the perturbation δn is comparable to n_e only near the wall. We use the approximation

$$\frac{\partial n}{\partial t}\approx\frac{\partial}{\partial t}n_e(r;n_0,\omega,T)=\frac{\partial n_e}{\partial n_0}\dot{n}_0+\frac{\partial n_e}{\partial\omega}\dot{\omega}+\frac{\partial n_e}{\partial T}\dot{T}. \tag{21}$$

One expects this approximation to be inaccurate only near the wall, where the density goes to zero.

In analogy with the initial conditions required for solution as a partial differential equation, we must specify which equilibrium the solution will be near: The equilibrium is characterized either by the parameters n_0, ω, T, or equivalently by the complete radial integrals N_e, L_e, W_e. The "initial conditions" on the profile $n(r)$ are then that it have the same number of particles, angular momentum, and energy as the equilibrium profile; that is, that it have integrals $N=N_e$, $L=L_e$, $W=W_e$. To the extent that the perturbation $\delta n(r)$ is negligible, these conditions specify the local conditions

$$\begin{aligned}
&n(0,t)=n_0,\\
&n''(0,t)=n_e''(0)=[2\pi e^2 n_0/T-m\omega(\Omega-\omega)/T]n_0,\\
&T(t)=T.
\end{aligned} \tag{22}$$

These local initial conditions are sufficiently accurate unless $R_p \gg \lambda_D$, and will be used in the following discussion for conceptual simplicity. The nature of the error from $\delta n(r)$ will be discussed later.

The functional form (21) for $\partial n/\partial t$ allows us to satisfy all the initial and boundary conditions on the transport equation (5). Recall that Eq. (5) is fourth order in $\partial/\partial r$. At $r=0$, the two boundary conditions (12) and the two initial conditions (22) specify n, n', n'', and n'''. With $\partial n/\partial t$ given for all r, Eq. (5) can then be integrated from $r=0$ to $r=R_w$. The solution $n(r)$ will satisfy the three wall conditions (13) and (14) only for an appropriate choice of the three parameters $\dot{n}_0, \dot{\omega}, \dot{T}$ in Eq. (21). The two conditions giving $\dot{L}$ and $\dot{W}$ in terms of $\dot{N}$ equivalently determine $\dot{\omega}$ and $\dot{T}$ in terms of $\dot{n}_0$. We then find the correct value for $\dot{n}_0$ by the "shooting method". The integral from 0 to R_w is repeated for various choices of $\dot{n}_0$ (and hence various $\dot{\omega}$ and $\dot{T}$) until the final boundary condition of $n(R_w)=0$ is satisfied. This procedure gives a density profile $n(r)$ which satisfies all the boundary conditions, and has an outward flux giving a loss rate $\dot{N}=2\pi R_w J(R_w)$.

Computational results for the scaled flux J are shown as the points in Fig. 1. Here, we have solved the transport equation for plasmas near equilibrium characterized by $\gamma=39$, 0.1, and 4×10^{-4} (having scaled radii $R_p/$ $R_p/\lambda_D=0.3$, 4, and 10) for various wall positions. As with the analytic estimate, the scaled flux essentially depends only on $n_e(R_w)/n_0$.

The adequacy of the approximation (21) for $\partial n/\partial t$ has been checked as follows. Having found a solution $n(r,t)$, we similarly find a second solution $n(r,t_2)$ appropriate to the plasma having evolved to a later time $t_2=t+\Delta t$. The function $[n(r,t_2)-n(r,t)]/\Delta t$ is then a closer approximation to $\partial n/\partial t$. Using this new "forcing function" in Eq. (5), we obtain a more accurate density profile and particle loss rate. We find that the two sets of results are essentially equal for small loss rates, and differ by no more than 10% for the largest rates of Fig. 1.

We now estimate the perturbation $\delta n(r)$ which will occur in the interior of a plasma which is many Debye lengths in radius. For simplicity, we consider the twice-integrated transport equation,

$$\left(\frac{3}{8}r_L^4\frac{\nu}{n}\right)2\pi r^3 n^2\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{n'}{n}-\frac{e\phi'}{T}\right)=-\tfrac{1}{2}\left[r^2\dot{N}(r,t)-\dot{L}(r,t)\right]. \tag{23}$$

If we define $n(r)\equiv n_e(r)+\delta n(r)$, linearize, and then approximate $n_e(r)\approx n_0$ for $r<R_p$, Eq. (23) becomes

$$\left(\frac{3}{8}r_L^4\frac{\nu}{n}\right)r^3 n_0^2\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{\delta n'}{n_0}-\frac{e}{T}\delta\phi'\right)=-\frac{\dot{n}_0 r^4}{8}. \tag{24}$$

Aside from the two homogeneous solutions which represent shifts in the equilibrium, the solution to Eq. (24) is given by

$$\delta n(r)=\frac{\dot{n}_0\lambda_D^2 r^2}{\frac{3}{2}r_L^4(\nu/n)n_0}\approx 2\left(\frac{R_w-R_p}{\lambda_D}\right)n_0\hat{J}\frac{r^2}{R_p^2}. \tag{25}$$

Here, we have used the approximate expression (19) for $\dot{n}_0$, and assumed $R_w-R_p\ll R_w$. This perturbation gives the flux $J(r)=-\dot{n}_0 r/2$, which is required to uniformly increase the interior density.

The perturbation remains a small fraction of n_0 for $0\le r\le R_p$ so long as $\hat{J}\ll 1$. However, the perturbation may have a significantly large second derivative at the origin: $\delta n''(0)$ is comparable to $n_e''(0)=-\gamma n_0/2\lambda_D^2$ whenever

$$\hat{J}\gtrsim\frac{\gamma}{8}\left(\frac{R_p}{\lambda_D}\right)^2\left(\frac{\lambda_D}{R_w-R_p}\right)\propto\left(\frac{R_p}{\lambda_D}\right)^{5/2}\exp\left(-\frac{R_p}{\lambda_D}\right). \tag{26}$$

For scaled fluxes larger than this limit, the local initial conditions of Eq. (22) do not properly specify a profile $n(r)$ which is near the desired equilibrium $n_e(r)$. In this case, the integral initial conditions must be used.

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