Finite length thermal equilibria of a pure electron plasma column

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The electrons of a pure electron plasma may be in thermal equilibrium with each other and still be confined by static magnetic and electric fields. Since the electrons make a significant contribution to the electric field, only certain density profiles are consistent with Poisson's equation. The class of such distributions for a finite length cylindrical column is investigated. In the limit where the Debye length is small compared with the dimensions of the column, the density is essentially constant out to some surface of revolution and then falls off abruptly. The falloff in density is a universal function when measured along the local normal to the surface of revolution and scaled in terms of the Debye length. The solution for the shape of the surface of revolution is simplified by passage to the limit of zero Debye length.

An interesting property of a pure electron plasma is that the electrons may come to thermal equilibrium with each other and still be confined by static electric and magnetic fields. Moreover, the long confinement times obtained in recent experiments[1] with pure electron plasmas suggest that the thermal equilibrium states may be obtainable in practice. For the confinement geometry of these experiments, we determine the class of thermal equilibrium density distributions which are consistent with Poisson's equation. Our work differs from previous work on this subject[2-4] in that we treat a column of finite length and give special attention to the limit of zero Debye length. Although this paper is self-contained, the reader may wish to refer to the preceding paper[2] for a detailed treatment of the infinitely long column.

The confinement geometry is shown in Fig. 1. A conducting cylinder is divided into three sections, with the two end sections having negative bias relative to the central section. A uniform magnetic field B is directed along the axis of the cylinder. The plasma resides in the central section, with axial confinement provided by the negatively biased end sections and radial confinement provided by the magnetic field. We denote the radius of the cylinder by r_c and the length of the central section by $2z_c$. The wall potential of the central section is denoted by V and that of the end sections by $V - \Delta V$.

Since the system has cylindrical symmetry, the electron canonical angular momentum and the electron energy enter the thermal equilibrium electron distribution function on equal footing,[4,5] as

$$f = n_0(m/2\pi T)^{3/2} \exp[-(1/T)(H - \omega p_\theta)]. \qquad (1)$$

The electron energy is given by $H = mv^2/2 - e\varphi(r,z)$, where φ is the electric potential and m, $-e$, and $\mathbf{v}$ are the electron mass, charge, and velocity. The electron canonical angular momentum is given by $p_\theta = mv_\theta r - (e/c)A_\theta(r)r$, where $A_\theta(r)$ is the θ component of the vector potential and c is the speed of light. The parameters n_0, T, and ω are determined by the total number of electrons, energy and canonical angular momentum in the system. The distribution can be rewritten as

$$f = n_0\left(\frac{m}{2\pi T}\right)^{3/2} \times \exp\left\{-\frac{1}{T}\left[\frac{m}{2}(\mathbf{v} - \omega r\hat{\theta})^2 - e\varphi(r,z) + \frac{m}{2}\omega(\Omega - \omega)r^2\right]\right\}, \qquad (2)$$

where we have used $A_\theta(r) = Br/2$ and have introduced the cyclotron frequency $\Omega = eB/mc$. (We neglect the diamagnetic field since all velocities are small compared with c and since the density is below the Brillouin limit.) The velocity dependence of f is a Maxwellian in a frame rotating with frequency ω. For sufficiently large magnetic field (i.e., large Ω), the last term in the exponential forces the distribution to zero at large r. We assume the conducting cylinder is outside the radius where the distribution becomes exponentially small. The electric potential makes the distribution exponentially small near the negatively biased end sections.

The electric potential must be determined self-consistently from Poisson's equation,

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} = 4\pi e n_0 \exp\{-T^{-1}[-e\varphi(r,z) + \tfrac{1}{2}m\,\omega(\Omega - \omega)r^2]\}, \qquad (3)$$

where the density on the right-hand side has been determined by integrating Eq. (2) over velocity. This equation must be solved subject to the boundary conditions on the cylinder. The solution depends on the parameters n_0, T, $\omega(\Omega - \omega)$, r_c, z_c, and ΔV. We assume that V is adjusted to make $\varphi(r,z)$ zero at the origin, in

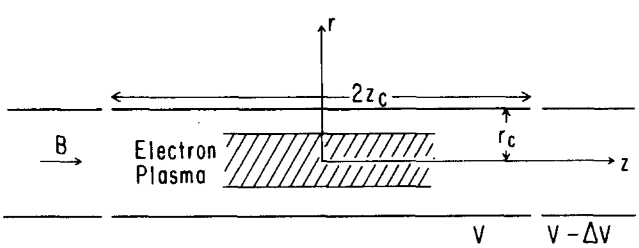

FIG. 1. The confinement geometry.

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which case n_0 represents the value of the density at the origin. By scaling Eq. (3), the number of parameters can be reduced. In terms of the variables

$$\psi \equiv \frac{e\varphi}{T} - \frac{m\omega(\Omega-\omega)r^2}{2T}, \quad \rho \equiv \frac{r}{\lambda_D}, \quad \zeta \equiv \frac{z}{\lambda_D}, \tag{4}$$

$$\lambda_D^2 \equiv \frac{T}{4\pi n_0 e^2}, \quad \gamma \equiv \frac{2m\omega(\Omega-\omega)}{4\pi n_0 e^2} - 1,$$

Eq. (3) takes the form

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial\zeta^2} = (e^{\psi}-1) - \gamma, \tag{5}$$

and the density is given by $n(\rho,\zeta) = n_0\exp[\psi(\rho,\zeta)]$. The solutions to Eq. (5) depend on the parameters γ, $e\Delta V/T$, $\rho_c = r_c/\lambda_D$, and $\zeta_c = z_c/\lambda_D$.

For any choice of these four parameters, the solution ψ is unique. To show this, let ψ_1 and ψ_2 be two solutions of Eq. (5) such that $\psi_1 = \psi_2 = 0$ at the origin. Since we have not specified the value of V, it is possible that ψ_1 differs from ψ_2 by a constant on the cylindrical boundary. To be specific, we assume that $\psi_2 \geq \psi_1$ on the boundary. Near the origin, Eq. (5) implies that $\nabla^2(\psi_2 - \psi_1) \simeq 0$. This implies that either $\psi_2 = \psi_1$ in a finite domain around the origin or there exists a neighboring domain where $\psi_2 > \psi_1$ and another where $\psi_2 < \psi_1$. We will assume the latter possibility and deduce a contradiction. The domain where $\psi_2 < \psi_1$ must have a boundary where $\psi_2 = \psi_1$, since $\psi_2 \geq \psi_1$ on the cylinder. From Eq. (5) and the divergence theorem one finds that $\int d\mathbf{S}\cdot\nabla(\psi_2 - \psi_1) = \int d^3r(e^{\psi_2} - e^{\psi_1})$, where the left-hand integral is over the boundary surface of the domain and the right-hand integral is over the volume of the domain. The left-hand integral is greater than or equal to zero and the right-hand integral is negative. Hence, there can be no domain where $\psi_2 < \psi_1$, and thus $\psi_2 = \psi_1$ over a finite domain around the origin. By extension, one sees that $\psi_1 = \psi_2$ throughout the cylinder.

For the case of an infinitely long column,[2-4] ψ is independent of ζ and is determined by a single parameter γ. To obtain a qualitative picture of the solutions for this case, we note that in the region where $\psi(\rho)$ is small one may replace $[\exp(\psi) - 1]$ by ψ in Eq. (5). The solution satisfying the boundary conditions $\psi(0) = \psi'(0) = 0$ is $\psi(\rho) = \gamma[1 - I_0(\rho)]$, where $I_0(\rho)$ is the Bessel function of imaginary argument. [The boundary condition $\psi'(0) = 0$ follows from cylindrical symmetry.] If γ is very small and positive, $\psi(\rho)$ remains small up to a large value of ρ, after which it rapidly becomes negative due to the exponential nature of $I_0(\rho)$. The rapid increase of $-\psi(\rho)$ continues even after $|\psi| > 1$, but in this region $\psi(\rho)$ varies as $-\rho^2$. The density, $n(\rho) = n_0\exp(\psi)$, remains constant up to a large value of ρ and then falls rapidly to zero. The width of the plasma column is given approximately by the value of ρ for which $\psi(\rho) \simeq -1$ (i.e., $\rho \simeq -\ln\gamma$). For very small γ, the falloff in density occurs at large ρ, and Eq. (5) can be approximated as $d^2\psi/d\rho^2 = \exp(\psi) - 1$ in this region. This equation is independent of γ and unchanged by a shift along the ρ axis. The boundary conditions at the beginning of the falloff region are $\psi(\rho) \simeq -\gamma I_0(\rho)$ and $\psi'(\rho) \simeq -\gamma I_0'(\rho)$. Taking into account the exponential nature of $I_0(\rho)$ for large ρ, the γ dependence in the boundary conditions can be accounted for by a shift along the ρ axis. In other words, the curves $\psi(\rho,\gamma)$ for different (but small) values of γ are nearly identical in the falloff region except for a slight shift along the ρ axis. This can be seen in Fig. 1 of the preceding paper[2] where numerical solutions for $\psi(\rho,\gamma)$ are plotted versus ρ for values of γ ranging from 1 to 10^{-5}. Figure 2 of this paper is a plot of the universal functions $\psi(\rho,\gamma \simeq 0)$ and $n(\rho) = n_0\exp(\psi)$.

A simple way to understand the density profile is to interpret the last term in the bracket on the right-hand side of Eq. (3) as the potential energy of an electron in a hypothetical cylinder of uniform positive charge. The electrons match their density to that of the hypothetical positive charge,

$$4\pi n_+ e = -\nabla^2\left[\frac{1}{(-e)}m\omega(\Omega-\omega)\frac{r^2}{2}\right] = \frac{2m}{e}\,\omega(\Omega-\omega), \tag{6}$$

neutralizing it out to some radius where the supply of electrons is exhausted. At that radius the electron density falls off on a scale set by the Debye length. Note that $n_0 = n_+$ is just the condition $\gamma = 0$.

For the case of a finite length column, the numerical solution of Eq. (5) is obtained by an iteration procedure. We choose a $\psi_1(\rho,\zeta)$ such that $\psi_1(0,0) = 0$ and calculate $\exp(\psi_1) - 1 - \gamma$. Using this for the right-hand side of Eq. (5), we solve Poisson's equation numerically[6] for a $\psi_2(\rho,\zeta)$ such that $\psi_2 = eV/T - (\gamma+1)\rho_c^2/4$ on the central cylinder and $\psi_2 = eV/T - e\Delta V/T - (\gamma+1)\rho_c^2/4$ on the end cylinders, with V adjusted to make $\psi_2(0,0) = 0$. To make the domain of the numerical integration finite in length, use is made of the fact that the potential is approximately uniform across the cylinder at an axial position sufficiently far into either of the end sections [i.e., $\psi(\rho) \simeq eV/T - e\Delta V/T - (\gamma+1)\rho^2/4$, for $|\zeta| - \zeta_c \gg \rho_c$]. After $\psi_2(\rho,\zeta)$ is obtained, the procedure is repeated until it converges.

For the Debye length small compared with the dimensions of the plasma, the picture that emerges from the numerical calculations is that of a uniform density plasma bounded by a surface of revolution where the density falls off rather abruptly, that is, on the scale of a Debye length. Such solutions occur in the limit of small γ. If we consider an arbitrary plane containing the ζ axis [i.e., a (ρ,ζ) plane], the curve defined by $\psi(\rho,\zeta) = -1$ lies in the region where the density falls off. At an arbitrary point along this curve, we can introduce a local orthogonal coordinate system (u, v), where u is tangent to the curve and v is normal to the curve. For small γ, the radius of curvature of the $\psi = -1$ curve should be large compared with unity, that is, large compared with the Debye length in unscaled variables. Consequently, Eq. (5) takes the form $\partial^2\psi/\partial^2 v = \exp(\psi) - 1 - \gamma$, and the fall off in density $\{$i.e., $n(v) = n_0\exp[\psi(v)]\}$ is of the same form as that shown in Fig. 2. As an illustration, we consider the case $\gamma = 0.0003$, $e\Delta V/T = 100$, $\rho_c = 16$, and $\zeta_c = 64$. Figure 3 shows the curve $\psi(\rho,\zeta) = -1$, with local orthogonal coordinate systems attached at the points a, b, and c. Figure 4 shows a comparison between $\psi(v)$ in these local coordinate systems and ψ

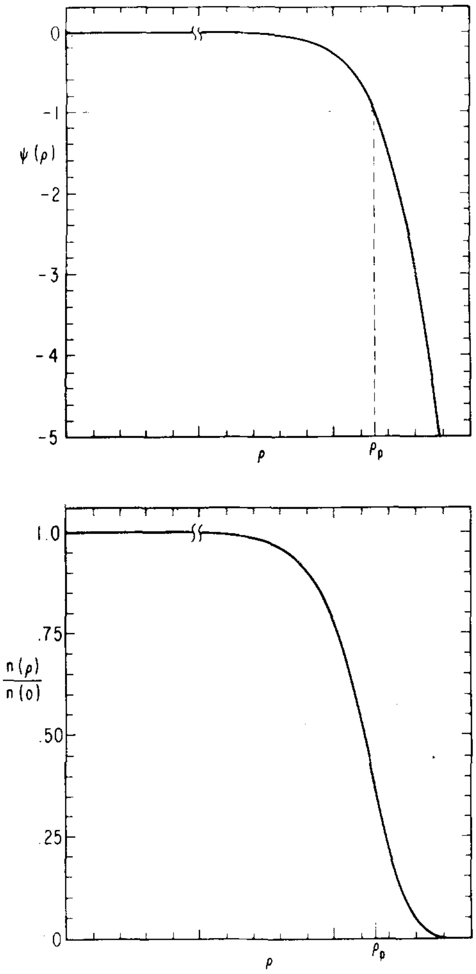

FIG. 2. The universal functions $\psi(\rho, \gamma \simeq 0)$ and $n(\rho) = n_0 \exp(\psi)$. The scaled radius ρ_p, where $\psi(\rho_p) = -1$, is given by $(2\pi\rho_p)^{1/2}\exp(-\rho_p) = \gamma$. The division marks show units of ρ.

from Fig. 2. The solid curve is from Fig. 2, and the circles, triangles, and crosses are values of $\psi(v)$ for the local systems a, b, and c, respectively.

In general, there are two length scales in the problem: the Debye length and the geometrical length scale. If we pass to the limit of zero Debye length (i.e., zero temperature), the plasma has a sharp boundary, and only the geometrical scale remains. This boundary is a surface of revolution inside of which $n = n_0$ and $\partial\varphi/\partial z = 0$. The statement $\partial\varphi/\partial z = 0$ follows from the result that $\psi = 0$ inside the plasma and simply expresses the fact that there can be no potential variation along a field line within a zero temperature plasma. Outside the surface, $n = 0$. The determination of the boundary surface reduces to a rather unusual problem in potential theory. One searches for a surface of revolution such that $\nabla^2\varphi = 4\pi n_0 e$ inside the surface, $\nabla^2\varphi = 0$ outside the surface, and $\partial\varphi/\partial z = 0$ on the surface. Of course, φ must also satisfy boundary conditions on the cylindrical wall. Taking into account the fact that φ is arbitrary up to an additive constant, we choose $\varphi = 0$ on the central cylinder and $\varphi = -\Delta V$ on the end cylinders. Note that the conditions $\nabla^2\varphi = 4\pi n_0 e$ inside the surface and $\partial\varphi/\partial z = 0$ on the surface together imply that $\partial\varphi/\partial z = 0$ inside the surface.

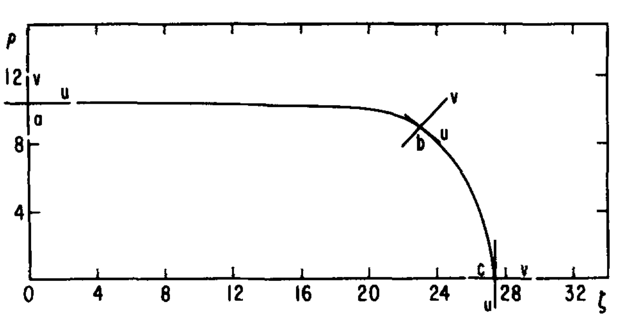

FIG. 3. The $\psi(\rho, \xi) = -1$ curve for $\gamma = 0.0003$, $e\Delta V/T = 100$, $\rho_c = 16$, and $\zeta_c = 64$. a, b, and c are three points where the local orthogonal coordinate systems (u, v) are introduced.

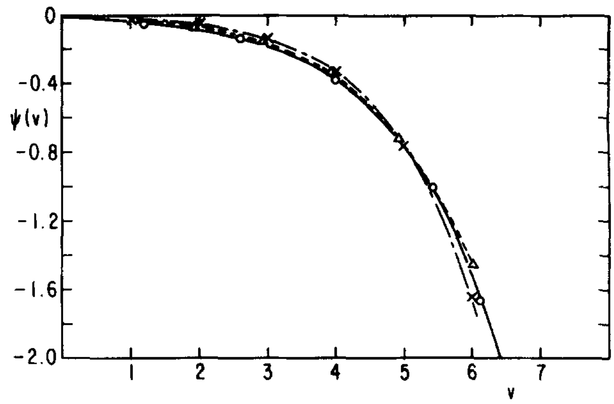

FIG. 4. Comparison of the fall-off of $\psi(v)$ at the points a (circles), b (triangles), and c (crosses) of Fig. 3. The solid curve is $\psi(\rho)$ from Fig. 2.

Passage to the limit of zero Debye length removes one of the parameters on which the solution depends. For the case of finite Debye length, the four parameters specifying the solution may be taken to be $e\Delta V/T$, $(r_c/\lambda_D)^2$ $(z_c/\lambda_D)^2$ and $(r_p/\lambda_D)^2$. Here, r_p is the radius of the plasma at $z = 0$; it is defined as $\psi(r_p/\lambda_D, \zeta = 0) = -1$ and must be given by a function of the form $r_p/\lambda_D = f(\gamma, e\Delta V/T, r_c/\lambda_D, z_c/\lambda_D)$. The four parameters all become infinite as λ_D^2 and T go to zero. Multiplying by (λ_D^2/r_c^2) yields the three well behaved parameters $(e\Delta V/n_0 e^2 r_c^2)$, $(z_c/r_c)^2$, $(r_p/r_c)^2$ and the constant, 1.

To construct a solution dependent on these three parameters we introduce the scaled lengths $R = r/r_c$ and $Z = z/r_c$ and the scaled potential $\Phi = e\varphi/n_0 e^2 r_c^2$. The edge of the plasma is a surface of revolution such that: (i) $\nabla^2\Phi$ is equal to 4π inside the surface and is equal to 0 outside the surface, and (ii) $\partial\Phi/\partial Z = 0$ on the surface. The boundary conditions at the wall are: (iii) $\Phi(1, Z) = 0$ for $|Z| < Z_c \equiv z_c/r_c$ and $\Phi(1, Z) = -e\Delta V/n_0 e^2 r_c^2$ for $|Z| > Z_c$.

We search for the boundary surface iteratively. Consider first an arbitrary surface of revolution $Z = \pm Z_s(R)$. We solve for a potential Φ satisfying conditions (i) and (iii). [In general, this Φ will not satisfy condition (ii).] Such a Φ can be written as

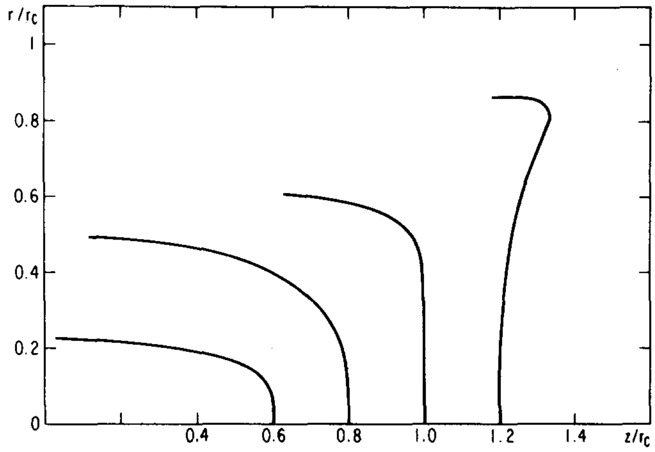

FIG. 5. The end shapes of the plasma column for different values of r_p/r_c. The zero of the abscissa has no physical significance.

$$\Phi(R,Z)=\Phi_0(R,Z) - \int_0^1 dR'\, 2\pi R' \int_{-Z_s(R')}^{Z_s(R')} dZ'\, G(R,Z;R',Z').$$

where $\nabla^2\Phi_0 = 0$ everywhere inside the cylinder and Φ_0 satisfies the boundary conditions (iii). The Green's function, $G(R,Z;R',Z')$, is zero on the cylindrical wall. One can easily show that

$$\Phi_0(R,Z) = -\frac{2\Delta V}{n_0 e r_c^2} \times \sum_n \frac{J_0(\mu_n R)}{\mu_n J_1(\mu_n)} \frac{\cosh \mu_n Z}{(\cosh \mu_n Z_c + \sinh \mu_n Z_c)}, \quad (7)$$

for $|Z| < Z_c$ and

$$G(R,Z;R',Z') = 2\sum_n \frac{J_0(\mu_n R')J_0(\mu_n R)}{\mu_n J_1^2(\mu_n)} \exp(-\mu_n |Z-Z'|),$$

where μ_n is defined by $J_0(\mu_n)=0$.

Let the surface be slightly deformed to $Z = \pm[Z_s(R) + \delta Z(R)]$. This changes $\partial\Phi/\partial Z$ evaluated at the edge of the plasma by an amount

$$\delta[\partial\Phi/\partial Z|_{Z_s(R)}] = \partial^2\Phi/\partial Z^2|_{Z_s(R)}\delta Z(R) - \int_0^1 dR'\, 2\pi R' \delta Z(R')\partial/\partial Z\{G[R,Z;R',Z_s(R')] + G[R,Z;R',-Z_s(R')]\}|_{Z_s(R)}. \quad (8)$$

This equation can be inverted numerically to yield $\delta Z(R)$ as a function of $\delta(\partial\Phi/\partial Z|_{Z_s(R)})$. One chooses $\delta(\partial\Phi/\partial Z|_{Z_s(R)})$ to be $-\alpha(\partial\Phi/\partial Z|_{Z_s(R)})$, where α is some small constant. By iterating, the surface deforms until $\partial\Phi/\partial Z = 0$ for all points on the surface. For this surface, Φ meets all of the conditions (i)–(iii). In other words, it is the boundary surface of the cold plasma.

In the limit of large Z_c and large $e\Delta V/n_0 e^2 r_c^2$, the potential Φ_0 is proportional to $\Delta V \exp[-\mu_1(Z_c - Z)]J_0(\mu_1 R)$ for $Z_c - Z \gg 1$. This means that changing ΔV does not change the shape of the surfaces of constant Φ_0, but merely causes a parallel displacement of them. As a consequence, the length of the plasma column is changed but not its end shape. The end shape is uniquely determined by the parameter r_p/r_c; Fig. 5 displays the end shapes for different values of r_p/r_c. It should be noted that the origin of the abscissa is arbitrary; the numbers on the abscissa are included only to indicate the scale length.

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