Effect of Nonlinear Collective Processes on the Confinement of a Pure-Electron Plasma

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For an electron plasma which is magnetically confined in a cylindrical field geometry, radial expansion occurs only if the angular momentum of the plasma changes. We discuss nonlinear collective processes by which perturbing, static, asymmetric fields can transfer angular momentum, but not energy, to the plasma and produce radial expansion. For example, the field asymmetry can act as a pump which excites daughter modes via the decay instability. Alternatively, the pump can drive a mode by induced scattering from particles.

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Recent experiments[1,2] have involved the confinement of a pure-electron plasma in a field geometry which is nominally cylindrical. The radial confinement is provided by an axial magnetic field and the axial confinement is provided by electrostatic fields which are due to negatively biased end cylinders; this arrangement is shown in Fig. 1. Because the column is nonneutral, there is a radial electric field, and this field, together with the axial magnetic field, produces an $\mathbf{E}\times\mathbf{B}$ drift rotation of the column. For typical experimental parameters the cyclotron frequency, plasma frequency, and rotation frequency satisfy $\omega_c \gg \omega_p \gg \omega_R$. Also, the Larmor radius is typically much smaller than the plasma radius.

To understand radial confinement in such a system, it is useful to introduce the total canonical angular momentum for the electrons

$$P_\theta = \sum_j \left[m v_{\theta_j} r_j - \frac{1}{2}\frac{e}{c} B r_j^2 \right], \tag{1}$$

where r_j is the radial position of the jth electron and v_{θ_j} is the azimuthal velocity of the jth electron. We have taken the vector potential $A_\theta(r) = \frac{1}{2}Br$, corresponding to a uniform axial magnetic field. For the ordering mentioned above, the mechanical part of the angular momentum is negligible compared to the vector potential part, and we can set $P_\theta \simeq (-eB/2c)\sum_j r_j^2$. To the extent that the field geometry has cylindrical symmetry, P_θ is conserved and there is a constraint on the allowed radial positions of the electrons: $\sum_j r_j^2 = \text{const}$. The mean square radius of the plasma can increase only if angular momentum is transferred to the plasma, that is, only if a torque is applied. This heuristic argument can be made rigorous.[3]

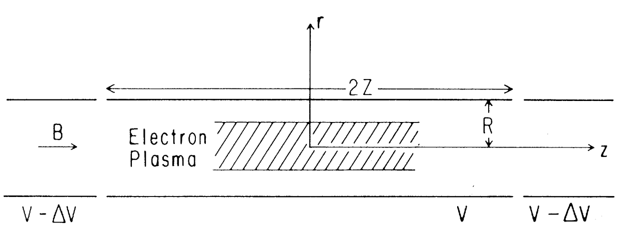

FIG. 1. Plasma confinement geometry.

Of course, in an experiment, the field geometry lacks perfect cylindrical symmetry; there are static, weak, perturbing fields which break the symmetry. In this paper, we discuss nonlinear collective processes whereby such fields can transfer angular momentum to the column and produce radial expansion, i.e., increase $\sum_j r_j^2$. Since recent experiments suggest that field errors are the dominant source of angular momentum,[1] we neglect all other sources of external torque, such as electron-neutral collisions.

The nonlinear processes are generalizations of a linear process which was recently discussed theoretically[4] and observed experimentally.[2] For a rotating column, a mode can occur which propagates backward (upstream) on the column and has zero frequency in the laboratory frame. A field asymmetry, with a Fourier component $e^{i(l\theta + kz)}$ which matches the mode, can resonantly (i.e., secularly) drive such a wave to large amplitude. Because the field asymmetry has zero frequency ($\omega = 0$) and nonzero azimuthal wave number ($l \neq 0$), it transfers angular momentum, but not energy, to the plasma. Correspondingly, the resonantly driven mode has zero energy and nonzero angular momentum, that is, $W = \omega N = 0$ and $L = lN \neq 0$, where W, L, and N are the mode energy, angular momentum, and action. As the mode grows secularly, the increasing angular momentum of the mode is associated with a radial expansion of the plasma, and this degrades the plasma confinement in various ways. If the mode amplitude becomes large enough, electrons may be driven into the wall by coherent wave motion. In addition, the large amplitude modes can produce an enhanced level of resonant particle transport.[2,5]

It is important to consider nonlinear generalizations of this linear process, since a zero frequency eigenmode will not generally appear in a bounded plasma for arbitrarily chosen values of density, magnetic field strength, column length, etc. The first generalization treats the field asymmetry as a pump which excites two daughter modes via the decay instability. The daughter modes (one positive energy and one negative) have zero net energy and nonzero net angular momentum. The existence of negative energy modes (or of zero energy modes) is due to the column rotation.

For the basic interaction between a particular error component of amplitude A_3 and two collective modes $A_1(t)$, $A_2(t)$, which satisfy resonance conditions $\omega_1 + \omega_2 + \Delta\omega = 0$ and $l_1 + l_2 = l_3$, we can anticipate dynamics of the form,[6]

$$\begin{aligned} iS_1\dot{A}_1 &= iS_1\gamma_1 A_1 + V^*_{3,2,1}A_3A_2^*, \\ iS_2\dot{A}_2 &= iS_2(\gamma_2 + i\Delta\omega)A_2 + V^*_{3,2,1}A_3A_1^*, \end{aligned} \tag{2}$$

where $\dot{A}_{1,2} = dA_{1/2}/dt$ and (*) denotes complex conjugation. In Eq. (2) $\gamma_{1,2}$ represents the linear damping, and a possible detuning from exact resonance (i.e., $\Delta\omega \neq 0$) has been absorbed into the dynamics of A_2.[7] $V_{3,2,1}$ is the three-wave coupling which involves overlap integrals for the mode wave functions, and possesses the usual symmetries.[8,9] (These symmetries permit the coupling terms to be written as shown.) S_1 and S_2 are energy sign factors, defined in terms of the real part of the dielectric function, $\epsilon'(\omega)$, by $S_\alpha = \mathrm{sgn}(\partial\epsilon'/\partial\omega|_{\omega_\alpha})$, where $\alpha = 1, 2$. The amplitudes are normalized such that the mode energy is $W_\alpha = S_\alpha\omega_\alpha|A_\alpha|^2$.

To determine the threshold of the decay instability, we solve Eq. (2) holding A_3 fixed. The linear eigenvalues, A_1, $A_2^* \sim e^{\lambda t}$, satisfy $(\lambda - \gamma_1)(\lambda - \gamma_2 + i\Delta\omega) = S_1S_2|V_{3,2,1}A_3|^2$. The threshold condition, $\mathrm{Re}\lambda = 0$, defines a critical pump amplitude, $|A_3|_c^2$,[10]

$$|A_3|_c^2 = \frac{\gamma_1\gamma_2}{S_1S_2}\frac{1 + [\Delta\omega/(\gamma_1+\gamma_2)]^2}{|V_{3,2,1}|^2}. \tag{3}$$

For damped modes ($\gamma_1\gamma_2 > 0$), $S_1S_2 = +1$ is a necessary condition for the instability. $S_1 = S_2$, together with the resonance condition $\omega_1 \approx -\omega_2$, implies one daughter is positive energy, and the other negative energy.

As in the case of linear resonance, the angular momentum of the column changes due to the daughter wave momentum, $P_\theta = S_1l_1|A_1|^2 + S_2l_2|A_2|^2$. Neglecting the damping, which simply transfers this angular momentum to the resonant particles, Eq. (2) implies $dP_\theta/dt = S_1(l_1+l_2)\,d|A_1|^2/dt$. Obviously the sign of $\dot{P}_\theta$ depends on S_1 which may be evaluated from simple physical considerations. The rotation corresponds to a beam in the azimuthal direction, and we expect waves traveling more slowly than the column but in the same direction to carry negative energy, $W < 0$.[11] For such modes, the angular phase velocity ($\dot\theta = \omega/l$) satisfies $\omega_R > \dot\theta > 0$, and the energy sign is $\mathrm{sgn}(W) = \mathrm{sgn}[(\omega - l\omega_R)\omega]$; thus $S = \mathrm{sgn}(\omega - l\omega_R)$. This prescription for S may be verified analytically. The sign of dP_θ/dt is then given by

$$S_1(l_1+l_2) = -S_1(\omega_1 - l_1\omega_R + \omega_2 - l_2\omega_R)/\omega_R = -(|\omega_1 - l_1\omega_R| + |\omega_2 - l_2\omega_R|)/\omega_R < 0, \tag{4}$$

where $S_1 = S_2$, $\omega_1 + \omega_2 = 0$ have been used. Thus, the angular momentum of the column is reduced, $\delta P_\theta < 0$, which implies that $\delta(\sum_j r_j^2) > 0$ from Eq. (1).

In a second generalization, one of the daughter modes is in effect replaced by resonant particles. The field asymmetry acts as a mode which undergoes induced scattering off of resonant particles into another mode, or equivalently, the field error and the other mode undergo nonlinear Landau damping by the resonant particles. For finite columns, which necessarily have discrete frequencies, this second mechanism can occur more generally than the decay instability because the resonance condition $\omega_1 + \omega_2 \approx 0$ is not required. These two processes are not exhaustive (higher-order analogs are obvious), but are illustrative of the kind of nonlinear effects that can degrade plasma confinement.

To analyze quantitatively such nonlinear effects, we have modeled the experimental plasma which was used to demonstrate the linear resonance between a field error and a mode.[2] For this model, we consider a cylindrically symmetric, quiescent electron column of length $2L$. The radial variation of the plasma occurs on a scale which is much larger than a Larmor radius, but is otherwise arbitrary. The column ends (at $z = \pm L$) are assumed to be flat, that is, at right angles to the applied field $\mathbf{B} = B\hat{\mathbf{z}}$. Consistent with the ordering $\omega_c \gg \omega_p \gg \omega_R$, the electron motion perpendicular to $\mathbf{B}$ is an $\mathbf{E}\times\mathbf{B}$ drift. Along field lines, the electrons stream freely subject to specular reflections at the ends of the column.

Prasad and O'Neil calculated the low-frequency ($\omega \lesssim \omega_p$) electrostatic modes of this model by use of an expansion in R/L where R is the radius of the containment vessel.[12] Because the equilibrium is invariant under $z \to -z$ and with respect to $v_z \to -v_z$, the modes may be assumed to be either even or odd functions of z. The ϵ_{zz} component of the dielectric tensor is of order $(\omega_p^2/\omega_R^2) \gg 1$, and this large response parallel to $\mathbf{B}$ forces the z component of the mode electric field to satisfy $\partial\phi/\partial z|_{|z| = L-\epsilon} \sim 0$ as $\epsilon \to 0$. In zeroth order, the even (odd) modes take the form, $\psi_{lkn}(r)e^{il\theta}\cos kz(\sin kz)$, appropriate to an infinite length column (varying n changes the number of radial nodes). The only effect of finite column length is the restriction of allowed k values required by the dielectric response, i.e., for $m = 0, 1, 2, \ldots,$ $k = m\pi/L$ for

$\cos kz$ modes and $k=(m+\frac{1}{2})\pi/L$ for $\sin kz$ modes. Thus to analyze the nonlinear couplings between these zero-order modes, or between these modes and external field asymmetries, it is necessary to discretize the k values correctly, but otherwise the column may be regarded as infinite.

Treating both field asymmetries and nonlinear couplings as perturbations of the linear modes, we have derived the resulting dynamics for the wave amplitudes. This derivation will be given in a longer publication.[8] In brief outline, we have followed earlier theories[13,14] by iteratively solving the electron drift kinetic equation to obtain the perturbed guiding center distribution function as an expansion in the electrostatic potential. When this distribution is used in Poisson's equation, a nonlinear equation is obtained for the potential. The solution of this nonlinear equation for the mode amplitudes is facilitated by expanding the radial dependence of the potential in the radial eigenfunctions of the plasma, denoted $\psi_{lkn}(r,\omega)$ above. In Poisson's equation, the presence of electrostatic field errors is included through the boundary condition on the potential at the wall. The dynamics of the slowly varying amplitudes of the even and odd modes, $A_{lkn}^{(+)}(t)$ and $A_{lkn}^{(-)}(t)$, respectively, may be cast into the form of Eq. (2)

$$iS_\alpha \dot{A}_\alpha^{(\pm)} = iS_\alpha\gamma_\alpha A_\alpha^{(\pm)} + \sum_{\alpha'\alpha''} \exp[i(\omega_\alpha-\omega_{\alpha'}-\omega_{\alpha''})t]\eta_{\alpha\alpha'\alpha''}V_{\alpha\alpha'\alpha''}(A_{\alpha'}^{(+)}A_{\alpha''}^{(\pm)} \mp A_{\alpha'}^{(-)}A_{\alpha''}^{(\mp)}) \tag{5}$$

where α denotes the triplet of indices (lkn) labeling the even or odd mode. In writing Eq. (5), the conversion from plane waves e^{ikz} to even and odd modes has introduced the combinatorial factor

$$\eta_{\alpha\alpha'\alpha''} = \delta_{l,l'+l''}\delta_{k,k'+k''}(1+\delta_{k',0})(1+\delta_{k'',0})/2(1+\delta_{k,0}).$$

The three-wave coupling is

$$V_{\alpha\alpha',\alpha''} = -\left[(4\pi e)\left(\frac{e}{m}\right)^2 M_{\alpha\alpha'\alpha''}\right]\left|\frac{\partial\epsilon'_\alpha}{\partial\omega}\frac{\partial\epsilon'_{\alpha'}}{\partial\omega}\frac{\partial\epsilon'_{\alpha''}}{\partial\omega}\right|^{-1/2}. \tag{6}$$

The matrix element is

$$M_{\alpha\alpha'\alpha''} = \frac{1}{2}\int_0^R r\,dr\,\psi_\alpha \int_C \frac{dv}{d_\alpha}(I_{\alpha'\alpha''}+I_{\alpha''\alpha'}), \tag{7}$$

where

$$I_{\alpha'\alpha''} \equiv \left[k''\psi_{\alpha''}\frac{\partial}{\partial v} + \frac{1}{r\omega_c}\left(l'\frac{\partial\psi_{\alpha''}}{\partial r} - l''\psi_{\alpha''}\frac{\partial}{\partial r}\right)\right]\left(\frac{\chi_{\alpha'}\psi_{\alpha'}}{d_{\alpha'}}\right) \tag{8}$$

and $\chi_{\alpha'} \equiv [k'\partial_v F_0 - (l'/r\omega_c)\partial_r F_0]$. $F_0(r,v)$ is the guiding center equilibrium distribution $(v\equiv v_z)$, $d_\alpha \equiv kv - (\omega_\alpha - l\omega_R)$, and the contour C is defined by $\omega+i0$. The dielectric function $\epsilon_\alpha = \epsilon'_\alpha + i\epsilon''_\alpha$, where

$$\epsilon_\alpha = \int_0^R r\,dr\left[\left(\frac{\partial\psi_\alpha}{\partial r}\right)^2 + \left(\frac{l^2}{r^2}+k^2-\frac{4\pi e^2}{m}\int_C \frac{dv[k\partial_v F_0 - (l/r\omega_c)\partial_r F_0]}{d_\alpha}\right)|\psi_\alpha|^2\right] \tag{9}$$

generalizes the previous result of Keinigs,[4] and determines ω_α, γ_α, and S_α by $\epsilon_\alpha(\omega_\alpha+i\gamma_\alpha)=0$ and

$$S_\alpha = \mathrm{sgn}\left(\frac{\partial\epsilon'_\alpha}{\partial\omega}\bigg|_{\omega_\alpha}\right) = \mathrm{sgn}\left[\mathrm{P}\int_{-\infty}^{\infty}\frac{dv F_0(r,v)}{(\omega_\alpha - l\omega_R - kv)^3} + O\left(\frac{\omega_R}{\omega_c}\right)\right]. \tag{10}$$

From this result for S_α, for F_0 monotonically decreasing in $|v|$, we recover the relation $S_\alpha = \mathrm{sgn}(\omega_\alpha - l\omega_R)$ proposed earlier on physical grounds. We omit here the lengthy formulas for the four-wave and induced scattering couplings.[8] Numerical investigations of the dispersion relation for this model in the cold fluid limit indicate that the resonance conditions for the decay instability can be satisfied at experimentally relevant parameters.

We have argued that nonlinear effects allow field asymmetries to exert torques on a rotating electron plasma column by exciting collective modes. The resulting transfer of angular momentum allows a bulk radial expansion of the column. As the physical basis of these mechanisms is quite general, they may be relevant to other forms of rotating plasma such as the central cell plasma of a tandem mirror.

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[1]J. H. Malmberg and C. F. Driscoll, Phys. Rev. Lett. **44**, 654 (1980); C. F. Driscoll and J. H. Malmberg, Phys. Rev. Lett. **50**, 167 (1983).

[2]D. L. Eggleston, T. M. O'Neil, and J. H. Malmberg, Phys. Rev. Lett. **53**, 982 (1984).

[3]T. M. O'Neil, Phys. Fluids **23**, 2216 (1980).

[4]R. Keinigs, Phys. Fluids **24**, 860 (1981), and **27**, 1427 (1984).

[5]J. R. Myra, X. S. Lee, and P. J. Catto, Bull. Am. Phys. Soc. **28**, 1035 (1983).

[6]R. C. Davidson, *Methods of Nonlinear Plasma Theory* (Academic, New York, 1972), Chap. 6.

[7]J. Weiland and H. Wilhelmsson, *Coherent Nonlinear Interaction of Waves in Plasmas* (Pergamon, New York, 1977), p. 265.

[8]J. D. Crawford and T. M. O'Neil, to be published.

[9]S. Johnston and Allan Kaufman, J. Plasma Phys. **22**, 105 (1979).

[10]In this formula, it is implicit that if $\gamma_1+\gamma_2=0$ then $\gamma_1=\gamma_2$; so the limit $(\gamma_1,\gamma_2)\rightarrow(0,0)$ must be taken as $\gamma_1=\gamma_2=\gamma$ and $\gamma\rightarrow 0$.

[11]P. Sturrock, J. Appl. Phys. **31**, 2052 (1960).

[12]S. A. Prasad and T. M. O'Neil, Phys. Fluids **27**, 206 (1984).

[13]M. Porkolab and R. P. H. Chang, Rev. Mod. Phys. **50**, 745 (1978).

[14]R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory* (Benjamin, New York, 1969).