Measurement of Screening Enhancement to Nuclear Reaction Rates using a Strongly Magnetized and Strongly Correlated Non-neutral Plasma

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An analogy is uncovered between the nuclear reaction rate in a dense neutral plasma and the energy equipartition rate in a strongly magnetized non-neutral plasma. In strong magnetic fields, cyclotron energy, like nuclear energy, is released only through rare close collisions between charges. The probability of such collisions is enhanced by plasma screening effects, just as in nuclear reactions. Enhancements of up to 10^{10} are measured in simulations of cyclotron energy equipartition and are compared to the theory of screened nuclear reactions.

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In the dense interiors of brown dwarfs, giant planets, and degenerate stars, nuclear fusion reactions are theorized to occur at rates that are significantly enhanced compared to the low density Gamow rates. The enhancement is caused by screening from surrounding nuclei and electrons that increases the probability of close collisions [1–5]. The enhancement is predicted to be very large when the plasma is strongly correlated. However, the twin requirements of high temperature (for measurable nuclear reaction rates) and high density (for strong correlation) put measurement of large screening enhancement factors out of reach in terrestrial settings.

This Letter examines an analogy between screened nuclear reactions in dense plasmas and energy equipartition in a strongly coupled non-neutral plasma. This analogy allows the measurement of the screening enhancement of nuclear reaction rates in regimes that are not accessible to present nuclear physics experiments. We explore the utility of this approach using molecular dynamics simulations.

When a non-neutral plasma in a strong magnetic field is at sufficiently low temperature, the cyclotron frequency $\Omega_c = eB/mc$ can be the highest dynamical frequency, higher even than the frequency $\overline{v}/b$ associated with typical collisions (here $b = e^2/T$ is the distance of closest approach and $\overline{v} = \sqrt{T/m}$ is the thermal speed). Therefore, in the strongly magnetized regime where $\kappa \equiv \sqrt{2}\Omega_c b/\overline{v} \gg 1$, cyclotron energy is an adiabatic invariant, so it is nearly constant. Cyclotron energy is shared with other degrees of freedom only through very rare close collisions that break the adiabatic invariant.

This was pointed out in a series of papers [6–8], and the rate ν_0 of equipartition of cyclotron temperature $T_\perp$ with the plasma temperature T was evaluated, assuming the plasma is weakly correlated. There it was shown that $\nu_0 = n\overline{v}b^2\overline{\nu}_0(\kappa)$ where n is the plasma density, and where $\overline{\nu}_0(\kappa)$ behaves for large κ as $\overline{\nu}_0(\kappa) \propto e^{-2.04\kappa^{2/5}}$. The exponential decrease in the equipartition rate as κ increases is caused by the exponentially small breaking of the cyclotron adiabatic invariant.

Here, we argue that the cyclotron energy is analogous to the energy stored in a nucleus, which can also be released only through close collisions. We show that when the non-neutral plasma is strongly coupled, plasma screening effects greatly enhance the probability of close collisions, increasing by orders of magnitude the energy equipartition rate ν compared to the weakly correlated rate ν_0. This effect is analogous to the screening enhancement of nuclear reactions in dense neutral plasmas.

The equipartition rate ν is the integral over a product of a Maxwellian distribution describing the relative energy of colliding pairs, and a function involving the change in the cyclotron adiabatic invariant for collisions of a given energy. This function is exponentially increasing as collision energy increases. Just as for nuclear reactions, the product of this exponentially increasing function and the exponentially decreasing Maxwellian leads to a Gamow peak in the differential rate, and this peak is shifted by plasma screening effects. Weak screening, strong screening, and "pycnonuclear" regimes can be identified that are analogous to those for nuclear reactions. In the weak and strong screening regimes we show that ν is the product of the rate without correlations, ν_0, and an enhancement factor $f(\Gamma)$ that depends only on the coupling parameter $\Gamma = e^2/aT$ (where a is the Wigner-Seitz radius). Furthermore, this enhancement factor is *identical* to the factor appearing in expressions for nuclear reaction rates in dense plasmas.

We now show that the energy equipartition rate ν is enhanced in the strongly coupled regime. Consider N strongly magnetized like charges for which the cyclotron motion has temperature $T_\perp$ and the motion parallel to the magnetic field has temperature T. The rate ν of energy equipartition is defined by

$$\frac{dT_\perp}{dt} = \nu(T - T_\perp). \tag{1}$$

The Green-Kubo expression for ν is

$$\nu = \frac{1}{2NT_\perp T}\int_{-\infty}^{\infty} dt\langle \dot{K}_\perp(t)\dot{K}_\perp(0)\rangle, \tag{2}$$

where $K_\perp = \sum_{i=1}^{N} m v_{\perp i}^2/2$ is the perpendicular kinetic energy, and the average is over a two-temperature distribution of initial conditions $\Lambda = (\mathbf{r}_1, \mathbf{p}_1, \ldots, \mathbf{r}_N, \mathbf{p}_N)$,

$$\rho = Z_N^{-1} e^{-K_\perp/T_\perp - (H-K_\perp)/T}, \tag{3}$$

where $H(\Lambda)$ is the system energy and Z_N is the partition function. The only assumption used in Eq. (3) is that $T_\perp(t)$ and $T(t)$ vary slowly in time compared to the correlation time of the fluctuations in $K_\perp$, as determined by $\langle \dot{K}_\perp(t)\dot{K}_\perp(0)\rangle$.

We evaluate this correlation function, *assuming that* it is dominated by close binary collisions. For two-body collisions, Eq. (2) becomes

$$\nu = \frac{1}{2NT_\perp T} \sum_{i>j}^{N} \int_{-\infty}^{\infty} dt \langle \dot{k}_{ij}(t)\dot{k}_{ij}(0)\rangle, \tag{4}$$

where $\dot{k}_{ij}(t) = e^2(\mathbf{r}_i - \mathbf{r}_j)\cdot(\mathbf{v}_{\perp i} - \mathbf{v}_{\perp j})/|\mathbf{r}_i - \mathbf{r}_j|^3$ is the rate of change of the relative perpendicular kinetic energy $k_{ij} \equiv \mu(\mathbf{v}_{\perp i} - \mathbf{v}_{\perp j})^2/2$ of particles i and j due to their collision (where $\mu = m/2$ is the reduced mass). By assumption, this collision is "close" so that the distance of closest approach in the collision, r_0, is small compared to the larger of a and the Debye length λ_D.

The parameter range for which this assumption is valid can be understood as follows: In Ref. [8], the Gamow peak in the equipartition rate was shown to occur for collisions with relative energy $E = T\kappa^{2/5}$ [this explains the $\kappa^{2/5}$ factor in $\overline{\nu}_0(\kappa)$]. Although such collisions are rare events when $\kappa \gg 1$, they dominate the rate. The Gamow energy E corresponds to a distance of closest approach $r_0 = e^2/E$ that is small compared to a only if

$$\Gamma < \kappa^{2/5}. \tag{5}$$

This inequality defines the regime where close binary collisions dominate. If, in addition, $\Gamma > 1$, we are in the strong shielding regime where shielding effects are expected to be important. If $\Gamma \ll 1$, we are in the weak-shielding regime.

The opposite regime $\Gamma > \kappa^{2/5} \gg 1$ is referred to here as the "pycnonuclear regime," in analogy to the high density regime considered in the theory of nuclear reactions [4]. In this regime the energy E of the Gamow peak is less than the Coulomb energy e^2/a, and collisions with impact parameters of order a are more important than close collisions. In the pycnonuclear regime, an analysis based on binary collisions may not apply, and currently there is no theory to describe the rate.

Previous theories of the pycnonuclear regime are inapplicable, because for nuclear reactions the Gamow energy is $E = T(b/r^*)^{1/3}$, where $r^* = \hbar^2/me^4$ is the nuclear Bohr radius [4]. The pycnonuclear regime, $E < e^2/a$, can then be expressed as $\hbar\omega_p > T$, where $\omega_p = \sqrt{4\pi e^2 n/m}$ is the ion plasma frequency, so for nuclear reactions the pycnonuclear regime requires that lattice vibrations be quantum mechanical. While this quantum regime is accessible in non-neutral plasmas, here we consider a classical pycnonuclear regime where $\hbar\omega_p \ll T$. This classical regime has no analogue in the theory of screened nuclear reactions.

For $N \gg 1$, Eq. (4) can be written as $\nu = N\langle \Delta k_{ij}\dot{k}_{ij}(0)\rangle/(4T_\perp T)$, where Δk_{ij} is the total change in k_{ij} during the collision. Writing the average explicitly in terms of an integral over phase space, we break this integral into one over relative initial position $\mathbf{r}_{ij}$ and momentum $\mathbf{p}_{ij}$ of particles i and j, and all other variables Λ':

$$\nu = \frac{N}{4T_\perp T} \int d\Lambda' \int d^3 r_{ij} d^3 p_{ij} \rho(H, K_\perp) \Delta k_{ij}(\Lambda)\dot{k}_{ij}(\Lambda). \tag{6}$$

Also, we break up the system energy, writing it as

$$H = H'(\Lambda') + \frac{1}{2}\mu v_{ij}^2 + \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|} + \phi(\mathbf{r}_i, \Lambda') + \phi(\mathbf{r}_j, \Lambda'), \tag{7}$$

where $H'(\Lambda')$ is the energy of a system of $N-2$ particles plus the kinetic energy of the center of mass of particles i and j, and $\phi(\mathbf{r}, \Lambda') = \sum_{\ell \neq i,j} e^2/|\mathbf{r} - \mathbf{r}_\ell|$ is the screening potential created by charges surrounding the colliding pair. Now, in a close collision with $r_0 \ll a$, $\mathbf{r}_i \approx \mathbf{r}_j \approx \mathbf{R}$ where $\mathbf{R} = (\mathbf{r}_i + \mathbf{r}_j)/2$ is the center of mass position. Then

$$H = H'(\Lambda') + 2\phi(\mathbf{R}; \Lambda') + \tfrac{1}{2}\mu v_{ij}^2 + e^2/|\mathbf{r}_i - \mathbf{r}_j|. \tag{8}$$

Noting that $\dot{k}_{ij}(\Lambda)$ is nonzero only for positions $\mathbf{r}_{ij}$ near the turning point, we may then replace the exact H in Eq. (6) by Eq. (8) and use this approximate Hamiltonian in determining Δk_{ij} and $\dot{k}_{ij}$. However, this binary collisional dynamics, and the resulting values of Δk_{ij} and $\dot{k}_{ij}$, are *identical* to those used in Ref. [8] for determining ν_0. Also, Δk_{ij} and $\dot{k}_{ij}$ depend only on $\mathbf{r}_{ij}$ and $\mathbf{p}_{ij}$.

The final step connecting Eq. (6) to the uncorrelated rate ν_0 is to assume that collisions are of the strongly magnetized form discussed in Ref. [8], where guiding centers interact as they move along the magnetic field. In this case we may write

$$\int d^3 r_{ij} d^3 p_{ij} \rho(H, K_\perp) \Delta k_{ij}(\mathbf{r}_{ij}, \mathbf{p}_{ij}) \dot{k}_{ij}(\mathbf{r}_{ij}, \mathbf{p}_{ij}) = \int d^2 \rho_{ij} d^2 p_{\perp ij} \frac{e^{-[H' + 2\phi(\mathbf{R};\Lambda') - K'_\perp]/T - K'_\perp/T_\perp}}{Z_N} \times \int dE_z dt e^{-E_z/T - \mu v_{\perp ij}^2/2T_\perp} \Delta k_{ij} \dot{k}_{ij}, \tag{9}$$

where $K'_\perp = K_\perp - \frac{1}{2}\mu v_{\perp ij}^2$, $E_z = \frac{1}{2}\mu v_{zij}^2 + e^2/|\mathbf{r}_i - \mathbf{r}_j|$ is the relative parallel energy, ρ_{ij} is the impact parameter of the collision, and we have used the identity $dp_{z_{ij}} dz_{ij} = dE_z dt$. Since E_z and Δk_{ij} are both constant along a given strongly magnetized collision trajectory, we can now perform the time integral, obtaining

$$\nu = \frac{N}{4T_\perp T}\int d\Lambda' d^2\rho_{ij} d^2 p_{\perp ij}\frac{e^{-(H'+2\phi-K'_\perp)/T-K'_\perp/T_\perp}}{Z_N} \times \int dE_z e^{-E_z/T-\mu v_{\perp ij}^2/2T_\perp}\Delta k_{ij}^2. \quad (10)$$

Comparing this expression to that for the uncorrelated rate ν_0 [Eq. (17) of Ref. [8]], we conclude that

$$\nu = \nu_0 f(\Gamma), \quad (11)$$

where

$$\nu_0 = \frac{n}{4T_\perp T}\int\frac{dE_z}{\mu} \times \int d^2\rho_{ij} d^2 v_{\perp ij}\frac{e^{-\mu v_{\perp ij}^2/2T_\perp - E_z/T}}{(2\pi T/\mu)^{1/2}2\pi T_\perp/\mu}\Delta k_{ij}^2, \quad (12)$$

and

$$f(\Gamma) = \frac{\mu^3 V}{(\frac{\mu}{2\pi T})^{1/2}(\frac{\mu}{2\pi T_\perp})}\int d\Lambda' \frac{e^{-(H'+2\phi-K'_\perp)/T-K'_\perp/T_\perp}}{Z_N} \quad (13)$$

is the screening enhancement factor, where V is the volume of the system. After performing the kinetic energy integrals, this enhancement factor becomes

$$f(\Gamma) = \frac{VZ_{U_{N-1}}(1)}{Z_{U_N}(0)}, \quad (14)$$

where $Z_{U_N}(n)$ is the configurational portion of the partition function (that part for which the kinetic energy term has been factored out) for a system of N charges, n of which have charge $2e$. This is precisely the same enhancement factor as appears in the theory of nuclear reaction rates in the strong and weak-shielding regimes [5]. Modifications to this theory have recently been proposed by several authors, due to "dynamical screening" effects [9]. For our system, no such modifications are found.

For weak shielding where $\Gamma \ll 1$, one can show that $f(\Gamma) = e^{\sqrt{3}\Gamma^{3/2}}$ [2]. For strong shielding where $\Gamma \gtrsim 1$, Ref. [10] provides semianalytic results for $f(\Gamma)$ based on Monte Carlo simulations:

$$\ln f(\Gamma) = 1.148\Gamma - 0.009\,44\Gamma\ln\Gamma - 0.000\,168\Gamma(\ln\Gamma)^2. \quad (15)$$

Figure 1 plots ν and ν_0 versus T in a strongly correlated plasma for which $\Omega_c/\omega_p = 12.4$. Note that both Γ and κ vary as T varies. In the scaled units used here, $\Gamma = 1.25/T$ and $\kappa = 42.4/T^{3/2}$. The strong shielding regime where Eqs. (5) and (11) apply is $T \gtrsim 0.041$ or $\Gamma \lesssim 30$. In this regime, enhancements $f(\Gamma)$ of up to 10 orders of magnitude are predicted.

To test these predictions, and to examine what happens in the classical pycnonuclear regime where Eq. (11) does not apply, we have performed molecular dynamics simulations of the equipartition of a Penning trap plasma consisting of $N = 200$ identical charges. The plasma is a prolate spheroid with $\Omega_c/\omega_p = 12.4$, and initially $T_\perp \gg T$. For these runs a 4th order Runga-Kutta method is used, with a fixed step size of $0.018\Omega_c^{-1}$. Over the course of a typical run of 10^7 time steps, energy is conserved to better than a part in 10^6. A decrease in the step size by a factor of two does not affect results. Periodically T is instantaneously increased by multiplying all parallel velocities by 2. This has no effect until $T \simeq 0.2$ (i.e., $\Gamma \simeq 6$), after which T and $T_\perp$ come to equilibrium (Fig. 2). The observed equipartition rate $\nu_{\rm exp}$ is then determined using $\nu_{\rm exp} = dT_\perp/dt/(T - T_\perp)$, and is plotted versus T in Fig. 1 (the sequence of red dots). We determine $dT_\perp/dt$ by fitting $T_\perp$ to line segments over short time intervals of order $20\omega_p^{-1}$. At early times where $T_\perp$ varies slowly and ν is small, we fit the evolution to a single straight line (the dashed line in Fig. 2), yielding a rate given by the thin horizontal red line in Fig. 1. As T increases during the simulation, the measured rate tracks the expected rate within a factor of 2–3.

As a further test, several simulations were also performed for which $T_\perp \ll T$ initially. Now T decreases as the temperatures equilibrate, but the measured rates (blue, green, and purple curves in Fig. 1) still agree with the theory within the measurement error.

The rate enhancement function $f(\Gamma)$ can be extracted from these data by simply taking the ratio of the measured

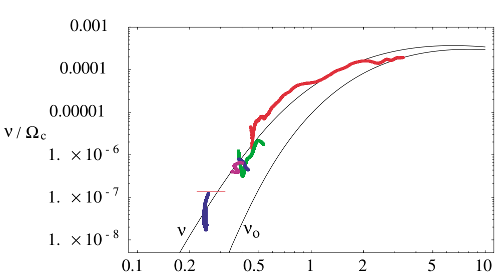

FIG. 1 (color). Equipartition rate ν versus scaled temperature T for $\Omega_c/\omega_p = 12.4$.

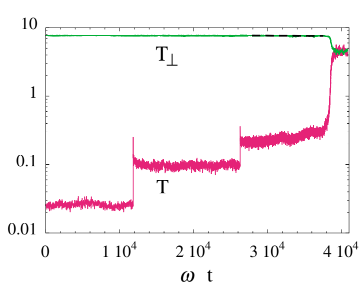

FIG. 2 (color). Temperatures versus time for the simulation shown by red lines in Fig. 1. The dashed line is a linear fit to $T_\perp(t)$ discussed in the text.

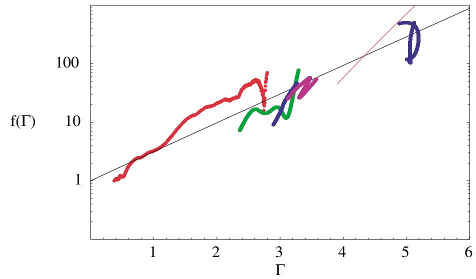

FIG. 3 (color). Enhancement factor for the simulations of Fig. 1 along with theory [Eq. (15)].

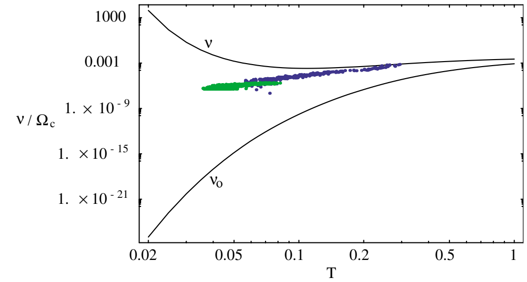

FIG. 4 (color). Equipartition rate versus scaled temperature T for $\Omega_c/\omega_p = 4.14$.

rate to ν_0. This is plotted in Fig. 3 and compared to Eq. (15). The measured rate enhancement agrees with theory over the full range of Γ up to $\Gamma = 5$, where enhancements of several hundred are predicted and observed. As far as we know, this is the first time that rate enhancements of this magnitude have been measured.

At larger Γ values, the rate is too small to be measurable in these simulations. However, a second set of simulations was performed for $\Omega_c/\omega_p = 4.14$ (3 times smaller than before), which allows us to measure equipartition rates for lower temperatures and hence larger Γ values, up to 40. Parameters are chosen so that $\Gamma = 1.25/T$ as before, but now $\kappa = 14.1/T^{3/2}$. For $T > 0.12$ Eq. (5) is satisfied, the simulation is in the strong shielding regime, and there is reasonable agreement between the measured and expected rates (see Fig. 4). However, for $T \lesssim 0.12$ the simulation is in the pycnonuclear regime where Eq. (11) no longer applies. Here the measured rate diverges noticeably from Eq. (11), which predicts an unphysical increase in the rate as T decreases. In the pycnonuclear regime existing theories [4,10,11] do not capture the observed behavior of the rate, which is enhanced by up to 10^{10} above the uncorrelated rate ν_0. As we discussed previously, this is because existing theories consider quantum effects that are not relevant to these simulations.

In conclusion, we have presented theory and simulations testing an analogy between the energy equipartition rate in a strongly magnetized ion plasma and the nuclear reaction rate in dense plasmas. In the strong and weak screening regimes the enhancement factor $f(\Gamma)$ for both rates is shown to be identical. Simulations testing the screening enhancement show factor-of-3 agreement with theory for screened nuclear reactions in the range $0.5 \lesssim \Gamma < 5$. As far as we know, this is the first time the theory of screening enhancement has been tested in the strong screening regime $\Gamma > 1$. These results open up the possibility of accurately measuring screening enhancement factors in Penning trap experiments with millions of ions whose parallel and perpendicular temperatures can be manipulated and probed using lasers. In fact, such experiments recently observed rapid parallel temperature increases that may have been caused by enhanced coupling to the relatively hot cyclotron motion of the ions [12]. Using such experiments, or simply larger simulations with lower noise, it should be possible to improve the measurement accuracy to the point where competing theories [10,13] for $f(\Gamma)$ can be tested (these theories differ by a factor of 2 in the range $1 \leq \Gamma \leq 10$). In the pycnonuclear regime we observe large rate enhancements of up to 10^{10} that cannot be explained by current theories, representing a challenge to which we return in future work.

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