

Stability theorem for a single species plasma in a toroidal magnetic configuration

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A stability theorem is developed for a single species plasma that is confined by a purely toroidal magnetic field. A toroidal conductor is assumed to bound the confinement region, and frequencies are ordered so that the cyclotron action and the toroidal action for each particle are good adiabatic invariants. The cross-field motion is described by toroidal-average drift dynamics. In this situation, it is possible to find plasma equilibria for which the energy is a maximum, as compared to all neighboring states that are accessible under general constraints on the dynamics. Since the energy is conserved, such states must be stable to small-amplitude perturbations. This theorem is developed formally using Arnold's method, and examples of stable equilibria are obtained.

I. INTRODUCTION

A recent paper¹ discussed a new stability theorem for a single species plasma column that is confined by a uniform axial magnetic field. The theorem guarantees stability against two-dimensional $\mathbf{E} \times \mathbf{B}$ drift perturbations such as diocotron modes. To understand the theorem, we first note that $\mathbf{E} \times \mathbf{B}$ drift dynamics in a uniform \mathbf{B} field conserves the electrostatic energy and generates an incompressible velocity flow. Thus, a given plasma equilibrium is stable against two-dimensional $\mathbf{E} \times \mathbf{B}$ drift perturbations if the electrostatic energy is a maximum, as compared to neighboring states that are accessible under incompressible flow. In the space of accessible states, the system trajectory must evolve along a contour of constant electrostatic energy, but at a maximum of this quantity such a contour shrinks to a point (the top of a peak), so no change in the state is possible; the state is a stable equilibrium.

There is support for this theorem in recent experiments.^{2,3} In these experiments, a pure electron plasma column was confined by a uniform axial magnetic field in a region of space that was bounded by a conducting cylinder. The cylinder was divided azimuthally into sectors, and azimuthally asymmetric equilibria were produced by holding different sectors at different values of the electric potential. These equilibria were observed to survive stably for long periods (seconds), and the observed cross-sectional shapes were predicted, at least approximately, by maximizing the electrostatic energy. A convincing qualitative feature was that an electron plasma distended toward a negatively biased sector, which is what one expects for a state of maximum electrostatic energy.

The purpose of this paper is to extend the theorem to the case where the plasma is confined in a toroidal magnetic field configuration. Figure 1 shows a schematic diagram of a toroidal trap used to confine an electron plasma in recent experiments.⁴ The confinement region is bounded by a toroidal conductor that is rectangular in cross section, and the magnetic field (i.e., $\mathbf{B} = B_0 R \hat{\theta} / \rho$) is purely toroidal. Here, B_0 and R are constants, and (z, ρ, θ) is the cylindrical coordinate

system that is defined in the figure. A suitable vector potential for this field is $A = \hat{z} A_z(\rho)$, where $A_z(\rho) = B_0 R \ln(R/\rho)$. For a non-neutral plasma, the rotational transform is produced by $\mathbf{E} \times \mathbf{B}$ drifts in the poloidal direction. Details of this configuration, such as the rectangular cross section of the conducting boundary, are not crucial to the arguments that follow, but we do assume that the magnetic field is purely toroidal and that the system is invariant under rotation in θ .

We assume that dynamical frequencies are ordered so that the plasma evolution can be followed by toroidal-average guiding center drift dynamics. To be specific, we assume that $\Omega_c \gg \omega_T \gg \omega_E$ and ω , where Ω_c is the cyclotron frequency, $\omega_T \sim \bar{v} / \rho$ is the characteristic toroidal rotation frequency for particles, $\omega_E \sim \omega_p^2 / \Omega_c$ is the characteristic frequency associated with the cross-field drift dynamics, and ω is the frequency of temporal variations in the electric potential. Here, \bar{v} is the thermal velocity, ω_p is the characteristic plasma frequency, and for the diocotron-like modes for which this analysis is designed, $\omega \sim \omega_E$. Also, we assume that parameters are ordered so that diamagnetic corrections to the magnetic field are negligible, that is, $(d^2 \omega_E \omega_p^2) / (\Omega_c^2 c^2) \sim d^2 \omega_E^2 / c^2 \ll 1$, where d is the characteristic minor radius of the toroidal plasma. This condition is very well satisfied for plasmas of interest.

For a non-neutral plasma, $\mathbf{E} \times \mathbf{B}$ drifts are much larger than curvature and gradient $|\mathbf{B}|$ drifts, so as a first approximation we retain only the $\mathbf{E} \times \mathbf{B}$ drifts and write the toroidal-average guiding center drift Hamiltonian as⁵

$$H = e \phi[z, \rho(p_z)], \quad (1)$$

where $p_z = (e/c) A_z(\rho)$ is the momentum conjugate to z and ϕ is the toroidal-average potential [i.e., $\phi(z, \rho) = (1/2\pi) \oint d\theta \phi(z, \rho, \theta)$]. One can easily check that this Hamiltonian gives the correct $\mathbf{E} \times \mathbf{B}$ drift equations. Also, the reader should note that the charge e carries a sign. In Appendix A, we will include the effect of curvature and gradient $|\mathbf{B}|$ drifts and argue that these small corrections typically do not change the answer to the question of whether or not the plasma is stable.

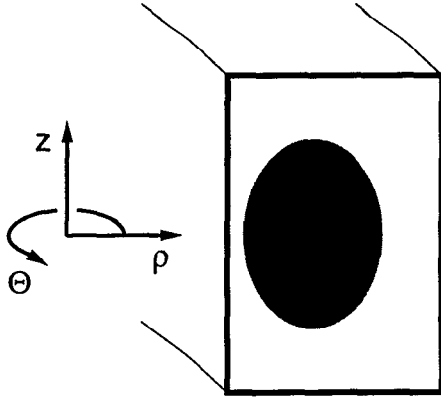


FIG. 1. Schematic diagram of a toroidal confinement device, showing a grounded conducting shell with rectangular cross section and the coordinates used in the text.

The toroidal-average potential satisfies Poisson's equation,

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) \phi = - \frac{2e|e|B_0 R f}{\rho^2 c}, \quad (2)$$

subject to the boundary conditions imposed on ϕ at the wall that surrounds the confinement region. Here, $f=f(z, p_z, t)$ is the toroidal-average particle distribution in (z, p_z) space; it is related to the toroidal-average density through the equation $n 2\pi\rho |dz dp_z| = f |dz dp_z|$.

For times that are short compared to the collisional time scale, the particle distribution evolves according to the equation

$$\frac{\partial f}{\partial t} + [f, H] = 0, \quad (3)$$

where $[f, H]$ is the Poisson bracket.⁶ This equation states that the flow is incompressible in (z, p_z) space. Since an element of area in this space is a constant times an element of magnetic flux (i.e., $|dz dp_z| = |e/c|B dz d\rho$), the equation simply expresses the well-known fact that the number of particles in a flux tube is conserved under $\mathbf{E} \times \mathbf{B}$ drift dynamics.

It is convenient to write the potential as the sum of an external potential and a space charge potential (i.e., $\phi = \phi_e + \phi_s$). The external potential satisfies Laplace's equation and is equal to the potential specified on the boundary wall; the space charge potential satisfies Eq. (2) and vanishes on the boundary wall. From Eqs. (1)–(3) it then follows that the electrostatic energy,

$$W = e \int dz dp_z \left(\phi_e + \frac{\phi_s}{2} \right) f, \quad (4)$$

is constant in time, provided that the potential specified on the boundary is constant in time.

We are now in a position to state the stability theorem for a toroidal trap. An equilibrium state is stable, to small-amplitude perturbations, under toroidal-average drift dynamics if W is a local maximum, as compared to neighboring states that are accessible under incompressible flow in a space where flux is the measure of area [e.g., (z, p_z) space].

The stability theorem is developed formally by using a variational analysis to establish that W is a local maximum subject to the constraint of incompressible flow. We will see that the first-order variation of W vanishes for any state, with a particle distribution of the form $f=f(e\phi)$. This is just the condition that the state is an equilibrium, that is, that $\partial f/\partial t$ vanishes in Eq. (3). For W to be a local maximum, the second-order variation of W must be negative (i.e., $\delta W < 0$ for all allowed δf).

This can be true only when $df/d(e\phi) > 0$ throughout the plasma, so we consider only such equilibria. If $df/d(e\phi)$ were negative near some equipotential contour, then an interchange of two flux tubes of plasma near the contour would lead to an increase in W . An eigenvalue problem can be associated with the variational problem, and this enables us to write the second-order variation of W as

$$\delta W = - \sum_j \lambda_j a_j^2, \quad (5)$$

where the first-order variation in the distribution has been expanded in the eigenfunctions (i.e., $\delta f = \sum_j a_j \delta f_j$) and the λ_j 's are the corresponding eigenvalues. Thus, δW is negative and the plasma is stable if the eigenvalues are all positive, or more simply, if the lowest eigenvalue is positive. The eigenvalue equation depends on $df/d(e\phi)$ and on the geometry of the wall that bounds the confinement region. In fact, we consider two forms of the eigenvalue equation by implementing the constraint of incompressible flow in two different ways. In the more general formulation, each of the eigenfunctions δf_j can be realized through incompressible flow, and the eigenfunctions can be thought of as a set of orthogonal vectors that locally span the space of accessible states.

There are many examples of this kind of stability theorem in the literature, and the review article by Holm *et al.* has an extensive bibliography.⁷ From the reasoning given in the first paragraph, one can see that stability is implied whenever the energy is either a maximum or a minimum, and for most examples in the literature the criterion that the energy be a minimum is established. Formally, this is the easier of the two cases. Nevertheless, both cases were discussed in the pioneering works by Lord Kelvin⁸ and Arnold.⁹

The well-known stability theorem of Davidson and Krall¹⁰ is an example where the criterion that the energy be a minimum was established. This theorem was developed for the case of a long single species column that is confined by a uniform axial magnetic field. The confinement region is assumed to be bounded by a conducting cylinder that is coaxial with the magnetic field. The theorem has the advantage that it applies to general collisionless dynamics, not just to $\mathbf{E} \times \mathbf{B}$ dynamics. On the other hand, the theorem relies on cylindrical symmetry about the direction of the magnetic field, and for the plasmas of interest here, such symmetry is spoiled by the toroidal curvature. We will see explicitly that the stability criterion of Davidson and Krall cannot be satisfied for these plasmas.

Before proceeding to the formal analysis, it is worth noting some effects that can circumvent the new theorem. The basic idea of the theorem is that the plasma cannot change

out of a state of maximum electrostatic energy because there is nowhere to deposit the energy that would be liberated under such a change. Within the context of $\mathbf{E} \times \mathbf{B}$ dynamics, the plasma kinetic energy cannot change. More generally, when magnetic drifts are included (see Appendix A), the kinetic energy is bound up in the adiabatic invariants,

$$\mu = \frac{mv_{\perp}^2}{2B} \quad \text{and} \quad I = \frac{1}{2\pi} \oint d\Theta \rho m v_{\parallel}, \quad (6)$$

and cannot accept the liberated energy. However, if there is some other energy sink in the system, the theorem can fail. For example, if there is finite resistance between sectors of the wall, a negative energy diocotron mode can grow exponentially by depositing the liberated electrostatic energy as heat in the resistor.^{11,12} If a non-neutral electron plasma is contaminated with a substantial number of nonadiabatic impurity ions and the ion motion resonates with a diocotron mode, the mode can grow by dumping electrostatic energy into ion kinetic energy.^{13,14}

In Sec. II, the variational analysis is developed. In Sec. III, sample equilibria are obtained for the toroidal trap shown schematically in Fig. 1, and one or the other of the eigenvalue problems is solved to prove stability. In Appendix A, the analysis is generalized to include the effect of magnetic drifts, and in Appendix B contact is made with previous results obtained for the case of a long column in a uniform magnetic field.¹⁻³ We also show how the current results extend those obtained earlier.

II. VARIATIONAL ANALYSIS

In this section, we adapt Arnold's variational analysis⁹ to the toroidal plasma, as described by Eqs. (1)–(4). Suppose that some equilibrium distribution $f = f(e\phi)$ undergoes the variation δf and that the space charge potential undergoes the variation $\delta\phi_s$. Here, $\delta\phi_s$ and δf are related through Eq. (2), subject to the boundary condition $\delta\phi_s = 0$ on the surrounding conductor. Of course, there is no variation in the external potential (i.e., $\delta\phi_e = 0$). From Eq. (4), it follows that the resulting change in the electrostatic energy W is given by

$$\delta W = e \int dz dp_z \phi \delta f + \frac{e}{2} \int dz dp_z \delta\phi \delta f, \quad (7)$$

where we have used integration by parts and have set $\phi = \phi_e + \phi_s$ and $\delta\phi = \delta\phi_s$. From Eq. (3), it follows that any functional of the form

$$\int dz dp_z K(f) \quad (8)$$

is time independent; this is a constraint associated with the fact that the flow is incompressible. Thus, any variation of δf that can be realized dynamically must be such that

$$0 = \int dz dp_z K'(f) \delta f + \frac{1}{2} \int dz dp_z K''(f) (\delta f)^2, \quad (9)$$

where terms up to second order in δf have been retained. Subtracting this equation from Eq. (7) yields the result

$$\begin{aligned} \delta W = & \int dz dp_z [e\phi - K'(f)] \delta f \\ & + \frac{1}{2} \int dz dp_z [e \delta\phi \delta f - K''(f) (\delta f)^2]. \end{aligned} \quad (10)$$

It is easy to see that the first-order variation vanishes for any equilibrium. By using the relation $f = f(e\phi)$, one can find a function $K(f)$, such that $e\phi = K'(f)$. For this choice the first integral in Eq. (10) vanishes, and the second reduces to the form

$$\delta W = \frac{1}{2} \int dz dp_z \left(e \delta\phi \delta f - \frac{d(e\phi)}{df} (\delta f)^2 \right). \quad (11)$$

To prove stability, one must show that this remaining second-order variation is either positive for all δf or negative for all δf .

To see that the first term in integral (11) is positive for all δf , we rewrite it as

$$\frac{1}{2} \int dz dp_z e \delta\phi \delta f = \int 2\pi\rho d\rho dz \frac{(\nabla\delta\phi)^2}{8\pi}, \quad (12)$$

where use has been made of Eq. (2) and of integration by parts. If $df/d(e\phi)$ were negative everywhere in the plasma, the second term in integral (11) would be positive for all δf , so that δW itself would be positive for all δf . In other words, the electrostatic energy would be a local minimum. This is the stability theorem of Davidson and Krall¹⁰ for the present situation. Unfortunately, the condition $df/d(e\phi) < 0$ cannot be satisfied everywhere in the plasma.

To understand why, consider a small equipotential contour that surrounds the peak value of $f(e\phi)$. The contour is to be chosen so that $f(e\phi)$ decreases as a point passes through the contour from the interior to the exterior. It follows that $\int \nabla f \cdot d\mathbf{S} < 0$, where the integral is over the flux tube that is defined by the magnetic field lines that pass through the contour and $d\mathbf{S}$ is directed outward. From Gauss' law, it follows that

$$0 > \int \nabla f \cdot d\mathbf{S} = \frac{df}{d(e\phi)} \int \nabla(e\phi) \cdot d\mathbf{S} = -4\pi e^2 N \frac{df}{d(e\phi)}, \quad (13)$$

where N is the number of charges within the flux tube. Thus, the criterion of Davidson and Krall⁸ [i.e., $df/d(e\phi) < 0$] is not satisfied at this potential contour.

Of course, the theorem of Davidson and Krall was originally developed for a system with cylindrical symmetry about the direction of the magnetic field. A cylindrically symmetrical equilibrium in such a system is stationary in any rotating frame, so the stability analysis can be carried out in a rapidly rotating frame. It is easy to see that the criterion $df/d(e\phi) < 0$ can be satisfied, where ϕ is the potential in the rotating frame.¹ For example, for a uniform axial magnetic field, $\phi = \phi + \omega B r^2 / 2c$, where $-\omega$ is the rotation frequency. The extra term is associated with the electric field that is induced by rotating through the magnetic field. For sufficiently large ωeB , this extra term makes it possible to satisfy the criterion $df/d(e\phi) < 0$ everywhere in the plasma. Physically, the extra term provides a potential well in which the

plasma can reside in a state of minimum energy. For a toroidal trap and for any trap that lacks symmetry about the direction of the magnetic field, the stability analysis must be carried out in the laboratory frame, because that is the only frame in which the equilibrium is stationary.

We take an approach that is opposite to that of Davidson and Krall; we assume that $df/d(e\phi) > 0$ everywhere in the plasma and then attempt to prove that W is a local maximum, that is, that $\delta W < 0$ for all allowed δf . Note that the condition $df/d(e\phi) > 0$ is natural for a non-neutral plasma, that is, in accord with Poisson's equation. The two terms in integral (11) have opposite sign, and it is necessary to show that the second term is larger in magnitude than the first. This is the reason that the case of maximum energy is more difficult formally than the case of minimum energy.

Since the second term in integral (11) is negative, we need only consider cases where the first term is nonzero. Since the sign of δW is not changed by multiplying δf by a real number, we can limit our consideration to δf that satisfy the normalization condition

$$1 = \int dz dp_z \frac{e \delta\phi \delta f}{2} = \int 2\pi\rho d\rho dz \frac{(\nabla\delta\phi)^2}{8\pi}. \quad (14)$$

This condition defines a manifold of functions $\{\delta f\}$, and δW is bounded from above on the manifold (i.e., $\delta W \leq 1$). We want to find the condition that the maximum value of δW on the manifold is negative. To this end, we consider the variational problem,

$$\delta(\delta W) = \int dz dp_z \left(e(1+\lambda)\delta\phi - \frac{d(e\phi)}{df} \delta f \right) \delta(\delta f), \quad (15)$$

where λ is an undetermined multiplier associated with constraint (14). The variation $\delta(\delta W)$ is zero, if δf satisfies the eigenvalue problem

$$e(1+\lambda)\delta\phi = \frac{d(e\phi)}{df} \delta f, \quad (16)$$

where $\delta\phi$ and δf are related through the variation of Eq. (2) and $\delta\phi = 0$ on the boundary surface. Eliminating δf in favor of $\delta\phi$ yields the single equation

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) \delta\phi = \frac{-2e^2 |e| B_0 R}{\rho^2 c} \frac{df}{d(e\phi)} (1+\lambda) \delta\phi. \quad (17)$$

From Eq. (16), it follows that eigenfunctions for different eigenvalues are orthogonal:

$$0 = (\lambda_i - \lambda_j) \int dz dp_z \delta f_i \delta\phi_j. \quad (18)$$

Degenerate eigenfunctions must be made orthogonal "by hand" in the usual manner. The eigenfunctions then satisfy the orthonormality condition

$$\delta_{ij} = \frac{e}{2} \int dz dp_z \delta f_i \delta\phi_j. \quad (19)$$

Substituting the expansion

$$\delta f = \sum_j a_j \delta f_j, \quad \delta\phi = \sum_j a_j \delta\phi_j \quad (20)$$

into Eq. (11), then yields the result

$$\delta W = - \sum_j \lambda_j a_j^2, \quad (21)$$

where $\sum_j a_j^2 = 1$. The lowest eigenfunction, say δf_1 , yields the maximum value of δW on the manifold. The second eigenfunction yields the maximum value of δW on that subset of the manifold that is orthogonal to δf_1 , and so on. Thus, δW is negative for all δf on the manifold if $\lambda_1 > 0$. If δf is known to be orthogonal to δf_1 , say as a requirement of incompressible flow, then δW is negative on the allowed portion of the manifold if $\lambda_2 > 0$. We will see in the next section that this latter case can arise, in practice.

The difficulty with this approach is that the eigenfunctions δf_j are not necessarily consistent with the constraint of incompressible flow. Some λ_j 's can be negative, even though the plasma is stable. As mentioned, it is sometimes possible to invoke incompressibility *a posteriori* by excluding δf_1 . However, a better and more general approach is to limit the manifold $\{\delta f_j\}$ at the outset to include only functions that are accessible through incompressible flow. To accomplish this we replace (z, p_z) by the action-angle variables (ψ, J) , where¹⁵

$$J = \frac{1}{2\pi} \oint dz p_z [e\phi, z], \quad (22)$$

$$\psi = \frac{\partial}{\partial J} \int_0^z dz' p_z [e\phi(J), z']. \quad (23)$$

Here, the function $e\phi(J)$ is obtained by inverting Eq. (22). This is a canonical transformation, so the flow is incompressible in the (ψ, J) space. Thus, to first order in smallness, δf can be expressed as

$$\delta f = \{ \delta h(\psi, J), f[e\phi(J)] \}, \quad (24)$$

where $\delta h(\psi, J)$ is a generating function for an infinitesimal canonical transformation.¹⁶ Note from Eq. (11) that δf need only be known to first order to obtain δW to second order. Equation (24) implies that δf satisfies the constraint

$$0 = \int d\psi dJ r(J) \delta f(\psi, J), \quad (25)$$

where $r(J)$ is an arbitrary function. Previously, we considered the manifold of functions $\{\delta f\}$ that satisfy normalization condition (14); here, we further restrict the manifold by excluding any functions that do not satisfy constraint (25).

Thus, δW is an extremum on this restricted manifold, provided that

$$e(1+\lambda)\delta\phi = \frac{d(e\phi)}{df} \delta f + r(J), \quad (26)$$

where $r(J)$ is introduced by use of the constraint. From the constraint, it follows that

$$r(J) = \frac{e(1+\lambda)}{2\pi} \int_0^{2\pi} d\psi \delta\phi(\psi, J), \quad (27)$$

so that the eigenfunction equation for $\delta\phi$ takes the form

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) \delta\phi = \frac{-2e^2 |e| B_0 R}{\rho^2 c} \frac{df}{d(e\phi)} (1+\lambda) \times \left(\delta\phi - \frac{1}{2\pi} \int_0^{2\pi} d\psi \delta\phi \right). \quad (28)$$

Alternatively, we can obtain an eigenfunction equation for δf . The Green's function $G(\rho, z | \rho', z')$ is defined as

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) G(z, \rho | z', \rho') = -4\pi \frac{\delta(\rho - \rho') \delta(z - z')}{2\pi\rho}, \quad (29)$$

where $G=0$ on the boundary surface, and Poisson's equation provides the relation

$$\delta\phi(\rho, z) = e \int dz' dp'_z \delta f(z', \rho'_z) G[z, \rho | z', \rho' (p'_z)]. \quad (30)$$

Using $dz dp_z = d\psi dJ$ together with Eqs. (26) and (27) then yields the result

$$e^2(1+\lambda) \frac{df}{d(e\phi)} \left[\left(1 - \frac{1}{2\pi} \int_0^{2\pi} d\psi \right) \times \int d\psi' dJ' G(\psi, J | \psi', J') \delta f(\psi', J') \right] = \delta f(\psi, J). \quad (31)$$

Since ψ is an angle variable, δf and G can be expanded in the Fourier series,

$$\delta f(\psi, J) = \sum_l \delta f_l(J) e^{il\psi},$$

$$G(\psi, J | \psi', J') = \sum_{l, l'} G_{l, l'}(J | J') e^{il\psi - il'\psi'}. \quad (32)$$

Constraint equation (25) implies that $\delta f_l = 0$ for $l=0$, and for $l \neq 0$, eigenfunction equation (31) reduces to

$$2\pi e^2(1+\lambda) \frac{df}{d(e\phi)} \sum_{l'} \int dJ' G_{l, l'}(J | J') \delta f_{l'}(J') = \delta f_l(J). \quad (33)$$

Equations (26) and (25) imply orthogonality condition (18), and with the proper normalization we obtain orthonormality condition (19). Substituting the expansion $\delta f = \sum_j a_j \delta f_j$ into Eq. (11) again yields Eq. (21). The plasma is stable if the lowest eigenvalue is positive.

III. EXAMPLES OF STABLE EQUILIBRIA

In this section, we obtain sample equilibria for a plasma that is confined in the toroidal configuration of Fig. 1, and we

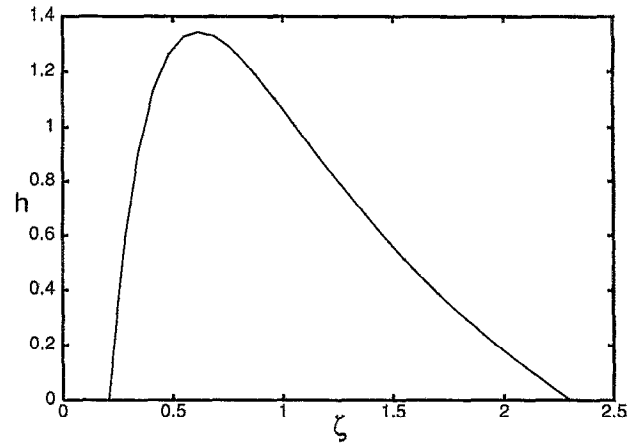


FIG. 2. The radial eigenfunction for the case of a linear function $f(e\phi)$, with other parameters given in the text.

solve one or the other of the eigenvalue problems to prove that the equilibria are local energy maxima. A particular equilibrium is determined by the choice of the functional form $f(e\phi)$, where ϕ is determined self-consistently from Poisson's equation [i.e., Eq. (2)]. For simplicity, we assume here that $\phi=0$ on the conducting boundary (i.e., $\phi_e=0$).

A simple choice for $f(e\phi)$ is the linear relationship

$$f(e\phi) = \left(\frac{2e|e|B_0R}{c} \right)^{-1} \gamma\phi, \quad (34)$$

where γ is a positive constant. This choice has been used previously for similar problems.¹³ Note that our condition $df/d(e\phi) > 0$ is satisfied, since $\gamma > 0$. Substituting into Eq. (2) yields a linear equation for the self-consistent potential,

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + \frac{\gamma}{\rho^2} \right) \phi = 0, \quad (35)$$

that must be satisfied everywhere within the conducting boundary. We want to find a solution for which $\phi=0$ on the boundary and for which $f \propto \phi$ is a non-negative interior to the boundary. Equation (35) is an eigenfunction equation for which γ is the eigenvalue, and we are looking for the lowest eigenfunction. The desired solution can be written as $\phi = C \cos(\pi z/2l) h(\zeta)$, where we have introduced the scaled variable $\zeta = (\pi/2l)\rho$, and the function $h(\zeta)$ satisfies the Bessel-like equation

$$\left(\frac{1}{\zeta} \frac{d}{d\zeta} \zeta \frac{d}{d\zeta} + \frac{\gamma}{\zeta^2} - 1 \right) h(\zeta) = 0. \quad (36)$$

Here γ must be chosen so that $h(\zeta_1) = h(\zeta_2) = 0$ and $h(\zeta) > 0$ for $\zeta_1 < \zeta < \zeta_2$, where $\zeta_1 = (\pi/2l)\rho_1$ and $\zeta_2 = (\pi/2l)\rho_2$. Incidentally, such a solution is possible only for $\gamma > 0$, which is consistent with our condition $df/d(e\phi) > 0$, but not with the condition $df/d(e\phi) < 0$.

As a specific numerical example, we take dimensions from the toroidal apparatus used in the recent experiments⁴ (i.e., $l=15$ cm, $\rho_1=2$ cm, and $\rho_2=22$ cm). A numerical solution of Eq. (35) yields the eigenvalue $\gamma=2.531$ and the solution for $h(\zeta)$ that is plotted in Fig. 2. Figure 3 shows a

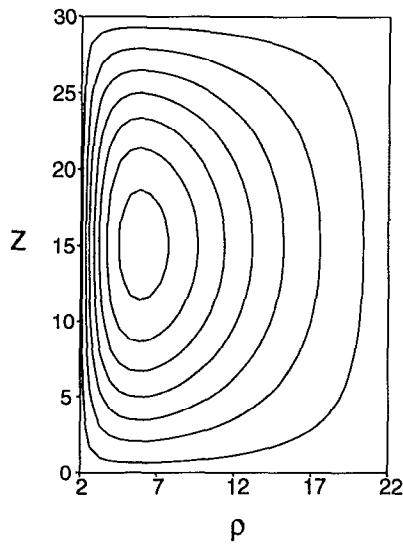


FIG. 3. Potential contours for the case of a linear function $f(e\phi)$.

contour plot for $\phi/C = \cos(\pi z/2l) \cdot h(\pi\rho/2l)$.

Substituting Eq. (34) into Eq. (17) yields the eigenfunction equation

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + (1+\lambda) \frac{\gamma}{\rho^2} \right) \delta\phi = 0, \quad (37)$$

where $\delta\phi=0$ on the conducting boundary. This equation differs from Eq. (36) only in that γ is replaced by $(1+\lambda)\gamma$. Thus, the equilibrium potential itself is the lowest eigenfunction, and it corresponds to the eigenvalue $\lambda=0$. The higher eigenvalues are all positive. For the dimensions used above, the second eigenvalue is $\lambda=0.610$.

The lowest eigenfunction can be excluded from the sum in Eq. (21) on the grounds that δf is orthogonal to the lowest eigenfunction. A first-order constraint of incompressible flow is that

$$0 = \int dz dp_z f \delta f. \quad (38)$$

This corresponds to the choice $K(f) = (f)^2$ in Eq. (8). Since f is proportional to ϕ , and ϕ is proportional to $\delta\phi_1$, Eq. (38) implies that

$$0 = \int dz dp_z \rho \phi_1 \delta f. \quad (39)$$

Note that the essential feature of this discussion is the linear relationship specified in Eq. (34). The rectangular cross section of the conducting boundary simplified the analysis, but was not crucial. One expects that the same result could be obtained for conducting boundaries of various shapes. It is only necessary that the eigenfunction for the lowest eigenvalue have a single peak, and this is typically the case.

Next we consider the waterbag model,

$$f(e\phi) = \left(\frac{2e|e|B_0R}{c} \right)^{-1} \left(\frac{E_1}{e} \right) U(E_2 - e\phi), \quad (40)$$

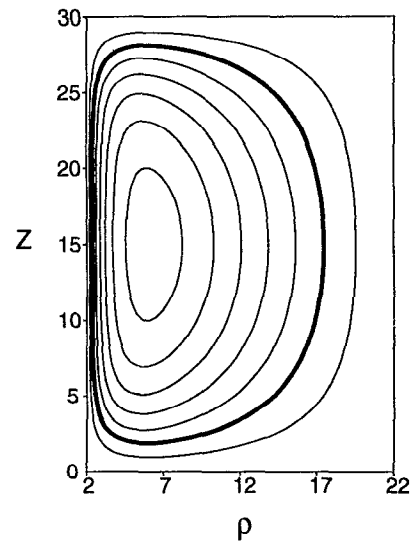


FIG. 4. Potential contours for the case of a "waterbag" equilibrium with $E_2/E_1=1.065$; the plasma boundary is bold.

where E_1 and E_2 are positive constants and $U(x)$ is a step function. The distribution is equal to a positive constant inside the potential contour $e\phi(z,\rho)=E_2$ and is zero outside. The potential is determined self-consistently from Poisson's equation

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{e\phi}{E_1} \right) = -\frac{1}{\rho^2} U \left[\frac{E_2}{E_1} - \left(\frac{e\phi}{E_1} \right) \right], \quad (41)$$

subject to the condition that ϕ vanish on the conducting boundary. Figure 4 shows a numerical solution for the contour levels of $(e\phi/E_1)$ for the case $E_2/E_1=1.065$ and the dimensions used above (i.e., $l=15$ cm, $\rho_1=2$ cm, and $\rho_2=22$ cm). The solid curve shows the special contour $e\phi=E_2$ that marks the outer boundary of the plasma. Figure 5 shows this contour for various values of the ratio E_2/E_1 . Since the boundary of the plasma is an equipotential contour, the tan-

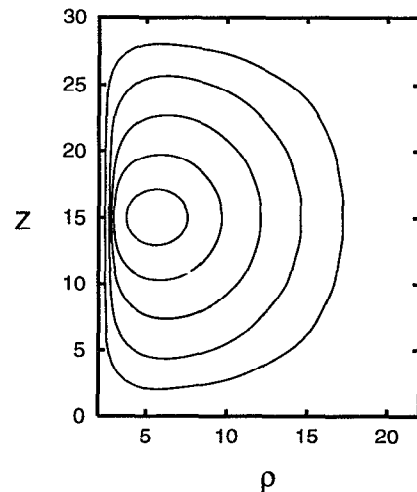


FIG. 5. Locations of the plasma boundary for the "waterbag" equilibria listed in Table I.

TABLE I. Calculated parameters for the “waterbag” equilibria, whose boundaries are shown in Fig. 5: Normalized flux, potential at the interface, and principal eigenvalue for the stability problem.

Φ (cm)	E_2/E_1	λ
2.4	0.532	0.091
8.8	1.127	0.110
18.2	1.395	0.214
29.2	1.381	0.490
44.6	1.065	0.965

gential component of the electric field must vanish on the boundary. Numerically, we found the boundary curve by minimizing the square of the tangential component integrated over the curve, while holding constant the flux linked by the plasma. Table I provides a listing of the normalized flux (i.e., $\Phi \equiv \int dz d\rho/\rho U$) for each value of E_2/E_1 . As Φ increases, E_2/E_1 first increases and then decreases.

To determine the stability of these equilibria, we use eigenfunction equation (33). Since $df/d(e\phi)$ is proportional to $\delta(e\phi_1/E_1 - E_2/E_1)$, $\delta f_l(J)$ must be of the form

$$\delta f_l(J) = \delta(e\phi/E_1 - E_2/E_1) \delta f_l. \quad (42)$$

Substituting this form into Eq. (33) reduces the integral equation to a matrix equation,

$$e^2(1+\lambda) \left(\frac{2e|e|B_0R}{c} \right)^{-1} \left(\frac{E_1}{e} \right) 2\pi \sum_{l'} G_{l,l'}(J_2|J_2) \delta f_{l'} = \delta f_l, \quad (43)$$

where J_2 is defined through the relation $e\phi(J_2) = E_2$. This equation must be complemented by the constraint associated with incompressible flow (i.e., $\delta f_l = 0$ for $l=0$).

To determine the eigenvalues for the matrix equation, one need only know the Green’s function for the field point and the source point both on the boundary contour of the plasma (i.e., for $J=J_2$ and $J'=J_2$). In terms of the variables (ρ, z) , the Green’s function can be written as a Fourier–Bessel expansion. To obtain numerical values for the matrix elements $G_{l,l'}(J_2, J_2)$, the equilibrium interface $J=J_2$ was first discretized into 64 uniformly spaced Hamilton–Jacobi angles. The Fourier–Bessel expansion was evaluated on this grid, retaining 32 terms at each point, and the matrix elements were then obtained by Fourier analyzing with respect to the angle variables. Standard QR routines from the EISPACK library were used to find the eigenvalues of the resulting discretized operator, and the computation was repeated at other resolutions to ensure that the results should be accurate to within a few percent. Table I reports the values of the lowest eigenvalues for each of the equilibria shown in Fig. 5. The eigenvalues are all positive, implying that the equilibria are local energy maxima.

Passage to the limit of a step function distribution greatly simplifies the eigenvalue problem; integral equation (33) reduces to matrix equation (43). However, in the passage to this limit, infinitely many eigenvalues are lost, and it is necessary to ask what happened to these eigenvalues. We will see in Appendix B that these eigenvalues are large and positive for a step function distribution with a slightly

rounded edge, and that they are pushed off to positive infinity in the limit of an exact step function. Consequently, the lost eigenfunctions are not important for the issue of stability. The lowest order eigenfunction for these equilibria is mainly a displacement along the z direction.

Finally, a speculation concerning the waterbag equilibria may point the way to a useful theorem. Our experience in searching numerically for equilibria suggests that the three conditions $f=f(e\phi)$, $df/d(e\phi)>0$, and fixed Φ uniquely determine an equilibrium, at least for the boundary condition used in the above examples. If a theorem guaranteed that the three conditions uniquely determine an equilibrium, then it would follow immediately that the equilibrium is a state of maximum energy. The point is that there must exist at least one maximum energy equilibrium that satisfies the three conditions. To generalize beyond the waterbag model, one would replace the condition fixed Φ with the condition fixed $\Phi(f)$, where $\Phi(f)$ is the flux linked by distribution with value greater than or equal to f . This condition, like the condition of fixed Φ , is a constraint of incompressible flow. We hasten to add that it is quite easy to invent boundary conditions for which the equilibrium is not uniquely determined by the three conditions. Consequently, the theorem, if it exists, is nontrivial, in the sense that it must take into account both the shape of the boundary wall and the potential specified on the wall.

ACKNOWLEDGMENTS

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APPENDIX A: INCLUSION OF THE CURVATURE AND GRADIENT $|\mathbf{B}|$ DRIFTS

In this appendix, we include the effect of the curvature and gradient $|\mathbf{B}|$ drifts. The toroidal-average drift Hamiltonian is given by⁵

$$H = \frac{P_\Theta^2}{2m\rho^2(p_z)} + \frac{\mu B_0 R}{\rho(p_z)} + e\phi[z, \rho(p_z)], \quad (A1)$$

where z and p_z are canonically conjugate and $p_z = (eB_0R/c)\ln(R/\rho)$. The magnetic moment,

$$\mu = \frac{mv_\perp^2}{2B} = \frac{mv_\perp^2 \rho}{2B_0 R} \quad (A2)$$

is an adiabatic invariant, and the toroidal angular momentum,

$$P_\Theta = \frac{1}{2\pi} \oint d\Theta P_\Theta = \frac{1}{2\pi} \oint d\Theta \rho m v_\parallel, \quad (A3)$$

is an adiabatic invariant. Here, v_\parallel and v_\perp are the velocity components parallel and perpendicular to the magnetic field. One can easily check that this Hamiltonian gives the correct guiding center drift equations of motion; the first term gives the curvature drift, the second the gradient $|\mathbf{B}|$ drift, and the third the $\mathbf{E} \times \mathbf{B}$ drift.

We introduce the distribution function $F = F(z, p_z, \mu, P_\Theta, t)$, which evolves according to the equation

$$\frac{\partial F}{\partial t} + [F, H] = 0. \quad (\text{A4})$$

This equation describes an incompressible flow in (z, p_z) space separately for each class of particles specified by values of the pair (μ, P_Θ) . Thus, the functional form,

$$\int d\Gamma K(F, \mu, P_\Theta), \quad (\text{A5})$$

is time independent under the flow for an arbitrary function K , where $d\Gamma = dz dp_z d\mu dP_\Theta$.

The electric potential must be determined self-consistently from Poisson's equation,

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) \phi = - \frac{2e|e|B_0 R}{\rho^2 c} \int F d\mu dP_\Theta, \quad (\text{A6})$$

subject to the boundary condition on the conducting wall. It is convenient to express the potential as the sum of an external potential and a space charge potential (i.e., $\phi = \phi_e + \phi_s$). The total energy (electrostatic plus kinetic) is then given by

$$W = \int d\Gamma F(\Gamma, t) \left(\frac{P_\Theta^2}{2m\rho^2} + \frac{\mu B_0 R}{\rho} + e\phi_e + \frac{e\phi_s}{2} \right), \quad (\text{A7})$$

and Eqs. (A1), (A4), and (A6) imply that W is conserved. Here, we assume that $\partial\phi_e/\partial t = 0$, that is, that the potential specified on the wall is time independent. Note that the electrostatic energy is not conserved separately when the curvature and gradient $|\mathbf{B}|$ drifts are included.

We are now in a position to generalize the stability theorem. A given equilibrium is stable, to small-amplitude perturbations, under toroidal-average drift dynamics if the total energy is a maximum, as compared to neighboring states that are accessible under incompressible flow. The flow must be incompressible in a space for which flux is the measure of area [e.g., (z, p_z) space], and it must be incompressible separately for each class of particles specified by values of the pair (μ, P_Θ) .

To implement this theorem formally, we proceed with a variational analysis as in Sec. II. The variation of W minus the variation of the functional in Eq. (A5) is given by

$$\delta W = \int d\Gamma \left[H - \frac{\partial K}{\partial F} \right] \delta F + \frac{1}{2} \int d\Gamma \left(e \delta\phi - \frac{\partial^2 K}{\partial F^2} \delta F \right) \delta F, \quad (\text{A8})$$

where δF and $\delta\phi$ are related through Poisson's equation. For any equilibrium $F = F(H, \mu, P_\Theta)$, one can find a function $K(F, \mu, P_\Theta)$, such that the bracket in the first integral vanishes. For this choice, Eq. (A8) reduces to

$$\delta W = \frac{1}{2} \int d\Gamma \left(e \delta\phi - \frac{\partial H}{\partial F} \delta F \right) \delta F. \quad (\text{A9})$$

We assume that $\partial F/\partial H > 0$ and try to show that $\delta W < 0$ for all allowed δF . The second term cannot be positive, so

we need only consider variations δF for which the first term is nonzero. The sign of δW is not effected when δF is multiplied by a real number, so we can limit our consideration to variations that satisfy the normalization

$$1 = \frac{e}{2} \int d\Gamma \delta F \delta\phi = \int 2\pi\rho d\rho dz \frac{(\nabla\delta\phi)^2}{8\pi}. \quad (\text{A10})$$

This normalization defines a manifold of functions $\{\delta F\}$.

Here δW is bounded from above on the manifold (i.e., $\delta W \leq 1$), and we want to determine the condition that the maximum value of δW on the manifold is negative. In addition, δW is an extremum on the manifold if

$$e(1 + \lambda) \delta\phi = \frac{\partial H}{\partial F} \delta F, \quad (\text{A11})$$

where λ is a Lagrange multiplier. Substituting for δF in Poisson's equation yields the eigenfunction equation,

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) \delta\phi = - \frac{2e^2|e|B_0 R(1 + \lambda) \delta\phi}{\rho^2 c} \times \int d\mu dP_\Theta \frac{\partial F}{\partial H}. \quad (\text{A12})$$

This is the generalization of eigenfunction equation (17).

Equation (A11) implies that eigenfunctions for different eigenvalues are orthogonal,

$$0 = (\lambda_i - \lambda_j) \int d\Gamma \delta F_i \delta\phi_j, \quad (\text{A13})$$

and we choose the degenerate eigenfunctions to be orthogonal. The eigenfunctions then satisfy the orthonormality condition,

$$\delta_{ij} = \frac{e}{2} \int d\Gamma \delta F_i \delta\phi_j. \quad (\text{A14})$$

Substituting the expansion

$$\delta F = \sum_j a_j \delta F_j, \quad \delta\phi = \sum_j a_j \delta\phi_j$$

into Eq. (A9) then yields the familiar result,

$$\delta W = - \sum_j \lambda_j a_j^2, \quad (\text{A15})$$

where $\sum_j a_j^2 = 1$.

To obtain the generalization of eigenfunction equation (33), we further restrict the manifold $\{\delta F\}$ to allow only functions that are accessible under incompressible flow. To this end, it is useful to introduce the action angle variables,

$$J = \frac{1}{2\pi} \oint dz p_z(z, \mu, P_\Theta, H), \quad (\text{A16})$$

$$\psi = \frac{\partial}{\partial J} \int_0^z dz' P_z[z', \mu, P_\Theta, H(J, \mu, P_\Theta)], \quad (\text{A17})$$

where $p_z(z, \mu, P_\Theta, H)$ is obtained by inverting Eq. (A1), and $H(J, \mu, P_\Theta)$ is obtained by inverting Eq. (A16). These equations define a canonical transformation from (z, p_z) to (ψ, J)

for each value of μ and P_Θ . Thus, the flow is incompressible in (ψ, J) space for each value of μ and P_Θ , and δF can be expressed as

$$\delta F = [\delta h(\psi, J, \mu, P_\Theta), F(J, \mu, P_\Theta)],$$

$$\delta F = \frac{\partial \delta h}{\partial \psi} \frac{\partial F}{\partial J}, \quad (\text{A18})$$

where $\delta h(\psi, J, \mu, P_\Theta)$ is a generating function.¹⁶ Thus, δF satisfies the equation

$$0 = \int d\Gamma r(J, \mu, P_\Theta) \delta F(\psi, J, \mu, P_\Theta), \quad (\text{A19})$$

where $d\Gamma = d\psi dJ d\mu dP_\Theta$ and $r(J, \mu, P_\Theta)$ is an arbitrary function. We further restrict the manifold $\{\delta F\}$ so that constraint (A19) is satisfied.

Thus, δW is an extremum on the manifold if

$$e(1+\lambda) \delta \phi = \frac{\partial H}{\partial F} \delta F + r(J, \mu, P_\Theta), \quad (\text{A20})$$

where λ and $r(J, \mu, P_\Theta)$ are Lagrange multipliers introduced through constraints (A10) and (A19). According to Eq. (A18), $r(J, \mu, P_\Theta)$ must be chosen, so that

$$r(J, \mu, P_\Theta) = e(1+\lambda) \frac{1}{2\pi} \int_0^{2\pi} d\psi \delta \phi(\psi, J, \mu, P_\Theta). \quad (\text{A21})$$

Relating $\delta \phi$ and δF through the Green's function then yields the eigenfunction equation

$$e^2(1+\lambda) \frac{\partial F}{\partial H} \left(1 - \frac{1}{2\pi} \int_0^{2\pi} d\psi \right)$$

$$\times \int d\Gamma' G(\Gamma|\Gamma') \delta F(\Gamma') = \delta F(\Gamma), \quad (\text{A22})$$

where $G(\Gamma|\Gamma') = G[\mathbf{r}(\Gamma')|\mathbf{r}(\Gamma)]$.

Since ψ is an angle variable, δF and G can be expanded in Fourier series,

$$\delta F = \sum_l \delta F_l(J, \mu, P_\Theta) e^{il\theta},$$

$$G = \sum_{l,l'} G_{l,l'}(J, \mu, P_\Theta | J', \mu', P'_\Theta) e^{il\psi - il'\psi'}. \quad (\text{A23})$$

Constraint equation (A19) implies that $\delta F_l = 0$ for $l=0$, and for $l \neq 0$ eigenfunction equation (A22) reduces to

$$2\pi e^2(1+\lambda) \frac{\partial F}{\partial H}(H, \mu, P_\Theta) \sum_{l'} \int dJ' d\mu' dP'_\Theta,$$

$$G_{l,l'}(J, \mu, P_\Theta | J', \mu', P'_\Theta) = \delta F_l(J, \mu, P_\Theta). \quad (\text{A24})$$

This equation is the generalization of Eq. (33). We can again choose the eigenfunctions to satisfy orthonormality condition (A14), and can obtain result (A15).

These results generalize the results in the body of the paper to include the effect of the curvature and gradient $|\mathbf{B}|$ drifts. Formally, these drifts arise from the two kinetic energy terms in the Hamiltonian [see Eq. (A1)]. For a non-

neutral plasma, these two terms tend to be small compared to the electric potential term, or more precisely, the cross-field variation of these terms is small compared to that of the electric potential. Consequently, the curvature and gradient $|\mathbf{B}|$ drifts provide only a small correction to the $\mathbf{E} \times \mathbf{B}$ drift, and one expects that this small correction typically does not change the answer as to whether or not the plasma is stable.

Formally, this is seen most easily by comparing eigenfunction equation (A12) with eigenfunction equation (17). To make the comparison, we first associate a reduced distribution,

$$f(e\bar{\phi}) = \int d\mu dP_\Theta F(e\bar{\phi}, \mu, P_\Theta), \quad (\text{A25})$$

with the distribution $F(H, \mu, P_\Theta)$. On the right-hand side, the kinetic energy terms in H have been set equal to zero. The electric potential for the reduced distribution, $\bar{\phi}$, must be determined self-consistently from Poisson's equation and the boundary condition on the wall. This potential differs only slightly from the self-consistent potential for the distribution $F(H, \mu, P_\Theta)$. We use $F(H, \mu, P_\Theta)$ and $f(e\bar{\phi})$ as the equilibrium distributions in Eqs. (A12) and (17), respectively. Since the kinetic energy terms are small, we can set

$$\int d\mu dP_\Theta \frac{\partial F}{\partial H} = \frac{df}{d(e\bar{\phi})} + \epsilon(z, p_z) \quad (\text{A26})$$

in Eq. (A12), where $\epsilon(z, p_z)$ is small. One can then see that the eigenvalues for the two equations are related by perturbation theory. If the lowest eigenvalue for Eq. (17) is sufficiently far above zero, the lowest eigenvalue for Eq. (A12) must also be positive. A similar comparison can be made between the eigenvalues of Eq. (A24) and Eq. (33).

APPENDIX B: LONG PLASMA COLUMN IN A UNIFORM AXIAL MAGNETIC FIELD

In this appendix, we make contact with and extend results that were obtained earlier for the case of a long plasma column that is confined by a uniform axial magnetic field.¹⁻³ Formally we reduce the toroidal geometry to this geometry by setting $\rho = R + r \cos \theta$ and taking the limit $R \rightarrow \infty$. Here, (r, θ) are polar coordinates in a plane that is orthogonal to the uniform magnetic field $\mathbf{B} = \hat{\Theta} B_0$. In passing to this limit, the quantity $n = (|e| B_0 / 2\pi R c) f$ is identified as the density per unit length along the column. Eigenfunction equation (33) reduces to the form

$$(1+\lambda) \frac{2\pi c|e|}{B_0} \frac{dn}{d(e\bar{\phi})} \sum_{l'} \int dJ' G_{l,l'}(J|J') 2\pi R \delta n_l(J')$$

$$= \delta n_l(J), \quad (\text{B1})$$

where the factor $(2\pi R)$ appears because δn_l refers to density per unit length and the Green's function, as defined in Eq. (29), is to be integrated over density.

Taking the limit $R \rightarrow \infty$ in Eq. (29) yields the equation

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\right) [2\pi R G(r, \theta | r', \theta')] = \frac{-4\pi \delta(r-r') \delta(\theta-\theta')}{r}. \quad (\text{B2})$$

In accord with the previous work,¹⁻³ we assume that the confinement region is bounded by a conducting cylinder that is coaxial with the magnetic field. The wall may be divided azimuthally into sectors, with the different sectors held at different values of the potential ϕ , say, to produce an asymmetric equilibrium. However, $\delta\phi$ must vanish at the wall, so we require that $G(r, \theta | r', \theta')$ vanish at the wall. For a wall at $r=a$, the method of images yields the well-known Green's function,¹⁷

$$2\pi R G(r, \theta | r', \theta') = \ln \left| \frac{r^2 r'^2 / a^2 + a^2 - 2rr' \cos(\theta - \theta')}{r^2 + r'^2 - 2rr' \cos(\theta - \theta')} \right|. \quad (\text{B3})$$

This function must be rewritten in terms of the action-angle variables J and ψ , as defined for the equilibrium under consideration, and the Fourier components,

$$(2\pi R) G_{l,l'}(J|J') = \int_0^{2\pi} \frac{d\psi}{2\pi} \int_0^{2\pi} \frac{d\psi'}{2\pi} e^{i l \psi - i l' \psi'} 2\pi R G \quad (\text{B4})$$

must be substituted into Eq. (B1).

There is no assumption here that the equilibrium is nearly cylindrically symmetrical. The previous work argues stability by continuity from a cylindrically symmetrical equilibrium and in that sense is limited to weak asymmetry.¹⁻³ The present formulation is an extension to allow for large asymmetry.

To make contact with the previous work, let us consider the case of a cylindrically symmetrical equilibrium. For this case, the action-angle variables are simply $J = (eB_0/2c)r^2$ and $\psi = \theta$. This follows from Eqs. (22) and (23) by taking the limit $R \rightarrow \infty$ and integrating around a contour of constant r . Alternatively, one can simply note that $dJ d\psi = (eB_0/c)r dr d\theta$ is e/c times the element of magnetic flux (so that J and ψ are canonically conjugate) and that J is constant along an equipotential contour. Since $\psi = \theta$ and $\psi' = \theta'$, the required Fourier components may be obtained directly from Eq. (B3),

$$(2\pi R) G_{l,l'}(r|r') = \frac{\delta_{l,l'}}{l} \left[\left(\frac{r_<}{r_>}\right)^l - \left(\frac{rr'}{a^2}\right)^l \right], \quad (\text{B5})$$

where $r_<$ is the smaller of r and r' and $r_>$ is the larger. Since $G_{l,l'}$ is diagonal in l and l' , the δn_l for different l decouple in Eq. (B1), which is what one expects for a cylindrically symmetrical equilibrium.

For the simple case of the step function equilibrium, $n_0(r) = n_0 U(r_0 - r)$, Eq. (B1) reduces to the form

$$-(1+\lambda) \frac{2\pi |e|c}{B_0} \frac{n_0 \delta(r-r_0)}{d(e\phi_0)/dr_0} \int dr' \left| \frac{dJ'}{dr'} \right| \times (2\pi R) G_{l,l}(r|r') \delta n_l(r') = \delta n_l(r). \quad (\text{B6})$$

The solution is given trivially by

$$\delta n_l(r) = \delta n_l \delta(r-r_0), \quad (\text{B7})$$

where

$$(1+\lambda) \left(\frac{-2\pi |e| n_0 c}{B_0 d(e\phi_0)/dr_0} \left| \frac{dJ}{dr_0} \right| \right) (2\pi R) G_{l,l}(r_0|r_0) = 1. \quad (\text{B8})$$

The quantity in large parentheses is unity, so we obtain

$$\lambda = -1 + l/[1 - (r_0/a)^{2l}] > 0. \quad (\text{B9})$$

This same inequality was obtained earlier by directly calculating the energy change associated with sinusoidal ripples on the circumference of the step function density distribution.³

Since the eigenvalues are all separated from zero by a finite amount, one can argue by continuity that a weakly asymmetric equilibrium $n(r, \theta) = n_0 U[e\phi_0(r, \theta) - E]$ also is stable. The θ dependence in $\phi_0(r, \theta)$ is produced by an asymmetric boundary condition at $r=a$. For example, if the boundary condition is $\phi_0(a, \theta) = \epsilon \cos \theta$, where ϵ is small, one expects the first-order correction to the Green's function to be of the form

$$2\pi R G_{l,l'}^{(1)}(J|J') \approx \delta_{l,l'} \pm 1 O(\epsilon). \quad (\text{B10})$$

By perturbation theory, it then follows that the corrections to the eigenvalues enter as $O(\epsilon^2)$.

In passing to the limit of a step function equilibrium, infinitely many eigenvalues are lost. To understand this point, it is useful to consider the differential form of the eigenvalue problem. In the limit $R \rightarrow \infty$, Eq. (28) takes the form

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{l^2}{r^2}\right) \delta \phi_l = -4\pi e^2 (1+\lambda) \times \frac{(dn/dr)}{[d(e\phi)/dr]} \delta \phi_l, \quad (\text{B11})$$

where we have assumed that the equilibrium is cylindrically symmetrical and have set $\delta \phi(r, \theta) = \delta \phi_l(r) \exp(il\theta)$. The constraint of incompressible flow excludes the $l=0$ modes. For the step function distribution, $n_0(r) = n_0 U(r_0 - r)$, Eq. (B11) reduces to the form

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{l^2}{r^2}\right) \delta \phi_l = \frac{-2(1+\lambda)}{r_0} \delta(r-r_0) \delta \phi_l. \quad (\text{B12})$$

We must require that $\delta \phi_l(a) = 0$, that $\delta \phi_l(r)$ be regular at $r=0$, and that $\delta \phi_l(r)$ be continuous at $r=r_0$. For each l , there is a single solution,

$$\delta \phi_l(r) = \begin{cases} r^l, & \text{for } r < r_0, \\ \left[\frac{(r/a)^l - (a/r)^l}{[(r_0/a)^l - (a/r_0)^l]} \right] a^l, & \text{for } r_0 < r < a. \end{cases} \quad (\text{B13})$$

This solution has a discontinuous derivative at $r=r_0$, and integrating both sides of the equation across the point $r=r_0$ yields the expression for λ given in Eq. (B9). From Sturm-Liouville theory, one expects that Eq. (B11) has infinitely many eigenvalues when $dn/d(e\phi)$ is smooth, but all except one of these is lost in passage to the step function limit.

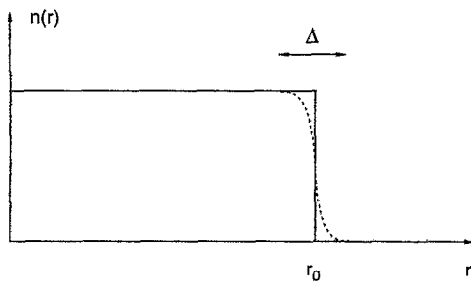


FIG. 6. Plasma density versus radius.

To find the missing eigenvalues, we consider a step function with a rounded edge of width Δ , where $\Delta \ll r_0$ (see Fig. 6). In Eq. (B12), the delta function is replaced by a smooth function of width Δ and of height $1/\Delta$. Eigenfunction (B13) is relatively unaffected by this change. According to Sturm–Liouville theory, this eigenfunction must be the lowest eigenfunction, since it has no zero between $r=0$ and $r=a$. The second eigenfunction must have one zero, and the third must have two zeros, etc. Since all these eigenfunctions must be of the form r^l to the left of the edge region and of the form $(r/a)^l - (a/r)^l$ to the right, the zeros must occur within the edge region. For an eigenfunction with n zeros, we set $\partial^2 \delta\phi_l / \partial r^2 \sim n^2 \Delta^2$ and $\delta(r-r_0) \sim 1/\Delta$ in Eq. (B12) to find the eigenvalue $\lambda_n \sim n^2/\Delta$. Thus, all these higher eigenvalues are pushed off to positive infinity, as we pass to the limit of

a step function distribution (i.e., $\Delta \rightarrow 0$). For a distribution with a smooth but reasonably sharp edge, these eigenfunctions are large and positive, and are not important for the issue of stability.

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