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Collisional damping of plasma waves on a pure electron plasma column

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The collisional damping of electron plasma waves (or Trivelpiece–Gould waves) on a pure electron plasma column is discussed. The damping in a pure electron plasma differs from that in a neutral plasma, since there are no ions to provide collisional drag. A dispersion relation for the complex wave frequency is derived from Poisson's equation and the drift-kinetic equation with the Dougherty collision operator—a Fokker–Planck operator that conserves particle number, momentum, and energy. For large phase velocity, where Landau damping is negligible, the dispersion relation yields the complex frequency $\omega=(k_z\omega_p/k)[1+(3/2)(k\lambda_D)^2(1+10i\alpha/9)(1+2i\alpha)^{-1}]$, where ω_p is the plasma frequency, k_z is the axial wavenumber, k is the total wavenumber, λ_D is the Debye length, ν is the collision frequency, and $\alpha\equiv\nu k/\omega_p k_z$. This expression spans from the weakly collisional regime ($\alpha\ll 1$) to the moderately collisional regime ($\alpha\sim 1$) and in the weakly collisional limit yields a damping rate which is smaller than that for a neutral plasma by the factor $k^2\lambda_D^2\ll 1$. In the strongly collisional limit ($\alpha\gg 1$), the damping is enhanced by long-range interactions that are not present in the kinetic theory (which assumes pointlike interactions); the effect of these long-range collisions on the damping is discussed. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2807220]

I. INTRODUCTION

This paper discusses the collisional damping of electron plasma waves that propagate on a pure electron plasma column. We have in mind a pure electron plasma (non-neutral plasma) in a Penning–Malmberg trap configuration.[1] As we will see, the collisional damping of electron plasma waves in a pure electron plasma is fundamentally different from that in a neutral plasma.

Theory on the collisional damping of electron plasma waves in a neutral plasma dates back to the pioneering work of Lenard and Bernstein and extends into recent literature.[2–5] Using an approximate Fokker–Planck collision operator, now called the Lenard–Bernstein (LB) collision operator, these authors solved the linearized Boltzmann and Poisson equations to obtain a dispersion relation for the complex wave frequency, ω. The dispersion relation admits a discrete infinity of roots, the least damped of which corresponds to the Landau (or Bohm–Gross) root of collisionless theory.[6] LB focused on the least damped root, finding the collisional damping decrement $\text{Im}(\omega)=-\nu/2$, where ν is a collision frequency. (Actually, the $\frac{1}{2}$ was omitted in the final step of the LB analysis, and the omission was corrected only recently.[4]) Recent work also showed that there is a complete set of kinetic eigenfunctions corresponding to the discrete infinity of roots, and these eigenfunctions replace the Van Kampen eigenfunctions of collisionless theory.[3,5,7]

The LB collision operator conserves particle number but not momentum and energy. Failure of the electron collision operator to conserve momentum and energy is acceptable for a neutral plasma, since momentum and energy can be transferred from the electrons to the ions by collisions. Indeed, the damping mechanism for a plasma wave involves just such a transfer of momentum (and, to a lesser extent, energy). The oscillating electrons experience a collisional transfer of momentum (and energy) to the relatively immobile ions, and this transfer (collisional drag) damps the wave. Although not mentioned explicitly in the LB analysis, ions play a crucial implicit role in the damping mechanism.

In contrast, the above mechanism is not available for the damping of plasma waves in a pure electron plasma, since there are no ions to provide the collisional drag. In a pure electron plasma, the damping results from the collisional interaction of electrons, so conservation of electron momentum and energy must be respected. In other words, for the case of a pure electron plasma, failure of the LB operator to conserve momentum and energy is a fatal flaw. Dougherty introduced a modified LB operator that conserves momentum and energy as well as particle number, and we will use this operator.[8–11]

The advantage of the LB and Dougherty operators is that they are analytically tractable. For example, the Hermite polynomials are a complete set of eigenfunctions of the 1D LB operator, and these polynomials are a convenient basis set for expansion of the velocity distribution when using this operator.[3] Here we use an analogous set of orthogonal functions as a basis for expansion of the velocity distribution.

In a Penning–Malmberg trap configuration, the plasma column is immersed in a large axial magnetic field. In the collisionless theory of electron plasma waves on such a column, the electron dynamics are described by the drift-kinetic equation. To study the effect of collisions on the waves, we append the Dougherty collision operator to the drift-kinetic equation, referring to the result as the Dougherty kinetic equation. To facilitate analytic solution, the electron density is taken to be constant inside the plasma column. The linearized Dougherty kinetic equation and Poisson's equation then yield a dispersion relation for the complex wave frequency. Like the LB dispersion relation, this dispersion relation admits a discrete infinity of roots for each wavenumber. We focus on the least damped root, which again corresponds to the Landau root of collisionless theory. For simplicity, we limit the discussion to waves with azimuthal mode number $m_\theta=0$. For large phase velocity [i.e., $\text{Re}(\omega)/k_z \gg v_{\text{th}}$, where v_{th} is the thermal velocity], Landau damping is negligible, and the least damped root of the dispersion equation is given by the simple approximate expression

$$\omega = \frac{k_z\omega_p}{k}\left[1+\frac{3}{2}k^2\lambda_D^2\left(\frac{1+10i\alpha/9}{1+2i\alpha}\right)\right], \tag{1}$$

where ω_p is the plasma frequency, k_z and $k_\perp$ are the wavenumbers along and transverse to the magnetic field, $k=\sqrt{k_z^2+k_\perp^2}$ is the total wavenumber, λ_D is the Debye length, ν is the collision frequency, and $\alpha \equiv \nu(k_z\omega_p/k)^{-1}$ is a parameter characterizing the strength of collisionality. For smaller phase velocity, the dispersion relation must be solved numerically, and Landau damping is recovered in the limit $\alpha \to 0$.

For weak collisionality (i.e., $\alpha \ll 1$), Eq. (1) reduces to the result

$$\text{Re}(\omega) \cong \frac{k_z\omega_p}{k}\left[1+\frac{3}{2}k^2\lambda_D^2\right], \tag{2}$$

$$\text{Im}(\omega) \cong -\tfrac{4}{3}\nu k^2\lambda_D^2. \tag{3}$$

Equation (2) is the well-known result from collisionless theory for the frequency of an electron plasma wave—or, more precisely, a Trivelpiece–Gould (TG) wave—in a pure electron plasma column.[12–14] Equation (3) gives the collisional damping rate of the wave. The damping rate in Eq. (3) is reduced from the damping rate for a plasma wave in a neutral plasma by the small factor $k^2\lambda_D^2 \ll 1$. This reduction is a reminder that the dominant damping mechanism in a neutral plasma is not available in a pure electron plasma.

Note from Eq. (1) that the ordering $k_z/k_\perp \ll 1$ implies that $\text{Re}(\omega) \ll \omega_p$. This is the typical wavenumber ordering for plasma wave experiments on a long column, and we assume this ordering here. In fact, this ordering is implicit in our use of a Fokker–Planck collision operator, since the derivation of such an operator requires the Bogoliubov ansatz,[15] that is, that $|\omega| \ll \omega_p$.

A weakly damped solution to the dispersion equation exists even in the limit of strong collisionality [i.e., $1 \ll \alpha \cong \nu/\text{Re}(\omega)$]. In this limit, Eq. (1) reduces to the result

$$\text{Re}(\omega) \cong \frac{k_z\omega_p}{k}\left(1+\frac{5}{6}k^2\lambda_D^2\right), \tag{4}$$

$$\text{Im}(\omega) \cong -\frac{1}{3}\left(\frac{v_{\text{th}}^2}{\nu}\right)k_z^2. \tag{5}$$

Here, we implicitly assume that the plasma is weakly correlated (i.e., $\nu \ll \omega_p$) even though the wave dynamics is strongly collisional, and this is possible since $\text{Re}(\omega) \ll \omega_p$.

Note that the Bohm–Gross correction to the real part of the frequency, that is the term $(3/2)k^2\lambda_D^2$ in the bracket of Eq. (2), has been replaced by $(5/6)k^2\lambda_D^2$ in Eq. (4). This change, which emerges automatically from the kinetic theory, is easy to understand from an elementary treatment that uses the adiabatic law of electron compression.[16] The numerical coefficient of the Bohm–Gross term is $(f+2)/(2f)$, where f is the number of degrees of freedom that share the compressional energy. For weak collisionality, there is negligible equipartition, so $f=1$ and $(f+2)/(2f)=3/2$. Whereas, for strong collisionality, there is nearly complete equipartition, so $f=3$ and $(f+2)/(2f)=5/6$.

For large phase velocity, the condition for strongly collisional wave dynamics is equivalent to the conditions for applicability of fluid theory (i.e., $\nu \gg |\omega|$ and $k_z v_{\text{th}}/\nu \ll 1$). Since the Dougherty collision operator conserves particle number, momentum, and energy, results from the kinetic theory should match onto fluid theory in the strongly collisional limit. Indeed, Eqs. (4) and (5) are recovered by fluid theory, provided that one uses the expressions for viscosity and thermal diffusivity that are predicted by the Dougherty operator.[11] From the fluid treatment, we find that the damping rate (5) results entirely from viscous momentum transport along the magnetic field; heat conduction contributes a higher-order correction to the damping.

Thus far, transport across the magnetic field has not been mentioned. The drift-kinetic equation only includes transport along the magnetic field. A naïve estimate suggests that the correction to the damping from cross-field transport is negligible. The classical transport coefficient for cross-field viscosity, $\zeta_\perp$, is of the order νr_c^2, where $r_c=v_{\text{th}}/\Omega_c$ is the cyclotron radius. From fluid theory, the corresponding correction to the damping decrement is $\text{Im}(\Delta\omega) \sim -\zeta_\perp k_\perp^2 \sim -\nu r_c^2 k_\perp^2$, which is negligible since r_c is very small.

However, recent theory and experiment have shown that the classical coefficients grossly underestimate cross-field transport in the parameter regime where $r_c \ll \lambda_D$, and this is the typical operating regime for non-neutral plasmas.[17,18] The new theory predicts larger transport coefficients (e.g., $\zeta_\perp \sim \nu\lambda_D^2$), and fluid theory then predicts a larger correction to the damping decrement, $\text{Im}(\Delta\omega) \sim -\nu\lambda_D^2 k_\perp^2$. As can be seen by comparison to Eqs. (3) and (5), this correction is not always negligible—cross-field transport can contribute to the wave damping.

The enhanced cross-field transport is associated with long-range interactions (with impact parameter of order λ_D) and is not captured by any kinetic theory that uses a Fokker–Planck collision operator.[18] A collision operator that describes interactions between particles on different field lines is necessary. Here, we simply use a fluid treatment that employs the new cross-field transport coefficients obtained previously. For the waves of interest, the transverse wavelength is large compared to the scale length for the transport (i.e., $k_\perp\lambda_D \ll 1$), so the cross-field transport is well described by local transport coefficients even in the limit of weak collisionality.

II. ELECTROSTATIC MODES IN PENNING-TRAP GEOMETRY

Following previous theoretical studies of Trivelpiece–Gould modes, we make several simplifying assumptions in order that Poisson's equation may be solved analytically.[13,14] In this section, these assumptions and the consequent simpli-

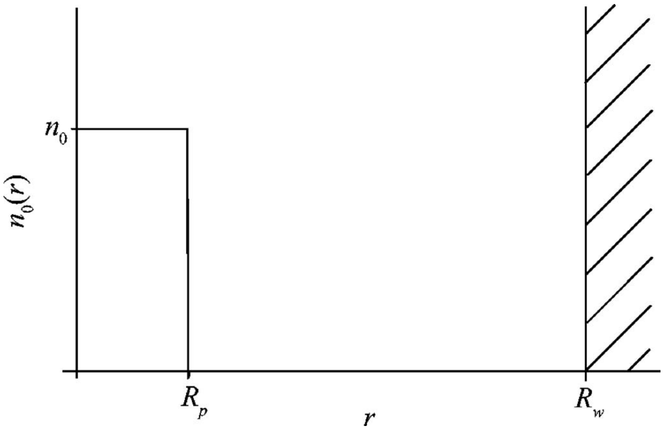

FIG. 1. Density of the unperturbed plasma column plotted as a function of radial coordinate r.

fication of Poisson's equation are discussed; the plasma dynamics will be addressed in the following section. We will consider a cylindrical Penning-trap and thus employ a cylindrical coordinate system, (r, θ, z), the z-axis of which coincides with the axis of the trap.

First, it is assumed that the ends of the plasma are flat and rigid, in the sense that each particle undergoes specular reflection in z at these ends. Under this assumption, one may equivalently consider an infinitely long plasma column which is periodic with period $2L_p$, where L_p is the length of the plasma. This periodic plasma admits axial wavenumbers given by

$$k_z = n\pi/L_p, \tag{6}$$

where n takes on nonzero integer values. In reality, of course, the ends of the plasma are rounded and the potential at these ends is not perfectly hard; as a result, the decoupling of the Fourier modes is not perfect. These effects are of order R_p/L_p, so the validity of our Fourier analysis requires that $R_p \ll L_p$.

In addition, it is assumed that the density of the unperturbed plasma is uniform, that is,

$$n_0(r) = \begin{cases} n_0 & \text{for } r < R_p \\ 0 & \text{for } R_p < r < R_w, \end{cases} \tag{7}$$

where R_p is the radius of the plasma and R_w is the radius of the conducting wall of the trap (see Fig. 1).

This second assumption allows us to look for eigenmodes of the form

$$\delta\varphi = \delta\hat{\varphi} J_0(k_\perp r) e^{i(k_z z - \omega t)}, \quad \delta f = \delta\hat{f} J_0(k_\perp r) e^{i(k_z z - \omega t)}, \tag{8}$$

inside the plasma (i.e., for $r < R_p$) and

$$\delta\varphi = \delta\hat{\varphi}[A I_0(k_z r) + B K_0(k_z r)] e^{i(k_z z - \omega t)}, \quad \delta f = 0, \tag{9}$$

outside the plasma (for $r > R_p$). Here $J_0(x)$ is a Bessel function of the first kind, $I_0(x)$ and $K_0(x)$ are modified Bessel functions of the first and second kinds, and A and B are constants specified by the requirements that the potential and the electric field be continuous across the radial boundary of the plasma column. The requirement that the potential vanish at the conducting wall imposes on the radial wavenumber, $k_\perp$, the well-known constraint[13,14]

$$k_z R_p \frac{I_0'(k_z R_p) K_0(k_z R_w) - K_0'(k_z R_p) I_0(k_z R_w)}{I_0(k_z R_p) K_0(k_z R_w) - K_0(k_z R_p) I_0(k_z R_w)} - k_\perp R_p \frac{J_0'(k_\perp R_p)}{J_0(k_\perp R_p)} = 0. \tag{10}$$

For each axial wavenumber k_z, this equation admits an infinite sequence of solutions for $k_\perp$, each corresponding to a different radial eigenmode.

Inside the plasma, for a perturbation of the above form, Poisson's equation reduces to

$$-k^2 \delta\hat{\varphi} = 4\pi e \int d\vec{v}\, \delta\hat{f}. \tag{11}$$

Outside the plasma, the perturbation satisfies Laplace's equation identically.

III. THE DOUGHERTY KINETIC EQUATION

The wavenumbers and frequencies of the waves under consideration are sufficiently small that the dynamics may be described using the drift approximation. In other words, we take f to be the distribution of guiding centers with parallel velocity v_z and cyclotron invariant $I_c = m v_\perp^2/2B$,

$$f = f(r, z, v_z, v_\perp^2). \tag{12}$$

The evolution of this distribution is governed by the drift-kinetic equation

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} + \frac{c\hat{z} \times \vec{\nabla}\varphi}{B} \cdot \vec{\nabla} f + \frac{e}{m} \frac{\partial \varphi}{\partial z} \frac{\partial f}{\partial v_z} = C(f). \tag{13}$$

We will assume that the effect of collisions on the distribution is given by the Dougherty collision operator, which in the drift approximation is given by

$$C_D(f) = \nu(n,T) \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} v_\perp \left[\frac{T[f]}{m} \frac{\partial f}{\partial v_\perp} + v_\perp f \right] + \nu(n,T) \frac{\partial}{\partial v_z} \left[\frac{T[f]}{m} \frac{\partial f}{\partial v_z} + (v_z - V_z[f]) f \right], \tag{14}$$

where $\nu(n,T)$ is a collision frequency and n, V_z, and T are given by the functionals

$$n[f] = \int dv_z dv_\perp 2\pi v_\perp f, \tag{15}$$

$$V_z[f] = \frac{1}{n} \int dv_z dv_\perp 2\pi v_\perp v_z f, \tag{16}$$

$$T[f] = \frac{1}{3n} \int dv_z dv_\perp 2\pi v_\perp [v_\perp^2 + (v_z - V_z[f])^2] f. \tag{17}$$

A comparison with the true Fokker–Planck operator suggests an approximate expression for the collision frequency,

$$\nu(n,T) \cong e^4 n m^{-1/2} T^{-3/2} \ln(r_c/b), \tag{18}$$

where $b = e^2/T$ is the classical distance of the closest approach.[2]

A steady-state solution to Eq. (13) is given by

$$f = f_0(r,v_z,v_\perp^2) = \frac{n_0(r)}{(2\pi T_0/m)^{3/2}} e^{-m(v_z^2+v_\perp^2)/2T_0} \tag{19}$$

and

$$\varphi = \varphi_0(r), \tag{20}$$

where $\varphi_0(r)$ is determined from $n_0(r)$ via Poisson's equation and $n_0(r)$ is given by Eq. (7). We consider a perturbation to this steady state of the form

$$f(r,z,v_z,v_\perp^2,t) = f_0(r,v_z,v_\perp^2) + \delta f(r,z,v_z,v_\perp^2,t), \tag{21}$$

$$\varphi(r,z,t) = \varphi_0(r) + \delta\varphi(r,z,t), \tag{22}$$

where δf and $\delta\varphi$ are assumed small and have the space and time dependence specified by Eqs. (8) and (9).

Substituting Eqs. (21) and (22) into Eq. (13), using Eq. (11) to eliminate $\delta\varphi$, and neglecting nonlinear terms, we obtain the linearized Dougherty kinetic equation

$$i\omega\delta\hat{f} = ik_z v_z \delta\hat{f} - C_D^{(1)}(\delta\hat{f}) + \frac{ik_z v_z f_0}{k^2\lambda_D^2 n_0}\int d\vec{v}\,\delta\hat{f}, \tag{23}$$

where λ_D is the Debye length in the unperturbed plasma and $C_D^{(1)}$ is the linearized Dougherty operator. This linearized operator takes a relatively convenient form when the perturbation is expressed as $\delta\hat{f}=f_0\phi$ and the thermal velocity, $v_{\rm th}=T_0/m$, and scaled velocity coordinates, $u_z=v_z/v_{\rm th}$ and $u_\perp=v_\perp/v_{\rm th}$, are introduced,

$$\begin{aligned} C_D^{(1)}(f_0\phi) = f_0\nu_0\Bigg[&\frac{1}{u_\perp}\frac{\partial}{\partial u_\perp}u_\perp\frac{\partial\phi}{\partial u_\perp} - u_\perp\frac{\partial\phi}{\partial u_\perp} + \frac{\partial^2\phi}{\partial u_z^2} - u_z\frac{\partial\phi}{\partial u_z} \\ &+ \frac{\delta T}{T_0}(u_\perp^2 - 2 + u_z^2 - 1) + \frac{\delta V_z}{v_{\rm th}}u_z\Bigg] \\ \equiv{}& \nu_0 f_0\chi(\phi), \end{aligned} \tag{24}$$

where

$$\delta V_z = n_0^{-1}v_{\rm th}^4\int du_z du_\perp 2\pi u_\perp u_z\delta f, \tag{25}$$

$$\delta T = (3n_0)^{-1}mv_{\rm th}^5\int du_z du_\perp 2\pi u_\perp[(u_z^2+u_\perp^2)-3]\delta f, \tag{26}$$

$$\nu_0 = \nu(n_0,T_0). \tag{27}$$

(Hereafter we omit the subscript on ν_0.) Equation (23) constitutes an eigenvalue problem; that is, $\delta\hat{f}$ is an eigenfunction with eigenvalue ω.

IV. DISPERSION RELATION

From Eq. (23) we can obtain a dispersion equation which relates the complex frequency, ω, to the axial and radial wavenumbers and the collisionality of the plasma. To this end, we will employ the complete set of orthogonal functions,

$$\phi_{mn} \equiv \frac{1}{\sqrt{m!}}\mathrm{He}_m(u_z)L_n(u_\perp^2/2), \tag{28}$$

where $\mathrm{He}_m(x)$ is the mth modified Hermite polynomial, $L_n(x)$ is the nth Laguerre polynomial, and m and n take on non-negative integer values. These functions satisfy the orthogonality relation

$$\begin{aligned}(\phi_{n_\perp n_z},\phi_{m_\perp m_z}) &\equiv (2\pi)^{-1/2}\int du_z du_\perp u_\perp \phi_{n_\perp n_z}\phi_{m_\perp m_z}e^{-u^2/2} \\ &= \delta_{n_\perp m_\perp}\delta_{n_z m_z}.\end{aligned} \tag{29}$$

The right-hand side of Eq. (23) takes a particularly simple form in this basis.

In a recent paper, a complete set of eigenfunctions of the linearized Dougherty operator was found, and it might seem that these eigenfunctions would constitute the most convenient basis for the problem at hand.[11] However, while the collision operator is diagonal in the basis of eigenfunctions, the streaming term takes a more complicated form in this basis than it does in the basis given by Eq. (28). As a result, the algebra required to obtain the desired dispersion relation is slightly more involved if the basis of eigenfunctions of χ is used.

We express the eigenfunction $\delta\hat{f}$ in the basis (28) as

$$\delta\hat{f}(u_z,u_\perp^2) = f_0\sum_{m,n=0}^{\infty} a_{mn}\phi_{mn}(u_z,u_\perp^2). \tag{30}$$

Substituting this series expansion into Eq. (23) and exploiting the orthogonality relation (29), one obtains an infinite-dimensional matrix equation of the form

$$\begin{aligned}\Omega a_{mn} = &\sum_{m',n'=0}^{\infty}(\phi_{mn},u_z\phi_{m'n'})a_{m'n'} \\ &+ i\mu\sum_{m',n'=0}^{\infty}(\phi_{mn},\chi\phi_{m'n'})a_{m'n'} + \frac{\delta_{1,m}\delta_{0,n}}{k^2\lambda_D^2}a_{00}\end{aligned} \tag{31}$$

for the coefficients a_{mn} and corresponding eigenvalue Ω; here we have introduced the scaled wave frequency $\Omega\equiv\omega/k_z v_{\rm th}$ and collision frequency $\mu\equiv\nu/k_z v_{\rm th}$, and the parentheses denote the inner product defined by Eq. (29). The first term on the right-hand side of Eq. (31) is given by

$$(\phi_{mn},u_z\phi_{m'n'}) = \delta_{m-1,m'}\delta_{n,n'}\sqrt{m} + \delta_{m,m'-1}\delta_{n,n'}\sqrt{m'}, \tag{32}$$

while the second term is given by

$$\chi(\phi_{10}) = 0, \tag{33}$$

$$\chi(\phi_{20}) = -\frac{4}{3}\phi_{20} - \frac{2\sqrt{2}}{3}\phi_{01}, \tag{34}$$

$$\chi(\phi_{01}) = -\frac{2\sqrt{2}}{3}\phi_{20} - \frac{2}{3}\phi_{01}, \tag{35}$$

and otherwise

$$\chi(\phi_{mn}) = -(m+2n). \tag{36}$$

In particular, for $m>2$, Eq. (31) reduces to the recursion relation[3]

$$[\Omega + i(2n+m)\mu]a_{mn} = \sqrt{m}a_{m-1,n} + \sqrt{m+1}a_{m+1,n}. \tag{37}$$

A necessary condition for the convergence of the series (30) is that for a given value of n, the coefficients a_{mn} must approach zero as m approaches infinity. With this assumption, Eq. (37) implies that[3]

$$\frac{a_{m+1,n}}{a_{mn}} \to -\frac{i}{\sqrt{m\mu}} \quad \text{as } m \to \infty. \tag{38}$$

Thus, at some sufficiently large value of m, $m_{\max}$, we "truncate" the recursion relation (37), setting

$$[\Omega + i(2n + m_{\max})\mu]a_{m_{\max},n} = \sqrt{m_{\max}}a_{m_{\max}-1,n}. \tag{39}$$

In addition, we will look for eigenfunctions for which $a_{mn}=0$ unless $n=0$ or $n=1$. The infinite-dimensional eigenvalue equation (31) then reduces to a $2(m_{\max}+1)$-dimensional eigenvalue equation.

We begin with Eq. (39) and iterate the recursion relation backwards for $n=0$ and $n=1$. For $n=0$, for example, Eq. (39) is solved for $a_{m_{\max},0}$, yielding

$$a_{m_{\max},0} = \frac{\sqrt{m_{\max}}}{\Omega + im_{\max}\mu}a_{m_{\max}-1,0}. \tag{40}$$

This expression is then substituted in the preceding equation,

$$[\Omega + i(m_{\max} - 1)\mu]a_{m_{\max}-1,0} = \sqrt{m_{\max} - 1}a_{m_{\max}-2,0} + \sqrt{m_{\max}}a_{m_{\max},0}, \tag{41}$$

which is then solved for $a_{m_{\max}-1,0}$, yielding

$$a_{m_{\max}-1,0} = \frac{\sqrt{m_{\max}-1}}{\Omega + i(m_{\max}-1)\mu - \dfrac{m_{\max}}{\Omega + im_{\max}\mu}}a_{m_{\max}-2,0}. \tag{42}$$

This expression is then substituted in the preceding equation, and so on. By means of these recursive substitutions, all but four of the coefficients a_{mn} can be eliminated. A by-product of this procedure is the development of continued fractions, the beginnings of which can be seen in Eq. (42). The set of $2(m_{\max}+1)$ equations given by Eq. (31) [with the truncation condition (39)] is thereby reduced to the four equations

$$-\Omega a_{00} + a_{10} = 0, \tag{43}$$

$$[1 + (k\lambda_D)^{-2}]a_{00} - \Omega a_{10} + \sqrt{2}a_{20} = 0, \tag{44}$$

$$3\sqrt{2}a_{10} - 3F_1(\Omega,\mu)a_{20} - 2\sqrt{2}i\mu a_{01} = 0, \tag{45}$$

$$-2\sqrt{2}i\mu a_{20} - 3F_2(\Omega,\mu)a_{01} = 0, \tag{46}$$

where $F_1(\Omega,\mu)$ and $F_2(\Omega,\mu)$ are the continued fractions

$$F_1(\Omega,\mu) \equiv \Omega + \frac{4}{3}i\mu - \cfrac{3}{\Omega + 3i\mu - \cfrac{4}{\Omega + 4i\mu - \cdots \cfrac{m_{\max}}{\Omega + m_{\max}i\mu}}}, \tag{47}$$

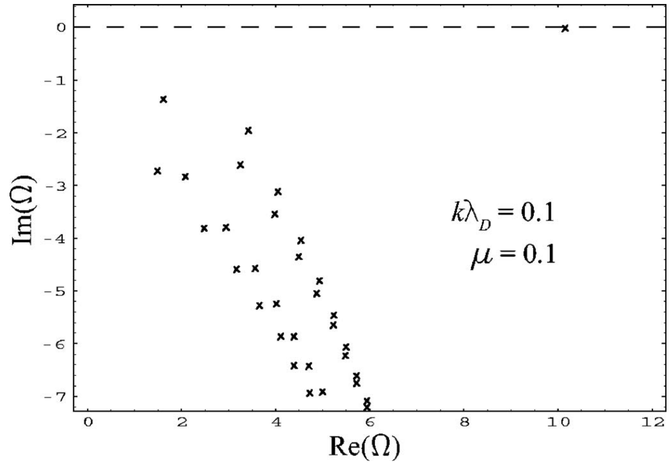

FIG. 2. Complex eigenvalues of the linearized Dougherty kinetic equation, for $k\lambda_D=0.1$ and $\mu=0.1$. The dashed line indicates the real-Ω axis. The eigenvalue with smallest imaginary part gives the complex frequency of the plasma wave (or Trivelpiece–Gould wave).

$$F_2(\Omega,\mu) \equiv \Omega + \frac{2}{3}i\mu - \cfrac{1}{\Omega + 3i\mu - \cfrac{2}{\Omega + 4i\mu - \cdots \cfrac{m_{\max}}{\Omega + (m_{\max}+2)i\mu}}}. \tag{48}$$

Finally, upon elimination of the coefficients a_{00}, a_{10}, a_{20}, and a_{01} from Eqs. (43)–(46), the following dispersion equation is obtained:

$$k^2\lambda_D^2 = \frac{F_1(\Omega,\mu)F_2(\Omega,\mu) + 8\mu^2/9}{[F_1(\Omega,\mu)F_2(\Omega,\mu) + 8\mu^2/9](\Omega^2 - 1) - 2F_2(\Omega,\mu)\Omega}. \tag{49}$$

This result becomes exact in the limit $m_{\max}\to\infty$.

In general, for given values of $k\lambda_D$ and μ, Eq. (49) must be solved numerically for the complex frequency Ω; in practice, this requires that the continued fractions $F_1(\Omega,\mu)$ and $F_2(\Omega,\mu)$ be evaluated approximately by carrying out a sufficiently large (but finite) number of iterations. The resulting dispersion relation is a polynomial equation, the number of roots of which increases with the number of iterations made in evaluating the continued fractions; each of these roots lies in the lower half of the complex Ω plane (see Fig. 2). In other words, there appears to be a countable infinite spectrum of damped eigenmodes, analogous to that found by Ng, Bhattacharjee, and Skiff for the one-dimensional LB kinetic equation.[3]

The least damped root of Eq. (49) approaches the Landau root of the collisionless dispersion relation in the limit $\mu\to 0$. In particular, in this limit, the imaginary part of this root does not approach zero exactly, but instead matches onto the Landau damping coefficient, as shown in Fig. 3 for $k\lambda_D=0.3$. Hereafter, we will focus on this least damped root, which we will refer to simply as the Landau root for the TG wave.

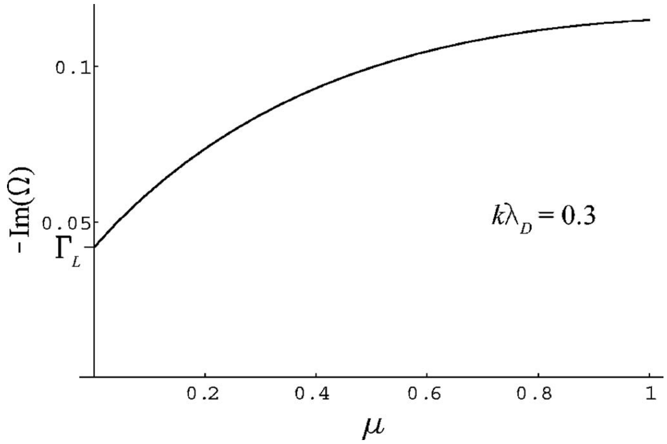

FIG. 3. Scaled damping rate $-\mathrm{Im}(\Omega)$ plotted as a function of μ, for $k\lambda_D=0.3$. The intercept at $\mu=0$ coincides with the Landau damping rate, Γ_L, of collisionless theory.

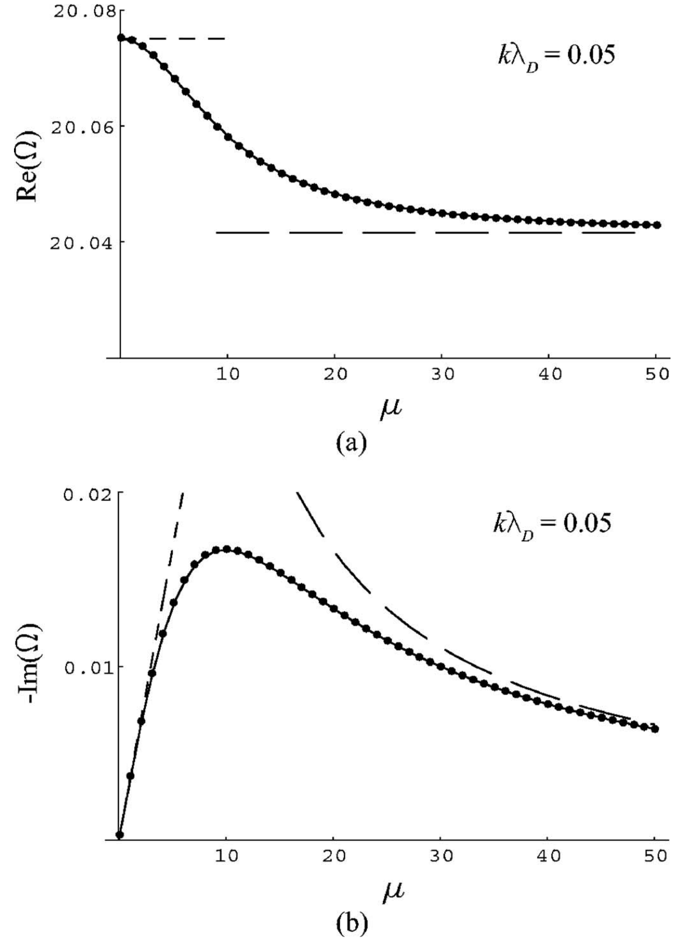

FIG. 4. Real (a) and imaginary (b) parts of the analytic approximation (51) to the Landau root, plotted (as solid curve) vs μ, for $k\lambda_D=0.05$. The solid circles represent the exact numerical solution of the dispersion equation (49). The short-dashed curves give the asymptotic forms (2) and (3), which are valid for $\mu \ll \Omega$, while the long-dashed curves give the asymptotic forms (4) and (5), valid for $\mu \gg \Omega$.

In order to isolate collisional effects from resonant particle effects, we restrict our attention to high-phase-velocity waves [i.e., $\mathrm{Re}(\Omega) \gg 1$], for which Landau damping is negligible. In this limit, a suitable approximation to Eq. (49) may be obtained by setting $F_1 \rightarrow \Omega + 4i\mu/3$ and $F_2 \rightarrow \Omega + 2i\mu/3$, since retaining the continued fractions only leads to corrections of higher order in Ω^{-1}. The resulting dispersion equation is

$$k^2\lambda_D^2 = \frac{\Omega^2 + 2i\mu\Omega}{\Omega^4 + 2i\mu\Omega^3 - 3\Omega^2 - 10i\mu\Omega/3}. \tag{50}$$

There exists a weakly damped root to this equation when $k\lambda_D \ll 1$, and this is the Landau root. An approximate expression for this root, valid in both the weakly collisional and strongly collisional limits, can be obtained by solving Eq. (50) using perturbation theory. More precisely, one assumes that $\mathrm{Re}(\Omega) \sim (k\lambda_D)^{-1}$ (this assumption is verified *a posteriori*) and takes $\mu \sim \mathrm{Re}(\Omega)$; Eq. (50) can then be solved order by order in the small parameter $k\lambda_D \ll 1$. When carried out to second order, this procedure yields the expression

$$\Omega \cong \frac{1}{k\lambda_D}\left[1 + \frac{3}{2}k^2\lambda_D^2\left(\frac{1 + 10i\mu k\lambda_D/9}{1 + 2i\mu k\lambda_D}\right)\right], \tag{51}$$

which is identical to Eq. (1) if the units are restored. In Fig. 4, this expression is plotted as a function of μ for $k\lambda_D=0.05$, and the exact numerical solution of Eq. (49) for the Landau root is shown for comparison, as are the limiting expressions given by Eqs. (2)–(5).

In the limit $\mu \ll 1$, we have evaluated the sum in Eq. (30) to determine the $u_\perp$-integrated eigenfunction, $\delta\tilde{f} \equiv 2\pi \times \int du_\perp u_\perp \delta\hat{f}$, corresponding to the Landau root for several values of the parameters μ and $k\lambda_D$. In Fig. 5, this function is plotted for $\mu=\sqrt{2}/40$ and $k\lambda_D=1/3$. (These values were chosen to facilitate comparison with Fig. 2 in Ref. 3. It should be noted that the scaled variables μ and Ω defined by these authors differ by a factor of $\sqrt{2}$ from those defined here.) We find that the eigenfunction exhibits the qualitative features of that determined by Ng *et al.* using the LB operator.[3] In the vicinity of the resonance (i.e., for $u_z \cong \Omega$), the eigenfunction deviates significantly from the collisionless expression $u_z e^{-u_z^2/2}[\sqrt{2\pi}(k\lambda_D)^2(\Omega - u_z)]^{-1}$, whereas far from the resonance, the collisionless expression is a good approximation. The width of the "boundary layer" surrounding the resonance increases with collisionality.

In the limit $\mu \gg \Omega$, all of the coefficients in the sum (30) are of the order of μ^{-1} or smaller, with the exceptions of a_{00}, a_{10}, a_{20}, and a_{01}; in this case, the eigenfunction is given by

$$\delta\hat{f}(u_z,u_\perp^2) = \frac{e^{-(u_z^2+u_\perp^2)/2}}{\sqrt{2\pi}}\Bigg\{1 + \Omega u_z + \frac{1}{2}[\Omega^2 - 1 - (k\lambda_D)^{-2}] \times (u_z^2 + u_\perp^2 - 3)\Bigg\} + O(\mu^{-1}). \tag{52}$$

Evidently, as a first approximation, the perturbation to the distribution is completely characterized by the perturbations in particle number, momentum, and energy. In other words, the distribution is simply a Maxwellian with perturbed density, drift velocity, and temperature, with all other components of the perturbation vanishing as μ^{-1}.

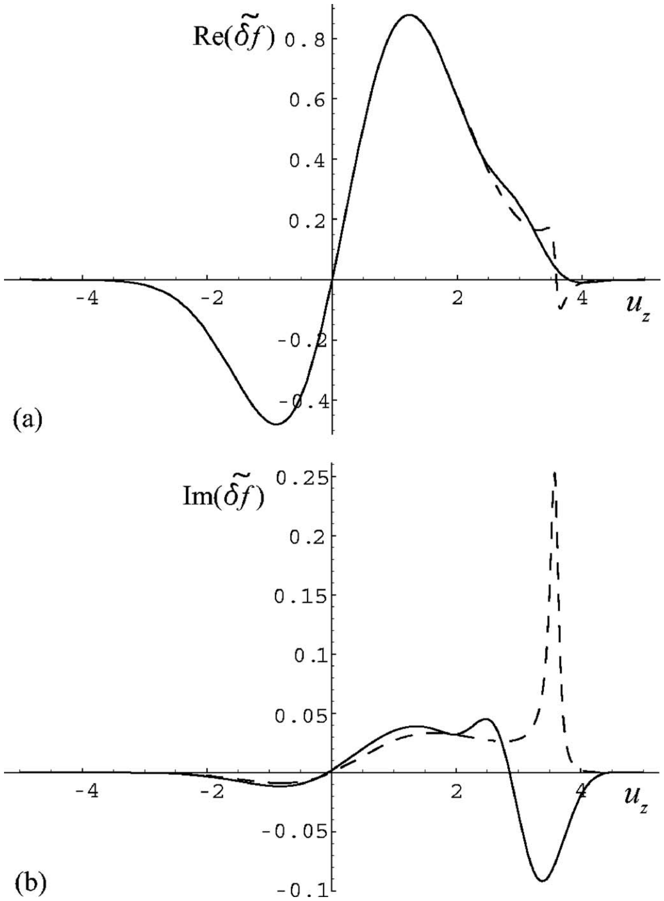

FIG. 5. Real (a) and imaginary (b) parts of the $u_\perp$-integrated eigenfunction, $\widetilde{\delta f} \equiv 2\pi \int du_\perp u_\perp \hat{\delta f}$, corresponding to the Landau root, Ω, for $\mu = \sqrt{2}/40$ and $k\lambda_D = 1/3$. The dashed curves give the real and imaginary parts of the expression $u_z e^{-u_z^2/2}[\sqrt{2\pi}(k\lambda_D)^2(\Omega - u_z)]^{-1}$.

V. FLUID THEORY OF THE TRIVELPIECE–GOULD WAVE

In this section we derive an expression for the complex frequency of the TG wave starting from fluid equations, the result being valid in the limit of strong collisionality. The drift-kinetic treatment of the preceding section leaves out cross-field transport, whereas fluid theory incorporates cross-field transport through the perpendicular viscosity and thermal diffusivity which appear in the fluid equations.

We assume as before that the unperturbed density and temperature are constant for $r < R_p$ and are zero for $r > R_p$. The unperturbed fluid velocity is then given by $\vec{V}_0 = r\omega_E \hat{\theta}$, where the $\vec{E} \times \vec{B}$ rotation frequency, ω_E, is a constant determined by n_0. With the ansatz that the density, fluid velocity, temperature, and potential perturbations share the parameter dependence

$$\delta n, \delta V_z, \delta T, \delta\varphi \sim J_0(k_\perp r) e^{i(k_z z - \omega t)}, \tag{53}$$

the linearized continuity, momentum, and heat transfer equations plus Poisson's equation reduce to

$$-i\omega \delta n + n_0 i k_z \delta V_z = 0, \tag{54}$$

$$-i\omega \delta V_z = i k_z \frac{e}{m}\delta\varphi - \frac{ik_z}{mn_0} T_0 \delta n - \frac{ik_z}{m}\delta T - \frac{4}{3}\zeta_z k_z^2 \delta V_z - \zeta_\perp k_\perp^2 \delta V_z, \tag{55}$$

$$-i\omega \delta T = -\frac{2}{3} T_0 i k_z \delta V_z - \chi_z k_z^2 \delta T - \chi_\perp k_\perp^2 \delta T, \tag{56}$$

$$-k^2 \delta\varphi = 4\pi e \delta n. \tag{57}$$

Here ζ_z and $\zeta_\perp$ are the parallel and perpendicular kinematic viscosities and χ_z and $\chi_\perp$ are the parallel and perpendicular thermal diffusivities. Elimination of the perturbed quantities from these linearized equations yields the dispersion relation

$$1 = \frac{(k\lambda_D)^{-2} + 1}{(-i\Omega + 4\bar{\zeta}_z/3 + \bar{\zeta}_\perp) i\Omega} - \frac{2}{3}\frac{1}{(-i\Omega + 4\bar{\zeta}_z/3 + \bar{\zeta}_\perp)(-i\Omega + \bar{\chi}_z + \bar{\chi}_\perp)}, \tag{58}$$

where again $\Omega = \omega / k_z v_{\rm th}$ is the scaled wave frequency and $\bar{\zeta}_z = \zeta_z k_z / v_{\rm th}$, $\bar{\zeta}_\perp = \zeta_\perp k_\perp^2 / k_z v_{\rm th}$, $\bar{\chi}_z = \chi_z k_z / v_{\rm th}$, and $\bar{\chi}_\perp = \chi_\perp k_\perp^2 / k_z v_{\rm th}$ are the scaled transport rates along and across the magnetic field. If one assumes further that $\bar{\zeta}_z$, $\bar{\zeta}_\perp$, $\bar{\chi}_z$, $\bar{\chi}_\perp \ll \Omega$, Eq. (58) takes the approximate form

$$\Omega^3 = \frac{1}{k^2\lambda_D^2}\Omega + \frac{5}{3}\Omega - i(4\bar{\zeta}_z/3 + \bar{\zeta}_\perp)\left(\frac{1}{k^2\lambda_D^2} + 1\right) - i\frac{2}{3}(4\bar{\zeta}_z/3 + \bar{\chi}_z + \bar{\zeta}_\perp + \bar{\chi}_\perp). \tag{59}$$

Finally, for $k\lambda_D \ll 1$, this dispersion equation can be solved perturbatively for the root corresponding to the TG wave, and one finds (after restoring the units)

$$\mathrm{Re}(\omega) = \frac{k_z \omega_p}{k}\left[1 + \frac{5}{6}k^2\lambda_D^2\right], \tag{60}$$

$$\mathrm{Im}(\omega) = -\left(\tfrac{2}{3}\zeta_z k_z^2 + \tfrac{1}{2}\zeta_\perp k_\perp^2\right). \tag{61}$$

Here the real part of the frequency is identical to that obtained by kinetic theory in the limit of strong collisionality, as one would expect. The damping, at this order in the perturbation theory, is due entirely to momentum transport, both along and across the magnetic field; from Eq. (59), one can see that the thermal diffusivity imparts a correction to the damping which is reduced by the factor $k^2\lambda_D^2$, and this reduction is a consequence of the large phase velocity of the wave. Equation (5) is recovered from Eq. (61) by setting $\bar{\zeta}_\perp = 0$ and $\bar{\zeta}_z = (2\mu)^{-1}$; these are the values of the parallel and perpendicular viscosity which the Dougherty operator predicts in the drift approximation.[11] However, we will see that long-range collisions (which are not encompassed by the Dougherty operator) result in an enhanced perpendicular viscosity that cannot be neglected in Eq. (61).

VI. LONG-RANGE INTERACTIONS AND CROSS-FIELD TRANSPORT

The Dougherty operator is well-suited to describe collisions in which the impact parameter is smaller than r_c; in particular, the velocity scattering in these collisions is nearly isotropic, since the effect of the magnetic field is negligible on scales small compared to r_c. Cross-field transport due to these collisions is omitted from the above treatment by the use of the drift approximation, and this omission is justified since the corresponding transport rate, which scales as $k_\perp^2 r_c^2$, is small. However, the typical operating regime for the plasmas under consideration is such that

$$b \ll r_c \ll \lambda_D, \tag{62}$$

where $b = e^2/T$ is the classical distance of the closest approach. In this parameter regime, there are long-range collisions with impact parameter of order λ_D that cannot be neglected. These collisions cannot be encompassed by any collision operator which assumes a pointlike interaction; in particular, they are outside the scope of the Dougherty operator.[18] The cross-field transport rate corresponding to these long-range collisions scales as $k_\perp^2 \lambda_D^2$ and is therefore non-negligible.[17] Indeed, in the limit of strong collisionality, the damping rate is significantly larger than that given by Eq. (5), and the enhancement is due to cross-field transport.

To incorporate the effect of cross-field transport on the damping rate, we estimate the perpendicular viscosity as

$$\zeta_\perp \cong n v_{\mathrm{th}} b^2 \lambda_D^2. \tag{63}$$

This expression is simply the product of the frequency of large-angle scattering events, $n v_{\mathrm{th}} b$,[2] with the square of the characteristic transport step size, λ_D. (A more rigorous kinetic calculation, analogous to the calculation of the perpendicular thermal conductivity by Dubin and O'Neil, corroborates this estimate.[19]) From Eq. (61), we infer that the contribution to the damping rate due to cross-field transport is

$$\mathrm{Im}(\Delta\omega) = -\tfrac{1}{2}\zeta_\perp k_\perp^2 \cong -\tfrac{1}{2} n v_{\mathrm{th}} b^2 k_\perp^2 \lambda_D^2. \tag{64}$$

The fluid theory from which Eq. (61) was derived is valid only in the limit of strong collisionality; otherwise, the assumption that transport *along* the magnetic field is local breaks down. However, the *cross-field* transport is local regardless of the strength of collisionality, provided that the transverse wavelength is large in comparison to the cross-field transport step size; for the weakly damped waves of interest, this requirement is clearly satisfied, since $k_\perp \lambda_D < k\lambda_D \ll 1$. Therefore, Eq. (64) gives the correct contribution from cross-field transport for arbitrary collisionality. (In the limit of weak collisionality, we have recovered this result from a perturbative kinetic theory which employs a nonlocal collision operator to describe long-range interactions.) Adding this expression to the right-hand side of Eq. (1), which, as it stands, only accounts for transport along the field, we find

$$\omega = \frac{k_z \omega_p}{k}\left[1 + \frac{3}{2}k^2\lambda_D^2\left(\frac{1 + 10i\alpha/9}{1 + 2i\alpha}\right)\right] - \frac{i}{2} n v_{\mathrm{th}} b^2 k_\perp^2 \lambda_D^2. \tag{65}$$

This expression is valid for arbitrary collisionality, provided that $k\lambda_D \ll 1$.

VII. DISCUSSION

In the limit of weak collisionality, the correction (64) is relatively small, so the order of magnitude of the damping rate is given by Eq. (3). The smallness of the damping rate Eq. (3) in comparison to that obtained by LB for a neutral plasma has been discussed in the Introduction. In addition to reducing the damping rate, the factor $k^2\lambda_D^2$ that appears in Eq. (3) clearly changes the scaling. Most noteworthy is the fact that the damping rate does not depend on density, since the density dependence in ν is cancelled by that in λ_D^2. There is also a partial cancellation of the temperature dependence in these two quantities, resulting in a $T^{-1/2}$ scaling, as opposed to the $T^{-3/2}$ scaling implicit in the LB rate. Finally, our rate goes as k^2, whereas that obtained by LB does not depend on wavenumber.

In the limit of strong collisionality, cross-field transport cannot be neglected. The ratio of the contributions to the damping from cross-field and parallel transport, respectively, is $3\alpha^2/2\ln(r_c/b)$, where α is the collisionality parameter. For $\alpha \gg 1$, this number is larger than one; thus, in the strongly collisional limit, the dominant contribution to the damping comes from cross-field transport, and the damping rate approaches that given by Eq. (64).

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