Waves in a cold pure electron plasma of finite length

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A theory for low-frequency electrostatic modes of a finite length pure electron plasma column is presented. The plasma is modeled as a cold uniform density cylinder with flat ends. An interesting result is that the diocotron mode can have an axial wavelength that is much larger than the plasma length. Also, for particular values of the plasma density, the axial magnetic field, and the dimensions of the plasma, the diocotron mode is degenerate with a plasma mode and this results in a strong mixing of the modes.

I. INTRODUCTION

An experimental effort[1] is underway to cool a pure electron plasma to very low tempertures (in the range of 1 K), the goal being to produce a pure electron liquid and a pure electron crystal.[2] The confinement geometry for these experiments is shown schematically in Fig. 1. A conducting cylinder is divided into three sections, the two end sections being held at a negative potential relative to the central section. There is a uniform magnetic field B directed along the axis of the cylinder. The electron plasma resides in the central section, with axial confinement provided by the negatively biased end sections and radial confinement by the magnetic field.

It is useful to have a theoretical description of modes in such a plasma. Here, we consider low-frequency electrostatic modes, paying particular attention to the finite length of the plasma column.

As a simple model, we assume that the unperturbed column is of uniform density and has the shape of a right circular cylinder (see Fig. 2). In accord with the experiments, we assume that the electron density is well below the Brillouin limit,[3] that is, that $\omega_p \ll \omega_c$. ω_p is the electron plasma frequency and ω_c is the electron cyclotron frequency. By low-frequency modes, we mean that the mode frequency is well below the plasma frequency; we will treat $\delta \sim \omega/\omega_p$ as a small parameter. Also, we will treat a/L as a small parameter, where a is the plasma radius and $2L$ is the length.

Since the mean free path between collisions is short, the plasma dynamics may be described by fluid theory. For the long wavelengths under consideration, pressure and viscosity are negligible, and, for the low frequencies under consideration, the dynamics perpendicular to the magnetic field reduces to the drift approximation. In equilibrium, the electron column executes a rigid body $\mathbf{E}\times\mathbf{B}$ rotation with the angular frequency $\omega_r = \omega_p^2/2\omega_c$. One finds that the mode potential satisfies the equation $\nabla\cdot\epsilon\cdot\nabla\phi = 0$, where ϵ is the cold plasma dielectric tensor as seen in the rotating frame.

The mode potential satisfies the boundary condition $\phi = 0$ at $r = R$ and at $z = \pm\infty$. Here, (r,θ,z) is a cylindrical coordinate system, and $r = R$ is the radius of the surrounding conducting cylinder. For the frequencies under consideration, the impedance between the three sections of the conducting wall (see Fig. 1) is negligible, and we can treat the wall as if it were continuous.

To solve for ϕ, we divide the volume of the tube into three regions i, ii, and iii with the plasma residing in region ii (see Fig. 2). In regions i and iii, we expand the mode potential in terms of the solutions of Laplace's equation, which vanishes at $r = R$ and at $z = \pm\infty$. These basis functions are $J_l(K_{lm}r)\exp(il\theta + K_{lm}z)$ in region i and $J_l(K_{lm}r)\times\exp(il\theta - K_{lm}z)$ in region iii, with $J_l(K_{lm}R) = 0$. In region ii, the mode potential is expanded in terms of the solutions for an infinitely long column. These basis functions can be classified into three types: (a) diocotron, (b) plasma, and (c) vacuum. The diocotron solution in the limit of zero axial wavelength is well known in non-neutral plasma literature.[3] Together with the plasma solutions it makes up the usual sinusoidal waves in the low-frequency regime supported by an infinitely long column (i.e., low-frequency Trivelpiece–Gould waves).[3,4] The vacuum-type solutions, unlike the other two, have significant amplitude only in the vacuum annulus between the plasma and the wall; they are associated with a purely imaginary axial wavenumber and hence are usually omitted in the treatment of infinitely long columns. However, these solutions must be retained here to have a complete set of basis functions (see Sec. IV).

The equation $\nabla\cdot\epsilon\cdot\nabla\phi = 0$ implies that ϕ and $\epsilon_{zz}(\partial\phi/\partial z)$ are both continuous at $z = \pm L$; here $\epsilon_{zz} = 1 - \omega_p^2/$

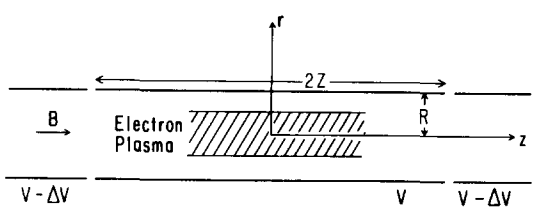

FIG. 1. The confinement geometry for electron plasmas.

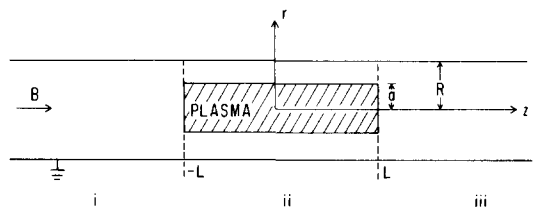

FIG. 2. Model for the wave theory.

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$(\omega - l\omega_r)^2$ is the zz component of the cold plasma dielectric tensor evaluated at the Doppler shifted frequency $\omega - l\omega_r$. These matching conditions and the orthogonality of the various basis functions when integrated over radii lead to a matrix equation $M \cdot A = 0$, where A is the vector of the coefficients of region ii and M is a matrix which depends on the frequency and on the parameters $(\omega_p, \omega_c, l, a, R, L)$. This equation can be solved only for certain discrete values of ω. The vectors A corresponding to these eigenfrequencies determine the eigenfunctions of the system. We solve the matrix equation by expansion in terms of the small parameters a/L and $\delta \sim \omega/\omega_p$.

A simple physical explanation of the solution can be given. From the matching conditions at $z = L$ and from the fact that $\partial\phi_{iii}/\partial z \simeq -\phi_{iii}/R \simeq -\phi_{iii}/a$, we see that

$$\left.\frac{\partial\phi_{ii}}{\partial z}\right|_L = \frac{1}{\epsilon_{zz}}\left.\frac{\partial\phi_{iii}}{\partial z}\right|_L \simeq \frac{\delta^2\phi_{iii}|_L}{a} = \frac{\delta^2\phi_{ii}|_L}{a}, \tag{1}$$

where $-1/\epsilon_{zz} \simeq \delta^2$. As an example, consider a mode even in z. The diocotron and plasma basis functions for such a mode are of the form $\cos(kz)$, where $k \simeq O(\delta/a)$. Equation (1) is satisfied if $-k \tan kL = (1/\phi_{ii})(\partial\phi_{ii}/\partial z)|_L \simeq \delta^2/a$, or equivalently, if $\tan kL \simeq -\delta$. To lowest order in δ, the plasma and diocotron basis functions are eigenfunctions of the finite length column if $kL = n\pi$. The ω's corresponding to these k's are the eigenfrequencies. Coupling of these functions to the vacuum basis functions and to each other (via the basis functions of regions i and iii) occurs in higher order in the small parameters δ and a/L.

An interesting feature of the solutions is that the column can support a diocotron-like mode which shows very little axial variation over the length of the column ($kL \simeq n\pi$ for $n = 0$); the effective wavelength is much larger than the length of the plasma. Analogous plasma-like modes showing little axial variation over the length of the column do not exist. The long wavelength diocotron-like mode has been experimentally observed.[5] Another interesting result is that for certain values of the plasma parameters, the diocotron-like mode is degenerate with a plasma-like mode and this results in a strong mixing of the modes.

Since this work is intended to apply to a highly collisional plasma, fluid theory is used and bounce motion plays no role. However, due to a fortuitous accident, the lowest order results should also be applicable to a collisionless plasma, provided $\bar{v} \ll L\omega_r$, where $\bar{v}$ is the electron thermal velocity. Rather than consider electrons which bounce back and forth between specular reflections at $z = \pm L$, we can imagine that the mode potential is periodically replicated on an infinitely long column (the replication being even about $z = \pm L, \pm 2L$,etc.) and then consider electrons which travel without reflection.[6] For the lowest-order eigenfunctions, $\partial\phi/\partial z = 0$ at $z = \pm L$; so the replication is equivalent to simply extending the sinusoidal dependence of these eigenfunctions to an infinitely long column. For a sinusoidal mode in an infinitely long column, the cold fluid equations and the Vlasov equation give the same electron response, provided that $\bar{v} \ll |\omega - l\omega_r|/k$. For the wavenumbers and frequencies under consideration here, this reduces to $\bar{v} \ll \omega_r L$. The experiment[5] referred to in the previous paragraph satisfies this inequality.

The paper is organized in the following way. In Sec. II, we derive the differential equation satisfied by the linearized mode potential. In Sec. III, we discuss the basis functions for the various regions. The orthonormality properties of these functions are considered in Sec. IV. We expand the wave potential of the finite length column in terms of these basis functions and obtain the dispersion equation in Sec. V. In Sec. VI, we solve the dispersion equation (for the eigenmodes of a finite length electron column) using a perturbation technique and discuss the interesting features of the solutions.

II. DIFFERENTIAL EQUATIONS FOR THE MODES

Since the unperturbed system is homogeneous in θ and t, we consider modes of the form $\phi(z,r,\theta,t) = \phi(z,r)\exp(il\theta - i\omega t)$. For the frequency ordering mentioned in the introduction (i.e., $\omega_r, \omega \ll \omega_p \ll \omega_c$), the velocity of an element of the cold electron fluid is determined by:

$$\frac{\partial v_z}{\partial t} = \frac{e}{m}\frac{\partial\phi}{\partial z}, \tag{2}$$

and

$$\mathbf{v}_\perp = r\omega_r\hat{\theta} + \frac{c}{B}\hat{z}\times\nabla\phi, \tag{3}$$

where $r\omega_r\hat{\theta} = c(E_{r0}/B)\hat{r}\times\hat{z}$ is the zero-order rotation velocity. The linearized continuity equation takes the form

$$\left(\frac{\partial}{\partial t} + \omega_r\frac{\partial}{\partial\theta}\right)n_1 + \frac{\partial}{\partial z}[n_0(r,z)v_z] + v_r\frac{\partial}{\partial r}n_0(r,z) = 0, \tag{4}$$

where $n_0(r,z)$ has the constant value n_0 in the region $(r < a, -L < z < L)$ and is zero outside. Combining these equations with Poisson's equation,

$$\nabla^2\phi = 4\pi e n_1 \tag{5}$$

yields the differential equation for the modes

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} - \frac{l^2\phi}{r^2} + \frac{\partial}{\partial z}\left[\left(1 - \frac{\omega_p^2(z,r)}{(\omega - l\omega_r)^2}\right)\frac{\partial\phi}{\partial z}\right] + \frac{2l\omega_r}{\omega - l\omega_r}\frac{\phi}{rn_0}\frac{\partial}{\partial r}n_0(r,z) = 0. \tag{6}$$

Equation (6) can be rewritten as:

$$\nabla\cdot\boldsymbol{\epsilon}\cdot\nabla\phi = 0, \tag{7}$$

where

$$\boldsymbol{\epsilon} = \begin{bmatrix} 1 & \dfrac{-i\omega_p^2}{(\omega - l\omega_r)\omega_c} & 0 \\ \dfrac{i\omega_p^2}{(\omega - l\omega_r)\omega_c} & 1 & 0 \\ 0 & 0 & 1 - \dfrac{\omega_p^2}{(\omega - l\omega_r)^2} \end{bmatrix} \tag{8}$$

is the dielectric tensor for a cold pure electron plasma immersed in a uniform magnetic field. Note that the tensor is evaluated at the Doppler-shifted frequency $\omega - l\omega_r$. This is to be expected, since the zero-order radial electric field is transformed out in the rotating frame.[3] As discussed in the

introduction, the mode potential satisfies the boundary conditions $\phi = 0$ at $r = R$ and at $z = \pm \infty$.

III. BASIS FUNCTIONS

The volume inside the conducting cylinder is divided into three regions, i, ii, and iii as shown in Fig. 2. The plasma resides in region ii. Our method is to expand the wave potential ϕ in terms of the solutions of Eq. (6) in these three regions and to determine the coefficients (and the corresponding eigenfrequencies) by matching ϕ and $\epsilon_{zz}(\partial\phi/\partial z)$ across the boundaries $z = \pm L$ between these regions.

In regions i and iii, Eq. (6) reduces to Laplace's equation. So the basis functions in these regions are the solutions of Laplace's equation which vanish at $r = R$ and at points far away from the plasma: $J_l(K_{lm}r)\exp(il\theta + K_{lm}z)$ in region i and $J_l(K_{lm}r)\exp(il\theta - K_{lm}z)$ in region iii, with K_{lm} being the solutions of $J_l(K_{lm}R) = 0$.

The basis functions of region ii are just the solutions of Eq. (6) for the case of an infinitely long column. Assuming a z dependence of the form $\phi \sim e^{ikz}$, Eq. (6) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} - \frac{l^2}{r^2}\phi - \epsilon_{zz}(\omega,r)k^2\phi + \frac{2l\omega_r}{\omega - l\omega_r}\frac{\partial n_0/\partial r}{rn_0}\phi = 0, \tag{9}$$

where the quantity $\epsilon_{zz}(\omega,r)$ is equal to $1 - [\omega_p^2(0)/(\omega - l\omega_r)^2] \equiv -\alpha^2$ inside the plasma ($r < a$) and is equal to 1 outside ($a < r < R$); also $(\partial n_0/\partial r)/n_0 = -\delta(r - a)$. The solutions of Eq. (9) are of the form $\phi_1 = J_l(\alpha kr)$ for $r < a$ and $\phi_2 = A[I_l(kr)K_l(kR) - K_l(kr)I_l(kR)]$ for $r > a$. In obtaining the form of ϕ, we have used the boundary conditions (i) $\phi(r = R) = 0$ and (ii) ϕ is regular at the origin. Applying the matching conditions

$$\phi_2|_{r=a} - \phi_1|_{r=a} = 0,$$

$$\left.\frac{\partial\phi_2}{\partial r}\right|_{r=a} - \left.\frac{\partial\phi_1}{\partial r}\right|_{r=a} = \frac{2l\omega_r}{\omega - l\omega_r}\frac{1}{a}\phi_1|_{r=a}, \tag{10}$$

[the last equation is obtained by integrating Eq. (9) across $r = a$] yields the dispersion relation[3,4]

$$ka\frac{I_l'(ka)K_l(kR) - K_l'(ka)I_l(kR)}{I_l(ka)K_l(kR) - K_l(ka)I_l(kR)} - \alpha ka\frac{J_l'(\alpha ka)}{J_l(\alpha ka)} = \frac{2l\omega_r}{\omega - l\omega_r}. \tag{11}$$

For a given ω, only certain discrete values of k can satisfy this equation. These k's constitute the various branches of the dispersion curve shown qualitatively in Fig. 3. Figure 4 presents these curves drawn to scale (for typical experimental parameters) in the region of interest to us.

A discussion of these branches and the corresponding wavefunctions is given in the Appendix. These wavefunctions $\phi_{lm}(\omega,r)\exp[ik_{lm}(\omega)z]$ are the basis functions for region ii, in terms of which the wave potential of the finite length column is expanded. They are of three types whose characteristics are summarized below (see Fig. 5).

(1) *Diocotron-type* [denoted by $\phi_{ld}(\omega,r)$]: It has a single maximum at $r = a$. The corresponding $k_{ld}(\omega)$ is real for $\omega < \omega_d$ and imaginary for $\omega \geqslant \omega_d$; $\omega_d = \omega_r[l - 1 + (a/R)^{2l}]$ is the diocotron frequency.

(2) *Plasma-type* [denoted by $\phi_{lp_i}(\omega,r)$]: They are oscillatory for $r < a$ and fall off monotonically to zero as $r \to R$. The corresponding $k_{lp_i}(\omega)$ are real.

(3) *Vacuum-type* [denoted by $\phi_{lv_i}(\omega,r)$]: They are essentially zero for $r < a$ and are oscillatory for $a < r < R$. The corresponding $k_{lv_i}(\omega)$ are imaginary.

IV. ORTHONORMALITY OF THE BASIS FUNCTIONS

The basis functions of region i (or region iii) are mutually orthogonal because of the orthogonality property of the Bessel functions. The basis functions are also complete in the

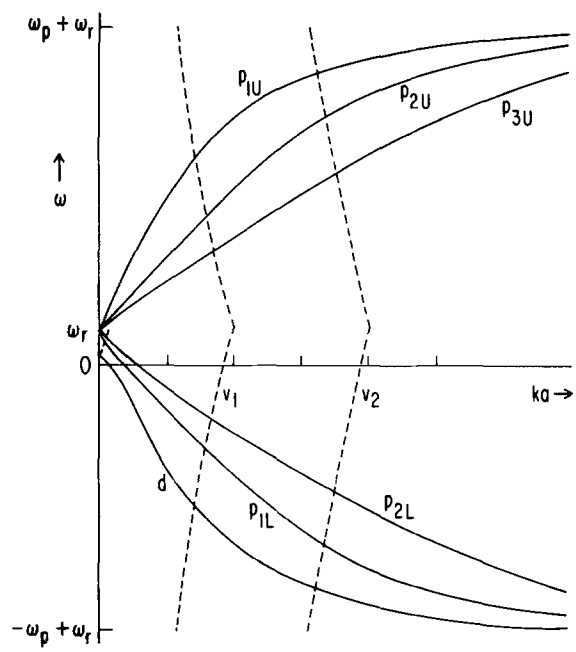

FIG. 3. Qualitative dispersion curves for $l = 1$. d stands for the diocotron branch, p for the plasma branch, and v for the vacuum-type solutions; dotted lines indicate that the corresponding k is purely imaginary. The subscripts L and U designate lower and upper branches. The curves for $l > 1$ are similar if all the frequencies are shifted down by $(l - 1)\omega_r$.

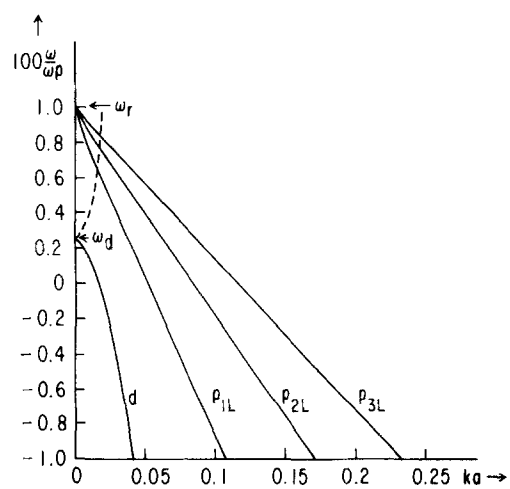

FIG. 4. Dispersion curves for $l = 1$ drawn to scale for typical experimental parameters: $a/R = 0.5$, $\omega_p/\omega_c = 0.02$ $(\omega_r/\omega_p = 0.01)$.

sense that any solution of Laplace's equation in region i (or region iii), satisfying the boundary conditions $\phi = 0$ at $r = R$ and at $|z| = \infty$ can be expanded uniquely in terms of these functions.

We now prove that the basis functions $\phi_{lm}(\omega,r)$ of region ii are also mutually orthogonal. We multiply by ϕ_{ln} the equation satisfied by ϕ_{lm} [i.e., Eq. (9)] and subtract ϕ_{lm} times the equation for ϕ_{ln} to obtain:

$$(k_{lm}^2 - k_{ln}^2)\int_0^R r\,dr\,\epsilon_{zz}(\omega,r)\phi_{lm}\phi_{ln} = \int_0^R dr\,\frac{\partial}{\partial r}\left(r\phi_{ln}\frac{\partial\phi_{lm}}{\partial r} - r\phi_{lm}\frac{\partial\phi_{ln}}{\partial r}\right). \tag{12}$$

The right-hand side is zero since ϕ_{ln} and ϕ_{lm} vanish at $r = R$. Thus we find the result

$$\int_0^R r\,dr\,\epsilon_{zz}(\omega,r)\phi_{lm}(\omega,r)\phi_{ln}(\omega,r) = 0 \quad \text{for } m \neq n, \tag{13}$$

i.e., the functions ϕ_{lm} are mutually orthogonal with the weight function $r\epsilon_{zz}(\omega,r) = r[1 - \omega_p^2(r)/(\omega - l\omega_r)^2]$. We assume that the functions ϕ_{lm} form a complete set of basis functions.

The normalization for the basis functions ϕ_{lm} is defined as follows:

$$\int_0^R r\,dr\,\epsilon_{zz}(\omega,r)\phi_{lm}^2(\omega,r) = N_m, \tag{14}$$

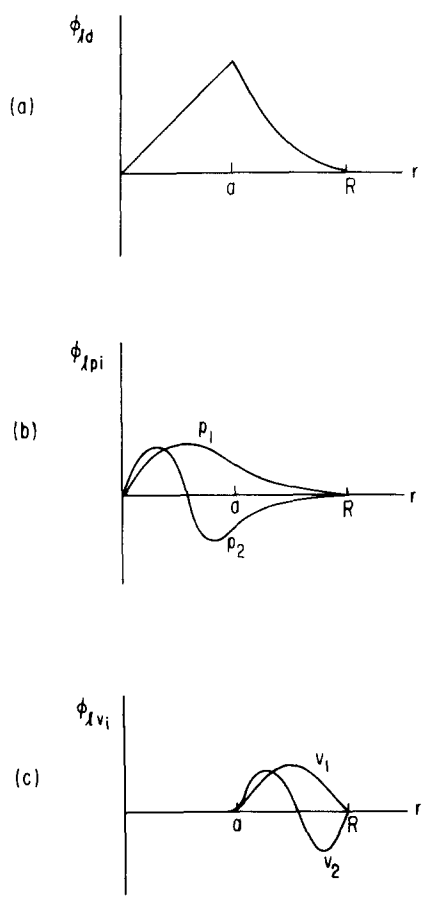

FIG. 5. Qualitative radial behavior of the basis functions for $l = 1$. Higher l functions are similar. a and R are the radius of the plasma and the radius of the conducting tube, respectively. d, p, and v stand for the diocotron-type, the plasma-type, and the vacuum-type basis functions.

where $N_m = -1$ for the diocotron-type and for the plasma-type functions (i.e., for $m = d, p_i$) and $N_m = +1$ for the vacuum-type functions ($m = v_i$). The basic reason for choosing differing normalizations is to keep the functions ϕ_{lm} real. For ϕ_{ld} and ϕ_{lp_i}, the large negative value of ϵ_{zz} for $r < a$ makes the integral (14) negative. Since $\phi_{lv_i} \simeq 0$ for $r < a$ and since $\epsilon_{zz} = 1$ for $r > a$, the integral for ϕ_{lv_i} is positive. The amplitudes of the basis functions set by the normalizing condition (14) are as follows:

$$\phi_{ld} \sim \phi_{lp_i} \sim O\left(\frac{\delta}{a}\right), \quad \phi_{lv_i} \sim O\left(\frac{1}{R-a}\right), \tag{15}$$

where we have set $|\epsilon_{zz}|^{-1} \sim \delta^2$.

V. DISPERSION EQUATION

Since the system is symmetric about the plane $z = 0$, the odd and the even modes in z can be treated separately. First, we consider modes of even parity in z [i.e., $\phi(-z) = \phi(z)$]. We expand the linearized wave potential in regions ii and iii in terms of their respective basis functions:

$$\phi_{ii}(r,z,l,\omega) = \sum_m A_m\phi_{lm}(\omega,r)\frac{\cos[k_{lm}(\omega)z]}{\cos[k_{lm}(\omega)L]}, \tag{16}$$

$$\phi_{iii}(r,z,l,\omega) = \sum_m B_m J_l(K_{lm}r)e^{-K_{lm}z}. \tag{17}$$

We shall not consider ϕ_i since ϕ_i is just the mirror image of ϕ_{iii} in the plane $z = 0$. The matching conditions at $z = L$ are:

$$\phi_{ii}(r,L,l,\omega) = \phi_{iii}(r,L,l,\omega), \tag{18}$$

and

$$\epsilon_{zz}(\omega,r)\frac{\partial}{\partial z}\phi_{ii}(r,z,l,\omega)|_L = \frac{\partial}{\partial z}\phi_{iii}(r,z,l,\omega)|_L, \tag{19}$$

or

$$\sum_m A_m\,\phi_{lm}(\omega,r) = \sum_m B_m J_l(K_{lm}r)e^{-K_{lm}L}, \tag{20}$$

and

$$\epsilon_{zz}(\omega,r)\sum_m A_m k_{lm}\phi_{lm}(\omega,r)\tan k_{lm}L = \sum_m B_m K_{lm}J_l(K_{lm}r)e^{-K_{lm}L}. \tag{21}$$

From Eq. (20) we obtain

$$B_m e^{-K_{lm}L}\int_0^R r\,dr\,J_l^2(K_{lm}r) = \sum_n A_n\int_0^R r\,dr\,\phi_{ln}(\omega,r)J_l(K_{lm}r). \tag{22}$$

We substitute this value of B_m in Eq. (21) and use the orthogonality of the function $\phi_{lm}(\omega r)$ to obtain:

$$A_n N_n k_{ln}\tan k_{ln}L = \sum_{n'} A_{n'}\sum_m K_{lm} \times \frac{\int_0^R r\,dr\,\phi_{ln}J_l(K_{lm}r)\int_0^R r\,dr\,\phi_{ln'}J_l(K_{lm}r)}{\int_0^R r\,dr\,J_l^2(K_{lm}r)}. \tag{23}$$

We note again that $N_n = -1$ for $n = d, p_i$ and $N_n = +1$ for $n = v_i$.

Equation (23) can be put into matrix form

$$M_{nn'}(\omega)A_{n'} = 0 \quad \text{or} \quad MA = 0, \tag{24}$$

where

$$M_{nn'}(\omega) = -\delta_{nn'}N_n k_{ln}\tan k_{ln}L + \sum_m K_{lm}\frac{\int_0^R r\,dr\,\phi_{ln}J_l(K_{lm}r)\int_0^R r\,dr\,\phi_{ln'}J_l(K_{lm}r)}{\int_0^R r\,dr\,J_l^2(K_{lm}r)}, \tag{25}$$

or

$$M_{nn'}(\omega) = D_{nn'}(\omega) + S_{nn'}(\omega). \tag{26}$$

Equation (24) determines the eigenfrequencies ω_a and the corresponding eigenfunctions.

VI. SOLUTION OF THE DISPERSION EQUATION

We now proceed with a perturbative solution of Eq. (23). From the dispersion relations in the Appendix, it follows that

$$k_{ld}, k_{lp_i} \sim O(\delta/a), \qquad k_{lv_i} \sim O(1/a), \tag{27}$$

and from the normalizaiton of the basis functions, it follows that:

$$S_{dd}, S_{dp_i}, S_{p_ip_j} \sim O(\delta^2/a), \quad S_{dv_i}, S_{p_iv_j} \sim O(\delta/a), \quad S_{v_iv_j} \sim O(1/a). \tag{28}$$

Thus, the elements of $M_{nn'}$ are of the form $[k\tan kL + O(\delta^2/a)]$, $O(\delta^2/a)$, $O(\delta/a)$, and $O(1/a)$. We look for a solution as an expansion in δ and a/L.

A. D_m modes

There is a class of modes, the D_m modes, for which the major component is a diocotron-type basis function. For these modes, we choose $A_d = 1$ and find that A_n (for $n \neq d$) is first order in δ or a/L.

Before presenting the formal solution, it is instructive to consider a schematic representation of the matrix equation in which the vector A has only three components: A_d, A_p, and A_v. For the choice $A_d = 1$, the matrix equation is of the form

$$0 = \begin{bmatrix} k_d\tan(k_dL) + O(\delta^2/a) & O(\delta^2/a) & O(\delta/a) \\ O(\delta^2/a) & k_p\tan(k_pL) + O(\delta^2/a) & O(\delta/a) \\ O(\delta/a) & O(\delta/a) & O(1/a) \end{bmatrix} \begin{bmatrix} 1 \\ A_p \\ A_v \end{bmatrix}. \tag{29}$$

In lowest order, the solution is $A_p^{(0)} = A_v^{(0)} = 0$ and $k_d^{(0)} \times \tan(k_d^{(0)}L) = 0$. This equation has many solutions (i.e., $k_d^{(0)}L = m\pi$), but we focus on the $m = 0$ solution, since this corresponds to the usual (i.e., $k = 0$) diocotron mode of an infinitely long column. The zero-order frequency of this mode (i.e., the D_0 mode) is given by $\omega^{(0)} = \omega_d$.

In the next order, the third row of Eq. (29) implies that $A_v^{(1)} \sim \delta$ and the second row implies that $A_p^{(1)} \sim \delta^2/[\delta^2 + ak_p \times \tan(k_pL)]$. Barring accidental degeneracy (see Sec. VID), $k_p\tan(k_pL)$ is not zero for the same value of the frequency for which $k_d\tan(k_dL)$ is zero. For $k_pa \sim \delta$ [see Eq. (27)], the expression for $A_p^{(1)}$ reduces to $A_p^{(1)} \sim \delta^2/[\delta^2 + \delta\tan(\delta L/a)]$; so $A_p^{(1)}$ is of order a/L for $\delta < a/L$ and of order δ for $\delta > a/L$. Using $A_p^{(1)}$ and $A_v^{(1)}$ in the first row yields $\omega^{(1)}\partial/\partial\omega[k_d\tan(k_dL)] \sim \delta^2/a$ which determines the correction $\omega^{(1)}$ to the frequency.

Returing to the full matrix equation, we obtain in lowest order: $0 = D_{dd} = k_{ld}^{(0)}\tan(k_{ld}^{(0)}L)$ or $\omega^{(0)} = \omega_d$. The components $A_{v_j}^{(1)}$ are given by

$$S_{v_jd} + \sum_{v_j} M_{v_iv_j}A_{v_j}^{(1)} = 0,$$

or

$$A_{v_i}^{(1)} = -\sum_{v_j}(M_{vv})_{v_iv_j}^{-1}S_{v_jd}, \tag{30}$$

where the matrix elements on the right-hand side are to be evaluated at the frequency $\omega^{(0)}$. The components of $A_{p_i}^{(1)}$ are given by

$$S_{p_id} + D_{p_ip_i}A_{p_i}^{(1)} + \sum_j S_{p_iv_j}A_{v_j}^{(1)} = 0,$$

or

$$A_{p_i}^{(1)} = \frac{-S_{p_id} - \Sigma_j S_{p_iv_j}A_{v_j}^{(1)}}{D_{p_ip_i}}, \tag{31}$$

and the frequency correction is given by

$$\left(\frac{dD_{dd}}{d\omega^{(0)}}\omega^{(1)} + S_{dd}\right) + \sum_{v_i} S_{dv_i}A_{v_i}^{(1)} = 0,$$

or

$$\omega^{(1)} = \frac{-S_{dd} - \Sigma_{v_i}S_{dv_i}A_{v_i}^{(1)}}{dD_{dd}/d\omega^{(0)}}. \tag{32}$$

From Eqs. (25), (26), and (14) one can easily show that for $\delta \ll 1$, $S_{DD}(\omega^0)$ can be written as $(1/a)(\omega_r^2/\omega_p^2)F_l(a/R)$, where $F_l(a/R)$ is a function of l and a/R [F_l is of order unity and hence this expession for $S_{DD}(\omega^0)$ is consistent with Eq. (28)]. A similar expression can be written for the second term in the numerator in Eq. (32). Also, from Eqs. (25), (26), and (A6) it follows that

$$\frac{dD_{dd}}{d\omega^{(0)}} = -\frac{2l(l+1)}{\omega_c}\frac{L}{a^2}.$$

Substituting these expressions in Eq. (32) and using Eq. (A5) we obtain

$$\omega^{(1)} = f_l(a/R)|\omega^{(0)} - l\omega_r|(a/L), \tag{33}$$

where $f_l(a/R)$ is a quantity which depends only on l and a/R.

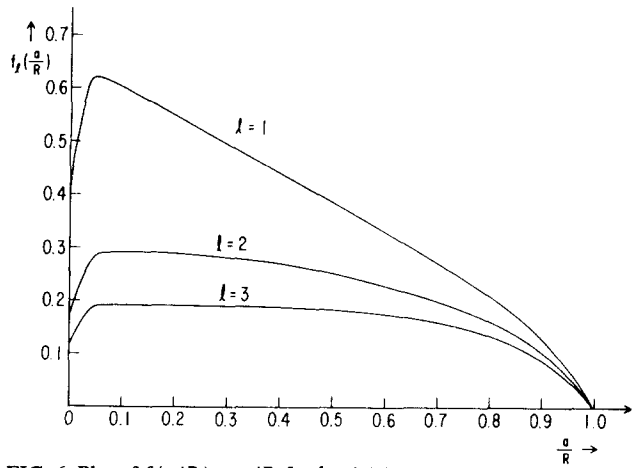

FIG. 6. Plot of $f_l(a/R)$ vs a/R for $l = 1,2,3$.

The dependence of f_l on a/R is plotted in Fig. 6 for the cases $l = 1,2,3$ with $\delta = 0.01$. The knee in the graphs, which is a consequence of the finite value of δ, moves to the left and disappears as $\delta \to 0$.

Since $\omega^{(1)} > 0$, the corresponding k_{ld} is imaginary. From the diocotron branch dispersion relation (A6), it follows that

$$k_{ld} = i\left(\frac{2l(l+1)|l\omega_r - \omega_d| f_l(a/R)}{\omega_c aL}\right)^{1/2}. \tag{34}$$

The fact that the wavenumber is imaginary simply means that the z dependence of the diocotron basis function goes over to $\cos(k_{ld}z) = \cosh(|k_{ld}|z)$. The magnitude of the wavenumber is of order $|k_{ld}| \sim (\delta^2 L/a)^{1/2}(1/L)$. This is in accord with recent experimental results[5] where the parameters were such that $a/L \sim \delta$ and where the measured axial phase variation of the $l = 1$ diocotron mode corresponded to an effective wavelength that was much larger than the plasma length (i.e., $|k| \ll 1/L$).

B. P modes

We call the mode which is predominantly the nth plasma-type basis function with axial wavenumber ($k = m\pi/L$) the P_{nm} mode. The perturbation solution for this mode proceeds parallel to that for the D_m modes. In zero order, we find $A_{n'} = \delta_{n'p_n}$ and $0 = D_{p_n p_n} = k_{lp_n}\tan(k_{lp_n}L)$ or $\omega^{(0)} = \omega_{p_n}$ ($k = m\pi/L$). The first-order corrections to A are given by

$$\begin{aligned} A^{(1)}_{v_i} &= -\sum_{v_j}(M_{vv})^{-1}_{v_i v_j} S_{v_j p_n}, \\ A^{(1)}_{d} &= \frac{-S_{dp_n} - \Sigma_{v_j} S_{dv_j} A^{(1)}_{v_j}}{D_{dd}}, \\ A^{(1)}_{p_{n'}} &= \frac{-S_{p_{n'}p_n} - \Sigma_{v_j} S_{p_{n'}v_j} A^{(1)}_{v_j}}{D_{p_{n'}p_{n'}}}, \end{aligned} \tag{35}$$

where the matrix elements on the right-hand sides are evaluated at $\omega^{(0)}$. The correction to the frequency is given by

$$\omega^{(1)} = \frac{-S_{p_n p_n} - \Sigma_{v_i} S_{p_n v_i} A^{(1)}_{v_i}}{dD_{p_n p_n}/d\omega^{(0)}}. \tag{36}$$

The $m = 0$ solutions are excluded from our theory. For these solutions, $\omega^{(0)} - l\omega_r = 0$ and all the particles are at resonance with the field. One expects these modes to be associated with heavy damping and nonlinearity. Also, there is an infinite degeneracy, with all the P_{no} modes having the same frequency $\omega^{(0)} = l\omega_r$. The present simple treatment is not capable of handling these difficulties.

C. Modes with odd parity

For modes which are odd under reflection in the $z = 0$ plane, the eigenmode expansion in region ii is replaced by

$$\phi_{ii}(r,z,l,\omega) = \sum_n A_n \phi_{lr}(\omega,r)\frac{\sin[k_{ln}(\omega)z]}{\sin[k_{ln}(\omega)L]}, \tag{37}$$

and the $\tan(k_{ln}L)$ terms in matrix M are replaced by $-\cot(k_{ln}L)$. In zero order, one finds modes that are predominantly made up of a diocotron-type basis function $\phi_{ld}(r)\sin(k_{ld}z)$, or of a plasma-type basis function $\phi_{lp_n}(r) \times \sin(k_{lp_n}z)$, where the wavenumber is equal to $(m + \frac{1}{2})\pi L$. We call these $D_{m+\frac{1}{2}}$ modes and $P_{n,m+\frac{1}{2}}$ modes. One can calculate the eigenvectors A and the eigenfrequencies for these modes as was done above.

D. Degeneracy

In the above treatment, it was assumed that $D_{dd}(\omega)$ and $D_{p_n p_n}(\omega)$ were not simultaneously zero. If this occurs, one must resort to degenerate perturbation theory. We set $A_d = 1$ and $A_n = \alpha$, where α is not necessarily small. In first order in δ, we obtain

$$S_{v_i d} + \alpha S_{v_i p_n} + \sum_{v_j} M_{v_i v_j} A^{(1)}_{v_j} = 0,$$

or

$$A^{(1)}_{v_j} = -\sum_{v_i}(M_{vv})^{-1}_{v_j v_i}(S_{v_i d} + \alpha S_{v_i p_n}), \tag{38}$$

where the matrix elements are evaluated at the frequency $\omega^{(0)}$ which is such that $D_{dd}(\omega^{(0)}) = D_{p_n p_n}(\omega^{(0)}) = 0$. In second order, we obtain

$$\left(\frac{dD_{dd}}{d\omega^{(0)}}\omega^{(1)} + S_{dd}\right) + \alpha S_{dp_n} + \sum_{v_i} S_{dv_i} A^{(1)}_{v_i} = 0, \tag{39}$$

$$\alpha\left(\frac{dD_{p_n p_n}}{d\omega^{(0)}}\omega^{(1)} + S_{p_n p_n}\right) + S_{p_n d} + \sum_{v_i} S_{p_n v_i} A^{(1)}_{v_i} = 0. \tag{40}$$

The condition that there is a solution for α leads to a quadratic equation in $\omega^{(1)}$. There are two solutions $\omega^{(1)}_a$ and $\omega^{(1)}_b$, and for these values of ω, one obtains α_a and α_b. Since these values of α tend to be of order unity, the modes are not diocotron-like or plasma-like, but rather have significant portions of both components.

The condition for degeneracy of the D_0 mode and the P_{nm} mode is $k_{lp_n}(\omega_d) = m\pi/L$. For $m \ll L/a$, this can be rewritten as

$$\frac{\omega_c}{\omega_p} = \frac{1}{2}\left(\frac{j_{l+1,n}}{m\pi}\right)\left(\frac{L}{a}\right)\left[1 - \left(\frac{a}{R}\right)^{2l}\right], \tag{41}$$

where use has been made of the fact that the right-hand side of Eq. (A2) is equal to l for $\omega = \omega_d$. As a numerical example, we note that for $l = m = n = 1$, $L/a = 40$, and $R/a = 2$ the condition reduces to $\omega_c/\omega_p \simeq 25$.

Thus far, we have considered exact degeneracy. In fact, the modes are mixed over a small range of parameters near the parameters for exact degeneracy. This occurs when $D_{p_n p_n}(\omega_d) \lesssim \delta^2/a$, which can be rewritten as $\Delta\omega/\omega_r \lesssim \delta^2(L/a)(1/m\pi)^2$, where $\Delta\omega$ is the mismatch between the zero order frequencies. As an example, suppose that the magnetic field is the only variable quantity and that $\omega_c^{(0)}$ is the value of the cyclotron frequency which produces exact degeneracy. When the magnetic field is swept, the Doppler-shifted frequency of the plasma mode does not change but that of the diocotron mode varies as $\omega_d - l\omega_r \propto \omega_r \propto \omega_c^{-1}$; so the range of $\Delta\omega_c$ over which the modes are mixed is given by $\Delta\omega_c/\omega_c^{(0)} \sim \delta^2(L/a)(1/m\pi)^2$.

The mixing can be detected by measuring the phase difference between the center and the end of the column. The phase at the end relative to that at the center is given by

$$\frac{\phi(z=L)}{\phi(z=0)} = \begin{cases} 1, & \text{for the } D_0 \text{ wave,} \\ (-1)^m, & \text{for the } P_{nm} \text{ wave,} \\ & (k_{lp_n}L = m\pi). \end{cases} \tag{42}$$

When the modes are degenerate, the above ratio becomes $[1+(-1)^m\alpha]/(1+\alpha)$. If the receiver tracks ω_d as the magnetic field ω_c varies, one would see the relative phase change from 1 to $[1+(-1)^m\alpha]/(1+\alpha)$ very sharply at $\omega_c = \omega_c^{(0)}$.

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APPENDIX: DISPERSION CURVES FOR AN INFINITELY LONG COLUMN

In this appendix, we shall make a study of the solutions of Eq. (11) (and the corresponding eigenfunctions) in various limits. Figure 3 presents a *qualitative* summary of the results. Figure 4 presents these curves drawn to scale in the region of interest to us.

For the sake of easy reference, we shall rewrite Eq. (11):

$$ka\frac{I_l'(ka)K_l(kR)-K_l'(ka)I_l(kR)}{I_l(ka)K_l(kR)-K_l(ka)I_l(kR)} - \alpha ka\frac{J_l'(\alpha ka)}{J_l(\alpha ka)} = \frac{2l\omega_r}{\omega - l\omega_r}, \tag{A1}$$

where $\alpha^2 = \omega_p^2(0)/(\omega - l\omega_r)^2 - 1$. From the form of Eq. (A1), it is clear that the equation is invariant under the transformations $(\omega,l)\to(-\omega,-l)$ and $k\to -k$.

In the limit $ka \ll 1$, Eq. (A1) reduces to

$$\alpha ka\frac{J_l'(\alpha ka)}{J_l(\alpha ka)} = l\frac{(a/R)^{2l}+1}{(a/R)^{2l}-1} - \frac{2l\omega_r}{\omega - l\omega_r}. \tag{A2}$$

Equation (A2) has two types of solutions. For the plasma-type solutions, $\omega - l\omega_r \simeq 0$. This means that $J_l(\alpha ka)\simeq 0$ or

$$\alpha ka \simeq j_{ln}, \tag{A3}$$

or

$$\omega - l\omega_r = \pm\,[\omega_p(0)/j_{ln}]\,ka, \tag{A4}$$

where j_{ln} is the nth root of $J_l(x) = 0$. The corresponding wave potential $\simeq J_l(j_{ln}r/a)$ for $r<a$ and $\simeq 0$ for $r \geqslant a$. These plasma-type solutions exist even when the plasma fills the tube.

For the second type of solution (the diocotron-type solution), $\omega - l\omega_r$ does not go to zero as $ka\to 0$. We can then use the small argument expansion for $J_l(\alpha ka)$ and obtain

$$\omega = \omega_r[l + (a/R)^{2l} - 1] \equiv \omega_d. \tag{A5}$$

If we include terms in k^2, we obtain

$$\omega = \omega_d - [\omega_c/2l(l+1)](ka)^2. \tag{A6}$$

The diocotron wave potential has the radial dependence (for $ka \ll 1$)

$$\phi \simeq \begin{cases} r^l, & r<a \\ \dfrac{a^{2l}}{r^l}\dfrac{r^{2l}-R^{2l}}{a^{2l}-R^{2l}}, & r>a. \end{cases}$$

When $|\omega - l\omega_r|$ is much larger than ω_r, the right-hand side of Eq. (A1) (which represents the effect of the surface charge at $r=a$) can be ignored. Since the left-hand side is an even function of $\omega - l\omega_r$, this implies that the lower branches of Fig. 3 are reflections of the upper branches about the line $\omega = l\omega_r$. In particular, the diocotron branch (d) is the reflection of the $n=1$ upper plasma branch (p_{1U}). In this regime, a plot of $|\omega|$ vs ka would show a lower branch running parallel to its corresponding upper branch; experimentally plotted dispersion curves exhibit this feature. The eigenfunctions of the two branches would be identical for any given value of ka.

For $ka \gg 1$, the first term of Eq. (A1) asymptotes to ka. So $J_l(\alpha ka)\to 0$ or

$$\alpha ka = j_{ln},$$

$$\omega - l\omega_r \simeq \pm\frac{\omega_p(0)}{[1+j_{ln}^2/(ka)^2]^{1/2}} \to \pm\,\omega_p(0). \tag{A7}$$

So all the branches of Fig. 3 asymptote to $l\omega_r \pm \omega_p(0)$. Comparison of Eqs. (A3) and (A7) shows that the eigenfunction of any upper branch is the same for $ka=0$ and $ka=\infty$. On the other hand, the eigenfunction of any lower branch is compressed in r with increasing ka; at the limit $ka=\infty$, the eigenfunction will have an extra node in the interval $(0,R)$. In this limit, the diocotron eigenfunction is $J_l[j_{l1}(r/a)]$ for $r \leqslant a$ and is zero for $r>a$.

We also note here the effect of varying a/R on the dispersion curves of Fig. 3. If we let $R\to a$, the lower branches move to the right (while the upper branches remain relatively unchanged) until at $R=a$, the diocotron branch takes the place of the first plasma branch and starts at $\omega = l\omega_r$; the nth lower branch takes the place of the $(n+1)$th.

In addition to the above solutions which have real k, Eq. (A1) allows solutions with purely imaginary k. These solutions are indicated in Fig. 3 by dotted curves. The first of these extends from $\omega = \omega_d$ (where $|ka|=0$) to $\omega = l\omega_r$ (where $|ka| = 2l\omega_r/\omega_p$). Since the eigenfunction for this branch has the characteristics of the diocotron, we shall classify it as a "diocotron-type" solution. The other solutions with imaginary k (the "vacuum-type" solutions) extend on either side of $\omega = l\omega_r$, where $|k_n| \simeq n\pi/(R-a)$; the value of $|k_n|$ at $\omega = l\omega_r \pm \omega_p(0)$ is $\simeq (n-\frac{1}{2})\pi/(R-a)$.

The solutions with imaginary k are ignored in a treatment of an infinite column because they increase without limit as z approaches $+\infty$ or $-\infty$. However, this consideration does not apply in the case of a finite length column and so these solutions have to be retained. As shown in Sec. IV, they are necessary components of a complete set of basis functions.

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