# Centrifugal separation of a multispecies pure ion plasma

T. M. O'Neil

*Department of Physics, University of California at San Diego, La Jolla, California 92093*
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Consider an unneutralized column of ions (a pure ion plasma) confined by an axial magnetic field. Because of space charge, there is a large radial electric field and a consequent rotation of the plasma column. For a multispecies ion plasma, the rotation tends to produce centrifugal separation of the plasma into its component species. Self-consistent thermal equilibrium states which exhibit various degrees of separation are discussed.

## I. INTRODUCTION

Recent experiments have achieved long confinement times for pure electron plasmas[1] and similar experiments are underway for pure ion plasmas.[2] These plasmas are columns with radial confinement provided by an axial magnetic field and axial confinement provided by electrostatic fields. Since the plasmas are nonneutral, there is a large radial electric field and a consequent rotation of the plasma. One can imagine producing a multispecies pure ion plasma, and, for such a plasma, the rotation will tend to produce centrifugal separation of the plasma into its component species.

To understand the mechanism which produces the separation, consider a multispecies ion plasma, which is in dynamical equilibrium, executing laminar rotational flow. Since the different species are characterized by different values of charge and mass, say $e_j$ and $m_j$, the species tend to rotate at different rates, at least for arbitrary density and pressure profiles. Because of the different rotation rates, collisional effects and possibly collective effects produce a momentum transfer in the $\theta$ direction between the species. Here, $(r, \theta, z)$ are the usual cylindrical coordinates. The equal and opposite momentum transfer between two species produces an inward radial flux of one species and an outward radial flux of the other, that is, the two species tend to separate. The plasma evolves until the density and pressure profiles are such that the plasma rotates as a rigid body. Alternatively, one may say that the plasma evolves until it comes to thermal equilibrium, one characteristic of thermal equilibrium being rigid body rotation.[3-5] This paper provides a theoretical discussion of the separation as exhibited in the thermal equilibrium states.

The degree of separation depends on the relative size of various scale lengths. A class of scale lengths, $l_{ij}(r)$, may be defined as

$$[l_{ij}(r)]^{-1} = \frac{d}{dr}\left[e_i \left|\frac{m_i}{e_i} - \frac{m_j}{e_j}\right| \left(\frac{\omega^2 r^2}{2kT}\right)\right], \tag{1}$$

where $\omega$ is the rotation frequency of the plasma, $k$ is Boltzmann's constant, and $T$ is the plasma temperature. For a plasma of radius $a$, centrifugal separation is a weak effect provided $l_{ij}(a) \gg a$ [see Eq. (23)]. This is simply a requirement that the difference between the centrifugal potentials for any two species be less than $kT$. If the scale lengths $l_{ij}$ are made smaller, say by reducing $kT$, the centrifugal separation becomes more pronounced. For the separation to be complete, that is, for the species to be arranged in separate concentric rings, it is necessary that the $l_{ij}$'s be small compared with the characteristic Debye length. All of the analysis assumes that the Debye length is small compared with the geometrical scale lengths such as the plasma radius or the thickness of a ring.

In Sec. II, the problem of finding thermal equilibrium distributions that are consistent with Poisson's equation is defined. In Sec. III, solutions are presented for the case of partial separation, and in Sec. IV, solutions are presented for the case of complete separation.

Before proceeding to the analysis, a word or two concerning possible applications of this work may be in order. It has been suggested that a nonneutral ion plasma might be a useful system in which to study the fusion of exotic fuels.[6] Since very long particle and energy confinement times would be required, the plasma would approach thermal equilibrium. Of course, centrifugal separation of the reacting species would reduce the reaction rates. It is unlikely that nonneutral plasmas would be useful for isotope separation, since the density of a nonneutral plasma is limited to rather low values (the Brillouin limit[7]). A more promising scheme for this purpose involves the use of a higher density nearly neutral plasma, which rotates because of a slight charge imbalance.[8]

## II. THERMAL EQUILIBRIUM

The thermal equilibrium ion distribution for the $j^{\text{th}}$ species is given by[3-5]

$$f_j = n_j(0)\left(\frac{m_j}{2\pi kT}\right)^{3/2} \exp\left(\frac{-1}{kT}(H_j + \omega P_{\theta_j})\right), \tag{2}$$

where

$$H_j = m_j v^2/2 + e_j \phi(r), \tag{3}$$

$$P_{\theta_j} = m_j v_\theta r + (e_j/c) A_\theta(r) r \tag{4}$$

are the energy and canonical angular momentum, respectively, for an ion of species $j$. These quantities enter the distribution on equal footing for a cylindrically symmetric confinement geometry. The quantities $\phi(r)$ and $A_\theta(r)$ are the electric potential and the $\theta$ component of the vector potential, respectively. For simplicity, we neglect the $z$ dependence in these potentials, that is, we treat the column as if it were infinitely long. For a uniform axial magnetic field, the vector potential is given by $A_\theta(r) = Br/2$, the diamagnetic field being negligible for the low ion densities and velocities that we have in mind here.

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By using Eqs. (3) and (4) and the relations $A_\theta = Br/2$, distribution (2) can be written as

$$f_i = n_j(r)\left(\frac{m_j}{2\pi kT}\right)^{3/2} \exp \frac{-m_j}{2kT}(\mathbf{v}+\omega r\hat{\theta})^2, \quad (5)$$

$$n_j(r) = n_j(0)\exp[-\psi_j(r)], \quad (6)$$

$$\psi_j(r) = \frac{e_j}{kT}\left(\phi(r) - \frac{m_j}{e_j}\frac{\omega^2 r^2}{2} + \frac{B\omega r^2}{2c}\right). \quad (7)$$

From the velocity dependence, one can see that the plasma rotates as a rigid body with angular frequency $-\omega$. The density distribution [i.e., $n_j(r) \equiv \int d^3\mathbf{v} f_j(r,\mathbf{v})$] is determined by three potentials: the electric potential, the centrifugal potential, and the potential associated with the electric field induced by rotation through the magnetic field [i.e., $B\omega r^2/2c$]. For large enough $B$, this last term forces the density to zero at large $r$, that is, it provides the radial confinement. Centrifugal separation comes into play when different species have different values of $m_j/e_j$. In the density distribution, the coefficient $n_j(0)$ represents the density of species $j$ at $r=0$, provided we choose the boundary condition $\phi = 0$ at $r=0$.

The electric potential is determined by Poisson's equation

$$\frac{1}{r}\frac{d}{dr} r \frac{d\phi}{dr} = -\sum_j 4\pi e_j n_j(r), \quad (8)$$

and the boundary conditions $\phi = d\phi/dr = 0$ at $r = 0$. The first of these is an arbitrary choice and the second follows from cylindrical symmetry. For future reference, we rewrite Poisson's equation in the form

$$\frac{1}{r}\frac{d}{dr} r \frac{d\psi_1}{dr} = \frac{2m_1\omega(\Omega_1-\omega)}{kT} - \sum_j \frac{4\pi e_1 e_j n_j(0)}{kT} \times \exp\left[-\frac{e_j}{e_1}\psi_1 + e_j\left(\frac{m_j}{e_j} - \frac{m_1}{e_1}\right)\left(\frac{\omega^2 r^2}{2kT}\right)\right], \quad (9)$$

where $\Omega_1 = e_1 B/m_1 c$ and use has been made of Eqs. (6) and (7). The boundary conditions satisfied by $\psi_1$ are $\psi_1 = d\psi_1/dr = 0$ at $r=0$.

## III. PARTIAL SEPARATION

In this section, solutions of Eq. (9) are obtained under the assumption that the left-hand side of the equation is small compared with either of the two terms on the right-hand side, at least within the body of the plasma. We will see that the two terms on the right-hand side are each of order $1/\lambda^2$, where $\lambda$ is the Debye length, and that $\psi_1(r)$ changes on the scale of the $l_{ij}$'s [see Eq. (1)]. Thus, in this section we assume that the $l_{ij}$'s are large compared with $\lambda$. In the next section, the opposite assumption is made.

It is convenient to express $\psi_1$ as the sum of two terms $\psi_1(r) = \eta(r) + \sigma(r)$, where $\eta(r)$ and $\sigma(r)$ satisfy the differential equations

$$\frac{1}{r}\frac{d}{dr} r \frac{d\eta}{dr} = \frac{2m_1\omega(\Omega_1-\omega)}{kT} - \sum_j \frac{4\pi e_1 e_j n_j(0)}{kT} \times \exp\left[-\frac{e_j}{e_1}\eta + e_j\left(\frac{m_j}{e_j} - \frac{m_1}{e_1}\right)\left(\frac{\omega^2 r^2}{2kT}\right)\right], \quad (10)$$

$$\frac{1}{r}\frac{d}{dr} r \frac{d\sigma}{dr} = -\sum_j \frac{4\pi e_1 e_j n_j(0)}{kT} \times \exp\left[-\frac{e_j}{e_1}\eta + e_j\left(\frac{m_j}{e_j} - \frac{m_1}{e_1}\right)\left(\frac{\omega^2 r^2}{2kT}\right)\right] \times \left[\exp\left(-\frac{e_j}{e_1}\sigma\right) - 1\right]. \quad (11)$$

The sum of these two equations reproduces Eq. (9).

We solve Eq. (10) by a perturbation expansion $\eta = \eta^{(0)} + \eta^{(1)} + ---$, treating the second derivative on the left-hand side as first order in the smallness parameter. To zero order, Eq. (10) reduces to

$$0 = \frac{2m_1\omega(\Omega_1-\omega)}{kT} - \sum_j \frac{4\pi e_1 e_j n_j(0)}{kT} \times \exp\left[-\frac{e_j}{e_1}\eta^{(0)} + e_j\left(\frac{m_j}{e_j} - \frac{m_1}{e_1}\right)\left(\frac{\omega^2 r^2}{2kT}\right)\right]. \quad (12)$$

The first order equation is

$$\frac{1}{r}\frac{d}{dr} r \frac{d\eta^{(0)}}{dr} = -\sum_j \frac{4\pi e_1 e_j n_j(0)}{kT} \times \exp\left[-\frac{e_j}{e_1}\eta^{(0)} + e_j\left(\frac{m_j}{e_1} - \frac{m_1}{e_1}\right)\left(\frac{\omega^2 r^2}{2kT}\right)\right] \times \left[\exp\left(-\frac{e_j}{e_1}\eta^{(1)}\right) - 1\right]. \quad (13)$$

This equation may be simplified by setting

$$\exp\left(-e_j\eta^{(1)}/e_1\right) \simeq 1 - e_j\eta^{(1)}/e_1, \quad (14)$$

and by using the approximation

$$\frac{1}{\lambda^2(r)} = \frac{1}{kT}\sum_j 4\pi e_j^2 n_j(r) \simeq \frac{1}{kT}\sum_j 4\pi e_j^2 n_j(0) \times \exp\left[-\frac{e_j}{e_1}\eta^{(0)} + e_j\left(\frac{m_j}{e_j} - \frac{m_1}{e_1}\right)\left(\frac{\omega^2 r^2}{2kT}\right)\right]. \quad (15)$$

In Eq. (15), we have anticipated the result that $\sigma(r)$ is exponentially small over nearly the whole radius of the plasma. With the aid of Eqs. (14) and (15), Eq. (13) reduces to the simple form

$$\eta^{(1)}(r) \simeq \lambda^2(r)\frac{1}{r}\frac{d}{dr} r \frac{d\eta^{(0)}}{dr}. \quad (16)$$

Since $\eta(r)$ satisfies the same differential equation as $\psi_1$ [i.e., Eq. (10) is the same as Eq. (9)], one might think that $\psi_1 = \eta$ provides an adequate solution of Eq. (9). However, the perturbation solution for $\eta(r)$ does not satisfy the proper boundary conditions (i.e., $\psi_1 = d\psi_1/dr = 0$ at $r=0$). The function $\sigma(r)$ ($\psi_1 = \eta + \sigma$) was introduced to allow the boundary conditions to be satisfied. Since

$\eta=\eta(r^2)$, it follows that $d\eta/dr=0$ at $r=0$. Thus, we must require that $\sigma=-\eta(0)$ and $d\sigma/dr=0$ at $r=0$.

If we restrict our attention to plasmas of radius large compared with the Debye length, $\sigma(r)$ is small compared with unity over nearly the whole radius of the plasma. Thus, $\exp(-e_j\sigma/e_1)$ may be approximated by $1-e_j\sigma/e_1$, and Eq. (11) may be reduced to the simple form

$$\frac{1}{r}\frac{d}{dr}r\frac{d\sigma}{dr}-\frac{1}{\lambda^2(r)}\sigma=0\,. \tag{17}$$

Near $r=0$, the solution is of the form

$$\sigma(r)=-\eta(0)I_0[r/\lambda(0)]\,, \tag{18}$$

where $I_0(x)$ is a Bessel function of imaginary argument. For larger $r$, one expects a WKB-like solution of the form

$$\sigma(r)\sim-\eta(0)\exp\int_0^r\frac{dr'}{\lambda(r')}. \tag{19}$$

If parameters are adjusted so that $-\eta(0)$ is an exponentially small (and positive) number, $\sigma(r)$ becomes of order unity at a value of $r$ that is large compared with the Debye length. This value of $r$ defines the outer edge of the plasma, since the $n_j(r)$ are forced to zero as $\sigma(r)$ grows past unity. Although Eq. (17) was derived under the assumption that $\sigma<1$, inspection of Eq. (11) shows that $\sigma$ continues to grow past $\sigma=1$, with the rate of growth continually increasing. It should be emphasized that the details of the solution for $\sigma(r)$ are not important. The important observations are that the parameters must be adjusted so that $\eta(0)\simeq 0$ and that $\sigma(r)$ may be neglected to within a few Debye lengths of the edge of the plasma.

As an illustration of these results, let us consider the case where centrifugal separation is sufficiently weak so that the exponentials in Eq. (12) may be Taylor expanded. Equations (12) and (16) then reduce to

$$\eta^{(0)}=\frac{\sum_j 4\pi e_j e_1 n_j(0)-2m_1\omega(\Omega_1-\omega)}{\sum_j 4\pi e_j^2 n_j(0)}\,, + \frac{\sum_j 4\pi e_j^2 n_j(0)e_1(m_j/e_j-m_1/e_1)(\omega^2r^2/2kT)}{\sum_j 4\pi e_j^2 n_j(0)}\,, \tag{20}$$

$$\eta^{(1)}=\frac{\sum_j 4\pi e_j^2 n_j(0)e_1(m_j/e_j-m_1/e_1)(2\omega^2)}{(\sum_j 4\pi e_j^2 n_j(0))^2}\,. \tag{21}$$

Thus, the condition $0\simeq\eta^{(0)}(0)+\eta^{(1)}(0)$ takes the form

$$\sum_j 4\pi e_j n_j(0)\simeq\frac{\sum_j e_j^2 n_j(0)(2\omega B/c-2m_j\omega^2/e_j)}{\sum_j e_j^2 n_j(0)}\,. \tag{22}$$

In the spirit of the Taylor expansions, the change in density $\Delta n_1(r)\equiv n_1(r)-n_1(0)$ is given by $\Delta n_1(r)\simeq-n_1(0)\psi_1(r)$, and in the region where $\sigma(r)$ is negligible $\psi_1(r)\simeq\eta^{(0)}(r)+\eta^{(1)}(r)$. Thus, the change in density is given by

$$\frac{\Delta n_1(r)}{n_1(0)}=\frac{\sum_j e_j^2 n_j(0)e_1(m_1/e_1-m_j/e_j)(\omega^2r^2/2kT)}{\sum_j e_j^2 n_j(0)}\,. \tag{23}$$

This is simply a weighted average of a difference in centrifugal potentials [i.e., $e_1(m_1/e_1-m_j/e_j)(\omega^2r^2/2kT)$] with the weighting function given by $e_j^2 n_j(0)$. To obtain $\Delta n_i(r)/n_i(0)$, one need only interchange the subscripts 1 and $i$ in Eq. (23). Of course, the analysis assumes that $|\Delta n_i(r)/n_i(0)|<1$ for all $i$.

The solution is valid out to within a few Debye lengths of the value of $r$, say $r=a$, where $\sigma(r)$ reaches unity. As $\sigma(r)$ passes through unity, the densities $n_j(r)$ drop to zero. Since $\sigma(r)$ enters the expression for $n_j(r)$ through the factor $\exp[-e_j\sigma(r)/e_1]$, the densities drop to zero sequentially in order of decreasing charge. The region where the densities drop to zero is scaled in terms of the Debye length, which is assumed to be small compared with the plasma radius (i.e., $\lambda\ll a$).

The parameters $n_j(0)$, $a$, and $\omega$ are determined by Eq. (22), the number of particles of species $j$ per unit length

$$N_j=\int_0^\infty 2\pi r\,dr\,n_j(r)\simeq\int_0^a 2\pi r\,dr[n_j(0)+\Delta n_j(r)]\,, \tag{24}$$

and the total canonical angular momentum per unit length

$$P_\theta=\sum_j\int_0^\infty 2\pi r\,dr\int d^3\mathrm{v}f_j(r,\mathrm{v})\left(m_jv_\theta r+\frac{e_jBr^2}{2c}\right) \simeq\sum_j m_j\left(\frac{\Omega_j}{2}-\omega\right)\int_0^\infty 2\pi r\,dr\,r^2[n_j(0)+\Delta n_j(r)]\,. \tag{25}$$

The $N_j$'s must be specified for all $j$.

When the centrifugal separation is sufficiently strong [i.e., $|\Delta n_i|/n_i(0)\geq 1$ for some $i$], the exponentials in Eq. (12) cannot be Taylor expanded. One is then forced to solve a transcendental equation to find $\eta^{(0)}(r)$. Nevertheless, this an easier task than solving the original differential equation. For the simple case of a two-component plasma (say with $m_2/e_2>m_1/e_1$), one can easily construct solutions in which there is a transition between a region where $n_1\gg n_2$ to a region where $n_1\ll n_2$. In the first region, $n_1(r)$ is very nearly constant and $n_2(r)$ grows exponentially {i.e., $n_2(r)\sim\exp[e_2(m_2/e_2-m_1/e_1)(\omega^2r^2/2kT)]$}. In the second region $n_2(r)$ is very nearly constant and $n_1(r)$ decreases exponentially {i.e., $n_1(r)\sim\exp[e_1(m_1/e_1-m_2/e_2)(\omega^2r^2)]$}. In other words, the density variation associated with the transition is exponential in nature. Of course, the function $\sigma(r)$ reaches unity at some value of $r$ and truncates the density of both species.

## IV. COMPLETE SEPARATION

In this section, we obtain solutions in which the various species are located in concentric rings, the density in the gap between two rings being exponentially small. The solutions are based on the assumption that $l_{j,i}\ll\lambda_j$, where $l_{j,i}$ is defined in Eq. (1) and $\lambda_j$ is the Debye length associated with species $j$ (i.e., $\lambda_j^2=kT/4\pi e_j^2 n_j$). The subscript $i$ refers to either of the two species adjacent to species $j$. If to be specific, we stipulate the ordering $m_{k+1}/e_{k+1}>m_k/e_k$, then $i$ takes the values $j-1$ and $j+1$.

We will find that $n_j(r)$ has the value $n_j$ in the region from $r=a_j$ to $r=b_j$ and drops to zero at the boundaries of this region. The drop in $n_j(r)$ occurs on the scale $\lambda_j$; of course, $a_1=0$ and $a_{j+1}>b_j$. The values of $a_j$, $b_j$, and $n_j$ will emerge from the analysis.

In the region $0<r<b_1$, only the charge density due to species 1 need be retained, and Eq. (9) reduces to

$$\frac{1}{r}\frac{d}{dr}r\frac{d\psi_1}{dr}=\frac{2m_1\omega(\Omega_1-\omega)}{kT}-\frac{4\pi e_1^2 n_1(0)}{kT}\exp(-\psi_1)\,. \qquad (26)$$

It is convenient to denote the maximum value of $n_j(0)\exp(-\psi_j)$ by $n_j$. We will see that $\psi_1(r)$ increases monotonically from the origin, where $\psi_1(0)=0$, so $n_1=n_1(0)$. By introducing the Debye length, $\lambda_1$, and the parameter

$$\gamma_1=[2m_1\omega(\Omega_1-\omega)]/(4\pi n_1 e_1^2)-1\,, \qquad (27)$$

Eq. (26) can be written as

$$\lambda_1^2\frac{1}{r}\frac{d}{dr}r\frac{d\psi_1}{dr}=1-\exp(-\psi_1)+\gamma_1\,. \qquad (28)$$

In the region where $|\psi_1|<1$, the exponential may be Taylor expanded, and Eq. (28) reduced to

$$\lambda_1^2\frac{1}{r}\frac{d}{dr}r\frac{d\psi_1}{dr}-\psi_1=\gamma_1\,. \qquad (29)$$

The solution subject to the boundary conditions $\psi_1=d\psi_1/dr=0$ at $r=0$ is given by $\psi_1=\gamma_1[I_0(r/\lambda_1)-1]$. We are interested in the class of solutions where $0<\gamma_1\ll 1$. For these solutions, $\psi_1$ remains small for large values of $r/\lambda_1$, and the asymptotic form of the Bessel function may be used: $\psi_1\simeq\gamma_1(2\pi r/\lambda_1)^{-1/2}\exp(r/\lambda_1)$. We identify the radius $r=b_1$ as the radius where $\psi_1=1$. A few Debye lengths inside $r=b_1$, $\psi_1$ is exponentially small and $n_1(r)\simeq n_1$. A few Debye lengths outside $r=b_1$, $\psi_1(r)$ is large compared with unity and $n_1(r)$ is exponentially small. Although Eq. (29) is valid only for $\psi_1<1$, inspection of Eq. (28) shows that $\psi_1(r)$ continues to grow past $\psi_1=1$. Since $\lambda_1\ll b_1$, we obtain the two relations

$$N_1\simeq n_1\pi b_1^2\,, \qquad (30)$$

$$4\pi e_1^2 n_1\simeq 2m_1\omega(\Omega_1-\omega)\,. \qquad (31)$$

An equation for $\psi_2(r)$ is obtained by simply interchanging the subscripts 1 and 2 in Eq. (9). Let $r_2$ be the radius where $\psi_2(r)$ reaches its minimum value. By hypothesis this point lies in the range $a_2<r<b_2$. Introducing $\Delta\psi_2=\psi_2(r)-\psi_2(r_2)$ and $n_2=n_2(0)\exp[-\psi_2(r_2)]$ allows Eq. (9) (with the subscripts 1 and 2 interchanged) to be written as

$$\lambda_2^2\frac{1}{r}\frac{d}{dr}r\frac{d\Delta\psi_2}{dr}=1-\exp(-\Delta\psi_2)+\gamma_2\,. \qquad (32)$$

Here, only the charge density due to species 2 has been retained. We are interested in the class of solutions where $0<\gamma_2\ll 1$ and $\Delta\psi_2$ remains small over a range that is large compared with $\lambda_2$. Taylor expanding the exponential and approximating the $r$ derivatives by $d^2/dr^2$ yields the equation

$$\lambda_2^2\frac{d^2\Delta\psi_2}{dr^2}-\Delta\psi_2=\gamma_2\,, \qquad (33)$$

and using the boundary conditions $\Delta\psi_2=d\Delta\psi_2/dr=0$ at $r=r_2$ yields the solution

$$\Delta\psi_2=\gamma_2\{\cosh[(r-r_2)/\lambda_2]-1\}\,. \qquad (34)$$

We identify the points $r=a_2$ and $r=b_2$ as the points where $\Delta\psi_2=1$. The density $n_2(r)$ is very nearly equal to $n_2$ for $a_2<r<b_2$ and drops to zero on the scale of $\lambda_2$ at the boundaries. Since $\lambda_2\ll b_2$, we obtain the two relations

$$N_2=n_2\pi(b_2^2-a_2^2)\,, \qquad (35)$$

$$4\pi e^2 n_2=m_2\omega(\Omega_2-\omega)\,. \qquad (36)$$

Next, let us determine the relation between $b_1$ and $a_2$. Over most of the region $b_1<r<a_2$, the charge density is negligible, and Eq. (9) (with the subscripts 1 and 2 interchanged) reduces to

$$\frac{1}{r}\frac{d}{dr}r\frac{d\psi_2}{dr}=\frac{2m_2\omega(\Omega_2-\omega)}{kT}\,. \qquad (37)$$

Integrating from $r=b_1$ to $r=a_2$ yields

$$a_2\frac{d\psi_2}{dr}\bigg|_{a_2}=b_1\frac{d\psi_2}{dr}\bigg|_{b_1}+\frac{m_2\omega(\Omega_2-\omega)}{kT}(a_2^2-b_1^2)\,, \qquad (38)$$

and using the relation $\psi_2=(e_2/e_1)\psi-e_2(m_2/e_2-m_1/e_1)\times(\omega^2r^2/2kT)$ yields

$$a_2\frac{d\psi_2}{dr}\bigg|_{a_2}=\frac{e_2}{e_1}b_1\frac{d\psi_1}{dr}\bigg|_{b_1}-e_2\left(\frac{m_2}{e_2}-\frac{m_1}{e_1}\right)\frac{\omega^2b_1^2}{kT}+\frac{m_2\omega(\Omega_2-\omega)}{kT}(a_2^2-b_1^2)\,. \qquad (39)$$

The two derivative terms are of order $1/\lambda$; whereas, the other two terms are of order $1/T\sim 1/\lambda^2$. Since $\lambda$ is a small parameter, the two derivative terms may be neglected. The errors involved in neglecting the charge density at the edges of the region $b_1<r<a_2$ are also of order $1/\lambda$. Thus, we obtain the relation

$$a_2^2-b_1^2=\frac{e_2(m_2/e_2-m_1/e_1)\omega^2b_1^2}{m_2\omega(\Omega_2-\omega)}\,. \qquad (40)$$

By continuing this procedure to successive regions, one obtains the relations

$$N_j=n_j\pi(b_j^2-a_j^2)\,, \qquad (41)$$

$$4\pi e_j^2 n_j=2m_j\omega(\Omega_j-\omega)\,, \qquad (42)$$

$$a_{j+1}^2-b_j^2=\frac{e_{j+1}(m_{j+1}/e_{j+1}-m_j/e_j)\omega^2b_j^2}{m_{j+1}\,\omega(\Omega_{j+1}-\omega)}\,. \qquad (43)$$

For completeness, we note that the parameter $\omega$ is determined by the total canonical angular momentum

$$P_\theta=\sum_j m_j(-\omega+\Omega_j/2)n_j\pi(b_j^4-a_j^4)/2\,. \qquad (44)$$

The requirement for complete separation is that $\lambda_j$, $\lambda_{j+1}\ll a_{j+1}-b_j$. For the case where this is most restrictive (i.e., $a_{j+1}-b_j\ll a_{j+1}$, $b_j$), Eqs. (42) and (43) can be used to rewrite the inequality as $l_{j,i}\ll\lambda_j$, where $l_{i,j}^{-1}=d/dr\,e_j|m_j/e_j-m_i/e_i|(\omega^2r^2/2kT)$ and $r$ is located in the gap between the species $j$ and the species $i$. This is the inequality mentioned in the first paragraph of this section as the basic approximation for the analysis to follow.

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