Cooling of a pure electron plasma by cyclotron radiation

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It has been suggested that a magnetically confined pure electron plasma might be cooled to the liquid and crystal states. Here, cyclotron radiation is considered as a possible cooling mechanism. The plasma and some cooled resisting medium are assumed to reside inside a conducting cavity. When the cyclotron motion of the electrons resonantly drives a cavity mode which is damped by the cooled medium, energy is transferred from the electrons to the medium. Attention is focused on the case where all of the electrons experience a sharp (i.e., high-Q) resonance with a single cavity mode. An interesting result is that the rate of energy loss per electron can exceed, by a factor of Q, the radiation rate for an electron executing cyclotron motion in unbounded space. The maximum value of Q is limited by cyclotron damping of the mode on the plasma itself and can be large for a non-neutral plasma at a density well below the Brillouin limit (i.e., for $\omega_p << \Omega$, where ω_p is the plasma frequency and Ω is the cyclotron frequency).

I. INTRODUCTION

Recent experiments have involved the confinement by static electric and magnetic fields of a collection of electrons of sufficient density to be called a plasma, that is, a pure electron plasma.[1] It has been suggested that such a plasma might be cooled to the liquid and crystal states,[2] and this paper considers cyclotron radiation as a possible cooling mechanism.

Motivated by the experiments of Ref. 1, the confinement geometry is assumed to be cylindrically symmetrical, with radial confinement of the electrons provided by an axial magnetic field and axial confinement provided by negatively biased end structures. For such a geometry, the electrons can be in thermal equilibrium with each other and still be confined. The N-electron thermal distribution is given by

$$F = Z^{-1} \exp[(-1/kT)(H - \omega_r P_\theta)], \tag{1}$$

where H is the Hamiltonian for the electrons and P_θ is the total canonical angular momentum for the electrons.[3-6] These quantities enter the distribution on equal footing for a cylindrically symmetric geometry. For a given number of electrons, the temperature T and the parameter ω_r, which will be identified as the rotation frequency for the system of electrons, are determined by the total electron energy and canonical angular momentum (i.e., by $\langle H\rangle$ and $\langle P_\theta\rangle$). We will see later that the distribution corresponds to a confined set of electrons. For a sufficiently low temperature, the distribution predicts that the electrons are in a liquid state, and for even lower temperature, in a crystal state.[2,7,8]

We assume that the confinement region is bounded by a cylindrically symmetric conducting wall and expand the radiation field in a set of cavity modes. There must be sections of the wall at either end that are biased negatively relative to the central section. These are the negatively biased end structures that provide the axial confinement. Nevertheless, the wall can appear continuous to the high frequency field. (This assumes that there is sufficient capacitance at the junction between the sections of the wall.)

A cooled resisting medium is assumed to reside somewhere inside the cavity. The interaction of the electrons with the cavity modes and of the cavity modes with the cooled resisting medium gradually transfers energy and angular momentum from the electrons to the medium. Assuming that this occurs slowly compared with the thermalization rate for the electrons, the electron distribution will remain very nearly of the thermal equilibrium form, but the parameters T and ω_r will evolve slowly in time. To determine dT/dt and $d\omega_r/dt$, we calculate $d\langle H\rangle/dt$ and $d\langle P_\theta\rangle/dt$.

In Sec. II, the dynamical equations for the electrons and for the cavity modes are developed. Within the context of a frequency ordering to be discussed shortly, the electron velocities and the amplitudes of the cavity modes satisfy coupled oscillator equations. When the cyclotron frequency for an electron matches the frequency of a particular mode, the mode amplitude is enhanced by resonance and a relatively large amount of energy and angular momentum are dissipated in the resisting medium. We consider the case where all of the electrons resonate with the same cavity mode. The mode amplitude relaxes to become a function of the electron velocities on a time scale which is the inverse of the frequency spread characterizing the width of the resonance. By writing the equation describing the rate of change of the total energy (i.e., electron energy plus mode energy), one obtains an expression for dH/dt in terms of the mode amplitude and the damping rate for the mode due to the resisting medium. A similar equation may be obtained for dP_θ/dt. By substituting for the mode amplitude as a function of electron velocities, expressions for dH/dt and dP_θ/dt are obtained in terms of the electron velocities.

In Sec. III, these expressions are averaged over the particle distribution to obtain $d\langle H\rangle/dt$ and $d\langle P_\theta\rangle/dt$ and, by implication, dT/dt and $d\omega_r/dt$. Under the condition that ω_r is small compared with the mode frequency, which is part of the frequency ordering, the change in angular momentum is negligible. The parameter ω_r remains essentially constant, and the temperature drops as energy flows to the cooled medium (i.e., $dT/dt \propto d\langle H\rangle/dt$).

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In Sec. IV, $d\langle H\rangle/dt$ is maximized. The maximum occurs when the three frequencies which determine the width of the resonance are comparable. The first two of these are the mode damping rate due to the resisting medium and the damping rate due to the plasma itself, that is, the cyclotron damping rate. This latter quantity arises naturally when solving the coupled oscillator equations. Matching the resistive and cyclotron damping rates is essentially matching the impedance of the cavity and the impedance of the plasma. The third quantity determining the resonance width is the spread in cyclotron frequencies due to the inhomogeneity in the magnetic field. Since the cyclotron damping rate varies inversely as the spread in cyclotron frequencies, that is, varies as $dN/d\Omega$, where $N(\Omega)$ is the distribution of cyclotron frequencies, the condition that the spread in cyclotron frequencies be comparable to the width of the resonance is simply a condition that the cyclotron damping rate be a minimum subject to the constraint that all electrons participate in the resonance. This maximizes the Q for the resonance, subject to the constraint that all electrons participate in the resonance. The maximum value for the energy loss rate is approximately $d\langle H\rangle/dt \simeq NkT(\pi e^2/mNV)^{1/2}$, where N is the number of electrons, k is Boltzmann's constant, e and m are the electron charge and mass, and V is the volume of the cavity.

It is interesting to compare this energy loss rate to that for the case where each electron experiences simultaneous resonance with many modes. In general, an electron can resonate with many modes if the dimensions of the cavity are large compared with c/Ω and the resistive broadening of the frequency for each mode is large compared with the frequency separation between modes. One expects that an electron in such a cavity will lose energy at the same rate as an electron executing cyclotron radiation in unbounded space, that is, at the Larmor radiation rate,[9] $2e^2\langle v_\perp^2\rangle\Omega^2/3c^3$. In Sec. IV, it is shown that the ratio of the two energy loss rates (i.e., single mode/many mode) can be as large as the value of Q defining the width of the single mode resonance. The maximum value of this quantity is approximately $Q \simeq \Omega(mV/\pi^2Ne^2)^{1/2}$. For a non-neutral plasma which is magnetically confined at a density that is well below the Brillouin limit[10] (i.e., $\omega_p \ll \Omega$), the value of Q can be large compared with unity.

The frequency ordering scheme assumes that the mode frequencies ω_λ and the characteristic cyclotron frequency Ω are the largest in the problem. Of course, the equality of Ω with some ω_λ defines the resonance. Next in size come the frequencies that determine the width of the resonance: the spread in cyclotron frequencies $\Delta\Omega$ and the damping rate for the mode $\nu_\lambda + \nu_p$. Here, ν_λ is the damping rate due to the resisting medium and ν_p is the damping rate due to the plasma (i.e., cyclotron damping). As mentioned earlier, the single mode resonance leads to maximum energy loss when $\Delta\Omega \sim \nu_p \sim \nu_\lambda$. The mode amplitude relaxes to become a function of the electron velocities on a time scale which is the inverse of the frequency width of the resonance. Effects such as electron-electron collisions and the guiding center motion of the electrons, which are neglected in the oscillator equation for the electron velocity, must correspond to frequencies that are small compared with the resonance width. The frequency associated with collisional effects is the collision frequency ν_c, and the frequencies associated with the guiding center motion are $l\omega_r$ and $(\Omega/c)(kT/m)^{1/2}$, where l is a characteristic azimuthal mode number and Ω/c is a characteristic axial mode number. For the low order modes considered here, l is of order unity. Thus, the frequencies are assumed to satisfy the ordering

$$\Omega, \omega_\lambda \gg \Delta\Omega, \quad (\nu_\lambda + \nu_p) \gg \nu_c, \quad \omega_r, \quad \Omega(kT/mc^2)^{1/2}. \tag{2}$$

This ordering is reasonable, provided the density is well below the Brillouin limit (i.e., $\omega_p \ll \Omega$) and all electron velocities are small compared with c. Recall that $\nu_c \lesssim \omega_p$ and $\omega_r \simeq \omega_p^2/2\Omega$. Finally, the inverse of the radiation time, $1/\tau_r$, is assumed to be small compared with all of these frequencies. This is reasonable for a large number of electrons, since we will find that [see Eq. (50)] $1/\tau_r \lesssim \nu_\lambda/N \ll \nu_\lambda$.

II. DYNAMICAL EQUATIONS FOR THE PARTICLES AND FIELDS

Let us start with a description of the electrons and the cavity modes in the absence of the cooled resisting medium. The Hamiltonian for the electrons is given by

$$H = \sum_j \frac{[\mathbf{p}_j + (e/c)\mathbf{A}(\mathbf{x}_j, t)]^2}{2m} + U(\mathbf{x}_1, \ldots, \mathbf{x}_N), \tag{3}$$

where $(\mathbf{x}_j, \mathbf{p}_j)$ are the position and momentum for the jth electron, $U(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ is the electrostatic energy of the electrons, and $\mathbf{A}(\mathbf{x}, t)$ is the vector potential.[11] We have in mind the Coulomb gauge; so, $U(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ is expressed in terms of the instantaneous electrostatic potential.

The vector potential can be expressed as $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1$, where $\mathbf{A}_0$ represents the static field (i.e., $\nabla \times \mathbf{A}_0 = \mathbf{B}$) and $\mathbf{A}_1$ represents the radiation field. For the low densities (i.e., $\omega_p \ll \Omega$) and low electron velocities (i.e., $v \ll c$) considered here, the diamagnetic field is negligible. The radiation field can be expressed as[11]

$$\mathbf{A}_1(\mathbf{x}, t) = \sum_\lambda \mathbf{A}_\lambda(\mathbf{x}) q_\lambda(t) + \text{c.c.}, \tag{4}$$

where the $\mathbf{A}_\lambda(\mathbf{x})$ are a complete set of cavity modes and the $q_\lambda(t)$ are complex time dependent amplitudes. The subscript λ represents the various eigennumbers characterizing each mode. The $\mathbf{A}_\lambda(\mathbf{x})$ must satisfy the Coulomb gauge condition (i.e., $\nabla \cdot \mathbf{A}_\lambda = 0$), the Helmholtz equation [i.e., $\nabla^2\mathbf{A}_\lambda + (\omega_\lambda^2/c^2)\mathbf{A}_\lambda = 0$], the boundary conditions on the conducting wall [i.e., $\hat{n} \times \mathbf{A}_\lambda = 0$ and $\hat{n} \cdot \nabla \times \mathbf{A}_\lambda = 0$, where $\hat{n}$ is the local normal to the wall], and they are traditionally normalized so that $\int d^3x\, \mathbf{A}_\lambda \cdot \mathbf{A}_\mu^* = 4\pi c^2 \delta_{\lambda,\mu}$. Under these conditions, the Hamiltonian for the radiation field is given by

$$H' = \sum_\lambda \frac{P_\lambda^2}{2} + \frac{\omega_\lambda^2 Q_\lambda^2}{2}, \tag{5}$$

where ω_λ is the frequency for mode λ and the canonical variables (Q_λ, P_λ) are related to the complex amplitudes through the relations

$$Q_\lambda = q_\lambda + q_\lambda^*, \quad P_\lambda = i\omega_\lambda(q_\lambda^* - q_\lambda). \tag{6}$$

The total Hamiltonian is the sum of the electron Hamiltonian and the field Hamiltonian (i.e., $H_T = H + H'$). The equation of motion for $q_\lambda(t)$ is given by

$$\frac{dq_\lambda}{dt} = \frac{d}{dt}\left(\frac{Q_\lambda}{2} - \frac{P_\lambda}{2i\omega_\lambda}\right) = \frac{1}{2}\frac{\partial H_T}{\partial P_\lambda} + \frac{1}{2i\omega_\lambda}\frac{\partial H_T}{\partial Q_\lambda}. \tag{7}$$

In evaluating the derivatives with respect to Q_λ and P_λ, one must be sure to take into account the fact that H depends on Q_λ and P_λ through $\mathbf{A}$. The result is

$$\frac{dq_\lambda}{dt} = -i\omega_\lambda q_\lambda + \sum_j \frac{e}{2ci\omega_\lambda}\mathbf{v}_j \cdot \mathbf{A}_\lambda^*(j), \tag{8}$$

where $\mathbf{v}_j = [\mathbf{p}_j + (e/c)\mathbf{A}(\mathbf{x}_j, t)]/(m)$ is the electron velocity and the notation $\mathbf{A}_\lambda(j) = \mathbf{A}_\lambda(\mathbf{x}_j)$ has been introduced. Equation (8) merely states that $q_\lambda(t)$ satisfies an oscillator equation which is driven by the projection of the current on $\mathbf{A}_\lambda(\mathbf{x})$. Let us assume that λ denotes the particular mode that is driven resonantly to large amplitude; so, the radiation field is of the form $\mathbf{E}_1 = (-1/c)(\partial \mathbf{A}_1/\partial t) \simeq (i\omega_\lambda/c)\mathbf{A}_\lambda(\mathbf{x})q_\lambda + \text{c.c.}$

This is a convenient point to introduce the effect of a cooled resisting medium. Suppose that some of the current on the right-hand side of Eq. (8) is in the resisting medium. This current is the sum of various parts. First, there is the driven current. The electric field $\mathbf{E}_1 \simeq (i\omega_\lambda/c)\mathbf{A}_\lambda(\mathbf{x})q_\lambda + \text{c.c.}$ produces terms on the right-hand side of Eq. (8) of the form[12] $-\nu_\lambda q_\lambda - \nu_\lambda' q_\lambda^*$. (If the dominant resistance is in what we have been calling the conducting wall, then the argument is slightly different but the result is the same.) Since q_λ^* is not resonant, we can drop this term. The electrostatic field also drives a current in the resisting medium, but we neglect this current, arguing that the electrostatic field is not enhanced by resonance. In addition to the driven current, there is a current due to the spontaneous thermal fluctuations in the resisting medium. However, this current can be neglected if the temperature of the resisting medium is much less than that of the plasma. Thus, including the effect of the cooled resisting medium modifies Eq. (8) by adding a damping term

$$\frac{dq_\lambda}{dt} = -(i\omega_\lambda + \nu_\lambda)q_\lambda + \sum_j \frac{e}{2ci\omega_\lambda}\mathbf{v}_j \cdot \mathbf{A}_\lambda^*(j). \tag{9}$$

At this point, the Hamiltonian description of the fields must be abandoned. The $q_\lambda(t)$ are simply time dependent amplitudes evolving according to Eq. (9).

Next, expressions for dH/dt and dP_θ/dt are obtained. From Eq. (3), we find

$$\frac{dH}{dt} = \frac{\partial H}{dt} = \sum_j \frac{e\mathbf{v}_j \cdot \mathbf{A}_\lambda(j)}{c}\frac{dq_\lambda}{dt} + \text{c.c.} \tag{10}$$

Adding this equation to the equation

$$\frac{d}{dt}2\omega_\lambda^2|q_\lambda|^2 = 2\omega_\lambda^2 q_\lambda^* \frac{dq_\lambda}{dt} + \text{c.c.} \tag{11}$$

and using Eq. (9) yields the result

$$\frac{d}{dt}(H + 2\omega_\lambda^2|q_\lambda|^2) = -2\nu_\lambda 2\omega_\lambda^2|q_\lambda|^2. \tag{12}$$

By recalling that $2\omega_\lambda^2|q_\lambda|^2 = P_\lambda^2/2 + \omega_\lambda^2 Q_\lambda^2/2$, one can see that Eq. (12) merely states that the combined energy in the electrons and in the field decreases at the rate at which energy is dissipated in the resisting medium.

Again from Eq. (3), one finds

$$\frac{dP_\theta}{dt} = -\sum_j \frac{\partial H}{\partial \theta_j} = -\sum_j \frac{e}{c}\mathbf{v}_j \cdot \mathbf{A}_\lambda(j) il q_\lambda + \text{c.c.} \tag{13}$$

Here, use has been made of the fact that the components of $\mathbf{A}_\lambda$ in cylindrical geometry have a θ dependence of the form $\exp(il\theta)$. The subscript λ contains l as one of the eigennumbers. Adding Eq. (13) to the equation

$$\frac{d}{dt}\frac{l}{\omega_\lambda}2\omega_\lambda^2|q_\lambda|^2 = \frac{l}{\omega_\lambda}2\omega_\lambda^2 q_\lambda^* \frac{dq_\lambda}{dt} + \text{c.c.} \tag{14}$$

and using Eq. (9) yields the result

$$\frac{d}{dt}\left(P_\theta + \frac{l}{\omega_\lambda}2\omega_\lambda^2|q_\lambda|^2\right) = -2\nu_\lambda \frac{l}{\omega_\lambda}2\omega_\lambda^2|q_\lambda|^2. \tag{15}$$

In other words, the combined angular momentum in the particles and the field decreases at the rate at which angular momentum is transferred to the resisting medium.

Once q_λ is known as a function of the electron velocities, Eqs. (12) and (15) can be averaged over the particle distribution to obtain $d\langle H\rangle/dt$ and $d\langle P_\theta\rangle/dt$. Within the context of the ordering scheme mentioned in the introduction, q_λ relaxes to become a function of the electron velocities on a time scale which is short compared with times associated with electron-electron collisions and with the guiding center motion of the electrons. By neglecting these effects, the equation for the high frequency component of the electron velocity takes the simple form

$$\frac{d}{dt}\delta\mathbf{v}_j = -\Omega_j \delta\mathbf{v}_j \times \hat{z} + \frac{e}{mc}\mathbf{A}_\lambda(j)(-i\omega_\lambda)q_\lambda + \text{c.c.}, \tag{16}$$

where $\Omega_j = eB(z_j)/mc$ is the local cyclotron frequency in the weakly inhomogeneous magnetic field and $\hat{z}$ is a unit vector in the z direction. The velocity has been expressed as the sum of a slowly varying component (guiding center motion) and a high frequency component (cyclotron motion). Equation (16) is the equation for the high frequency component, which is the only part that can resonate with the cavity modes. In solving Eq. (16), the guiding center motion is neglected, that is, Ω_j and $\mathbf{A}_\lambda(j)$ are treated as time independent. Of course, Eq. (16) can be derived from the Hamiltonian in Eq. (3); it is simply easier to write down the Lorentz force equation directly.

By introducing the quantity $u_j = (\hat{\mathbf{r}} - i\hat{\boldsymbol{\theta}})_j \cdot \delta\mathbf{v}_j$, Eq. (16) takes the form

$$\frac{du_j}{dt} = -i\Omega_j u_j - \frac{e}{mc}(\hat{\mathbf{r}} - i\hat{\boldsymbol{\theta}})_j \cdot [\mathbf{A}_\lambda(j) i\omega_\lambda q_\lambda + \text{c.c.}]. \tag{17}$$

This equation must be complimented by an equation for dq_λ/dt. Substituting $\delta\mathbf{v}_j = (\hat{\mathbf{r}} + i\hat{\boldsymbol{\theta}})_j u_j/2 + \text{c.c.}$ into Eq. (9) yields the desired result

$$\frac{dq_\lambda}{dt} = -(i\omega_\lambda + \nu_\lambda)q_\lambda + \sum_j \frac{e}{4ci\omega_\lambda}\mathbf{A}_\lambda^*(j) \cdot [(\hat{\mathbf{r}} + i\hat{\boldsymbol{\theta}})u_j + \text{c.c.}]. \tag{18}$$

In each equation, the complex conjugate term appearing in the bracket may be dropped because it cannot produce resonance.

To solve these equations, we start by integrating Eq. (17) forward from some arbitrary time, $t=0$, up to time t. The result is

$$u_k(t)=u_k(0)\exp(-i\Omega_k t)-(i\omega_\lambda e/mc)(\hat{\mathbf{r}}-i\hat{\boldsymbol{\theta}})_k\cdot\mathbf{A}_\lambda(k)\times\int_0^t dt' q_\lambda(t')\exp[i\Omega_k(t'-t)]. \quad (19)$$

Substituting this expression into Eq. (18) yields the equation

$$\frac{dq_\lambda}{dt}+(i\omega_\lambda+\nu_\lambda)q_\lambda+\sum_k\frac{e^2}{4mc^2}|(\hat{\mathbf{r}}-i\hat{\boldsymbol{\theta}})\cdot\mathbf{A}_\lambda(k)|^2\times\int_0^t dt' q_\lambda(t')\exp[i\Omega_k(t'-t)]=\sum_j\frac{e}{4ci\omega_\lambda}(\hat{\mathbf{r}}+i\hat{\boldsymbol{\theta}})_j\cdot\mathbf{A}_\lambda^*(j)u_j(0)\exp(-i\Omega_k t). \quad (20)$$

The third term on the left represents the correction to the mode frequency and damping rate due to the plasma loading of the cavity.

The solution for $q_\lambda(t)$ consists of an initial transient which is negligible provided t multiplied by the total damping rate is large compared with unity and of terms which are driven by the right-hand side. The driven terms are given by

$$q_\lambda(t)=\frac{e}{4i\omega_\lambda c}\times\sum_j\frac{(\hat{\mathbf{r}}+i\hat{\boldsymbol{\theta}})_j\cdot\mathbf{A}_\lambda^*(j)u_j(0)\exp(-i\Omega_j t)}{[i(\omega_\lambda-\Omega_j)+\nu_\lambda+i\Delta_p(\lambda,j)+\nu_p(\lambda,j)]}, \quad (21)$$

where

$$i\Delta_p(\lambda,j)+\nu_p(\lambda,j)=\frac{e^2}{4mc^2}\sum_k|(\hat{\mathbf{r}}-i\hat{\boldsymbol{\theta}})_k\cdot\mathbf{A}_\lambda(k)|^2\times\int_0^t dt'\exp[i(\Omega_k-\Omega)(t'-t)]. \quad (22)$$

Since the number of electrons is assumed to be large, the sum over k can be expressed as an integral

$$\sum_k=\int_{-\infty}^{+\infty}dz\int_0^\infty 2\pi r\,dr\,n(r,z),$$

where $n(r,z)$ is the density of electrons. For $t\Delta\Omega>1$, where $\Delta\Omega$ is the spread in cyclotron frequencies, the frequency shift and damping rate take the form[13]

$$\Delta_p(\lambda,j)=-\frac{-e^2}{4mc^2}\int_{-\infty}^{+\infty}dz\int_0^\infty 2\pi r\,dr\,n(r,z)\times|(\hat{\mathbf{r}}-i\hat{\boldsymbol{\theta}})\cdot\mathbf{A}_\lambda(r,\theta,z)|^2\frac{P}{\Omega(z)-\Omega_j}, \quad (23)$$

$$\nu_p(\lambda,j)=\frac{\pi e^2}{4mc^2}\left|\frac{d\Omega}{dz}\right|^{-1}\int_0^\infty 2\pi r\,dr\,n(r,z)\times|(\hat{\mathbf{r}}-i\hat{\boldsymbol{\theta}})\cdot\mathbf{A}_\lambda(r,\theta,z)|^2\Big|_{\Omega(z)=\Omega_j}, \quad (24)$$

where the symbol P in Eq. (23) indicates that the principal part is to be taken. As mentioned earlier, the radiation turns out to be maximum for $\nu_p\sim\Delta\Omega\sim\nu_\lambda$; so one is not interested in cases where $d\Omega/dz$ vanishes. Equations (23) and (24) have a rather obvious interpretation in terms of the plasma dielectric for cyclotron waves, and the damping is clearly cyclotron damping.[13]

The reaction of q_λ back on $u_j(t)$ is significant after a time τ_r, the radiation time. For $t\ll\tau_r$, we can set $u_j(0)\exp(-i\Omega_j t)\simeq u_j(t)$ on the right-hand side of Eq. (21). Note that the assumptions $(\Delta\Omega)^{-1}$, $(\nu_\lambda+\nu_p)^{-}\ll t\ll\tau_r$ are consistent with the frequency ordering mentioned in the introduction. By re-introducing the notation $u_j=(\hat{\mathbf{r}}-i\hat{\boldsymbol{\theta}})_j\cdot\delta\mathbf{v}_j$, Eq. (21) takes the form

$$q_\lambda=\frac{e}{4i\omega_\lambda c}\sum_j\frac{(\hat{\mathbf{r}}+i\hat{\boldsymbol{\theta}})_j\cdot\mathbf{A}_\lambda^*(j)(\hat{\mathbf{r}}-i\hat{\boldsymbol{\theta}})_j\cdot\delta\mathbf{v}_j}{[i(\omega_\lambda-\Omega_j)+\nu_\lambda+i\Delta_p(\lambda,j)+\nu_p(\lambda,j)]}. \quad (25)$$

Effects such as the electron-electron scattering of $\delta\mathbf{v}_j$ and the guiding center motion included in $\mathbf{A}_\lambda(j)$ are followed adiabatically by this equation. Of course, this assumes that $(\nu_\lambda+\nu_p)$, $\Delta\Omega\gg\nu_c$, ω_r, $\Omega(kT/mc^2)^{1/2}$, as mentioned in the introduction.

III. DETERMINATION OF dT/dt and $d\omega_r/dt$

Although Eqs. (17) and (18) are adequate to determine the relaxation of q_λ to become a function of the $\delta\mathbf{v}_j$'s, the equations are not adequate to predict the evolution of the $\delta\mathbf{v}_j$'s over a radiation time. The radiation time is long compared with a collision time, and collisions have been neglected in Eq. (17). At this point one must introduce statistical mechanics. By assuming that the electron distribution remains nearly of the thermal equilibrium form [i.e., Eq. (1)], the parameters characterizing the distribution (i.e., T and ω_r) can be projected ahead over a radiation time, even though the detailed electron dynamics cannot. This is accomplished by substituting Eq. (25) into Eqs. (12) and (15) to obtain expressions for dH/dt and dP_θ/dt in terms of the electron velocities. Averaging these expressions over the particle distribution yields $d\langle H\rangle/dt$ and $d\langle P_\theta\rangle/dt$, and by implication, dT/dt and $d\omega_r/dt$. These expressions are valid on a long time scale, because Eqs. (12) and (15) do not involve approximations concerning the electron dynamics. By setting

$$H=\sum_j\frac{mv_j^2}{2}+U(\mathbf{x}_1,\ldots,\mathbf{x}_N)$$

and

$$P_\theta=\sum_j mv_{\theta_j}r_j-\frac{m\Omega r_j^2}{2},$$

the thermal distribution can be written as

$$F=\frac{1}{Z}\exp\left[-\frac{1}{kT}\left(\sum_j\frac{m}{2}(\mathbf{v}_j-r\omega_r\hat{\boldsymbol{\theta}}_j)^2+U(\mathbf{x}_1,\ldots,\mathbf{x}_N)-\frac{m\omega_r}{2}(\Omega-\omega_r)\sum_j r_j^2\right)\right]. \quad (26)$$

Before carrying out the averages, it is useful to recount a few results from studies of the thermal equilibrium states.[3] To understand these results intuitively, note that the last term in the bracket in Eq. (26) can be interpreted as the potential energy of the electrons in a hypothetical cylinder of uniform positive charge. This positive charge plus the negatively biased end cylinders produce a potential well in which the electrons reside. In the limit of weak correlation and Debye length small compared with the dimensions of the plasma, the electrons fill the well out to some surface of revolution where the electron density drops off on the scale of the Debye length. Inside the surface of revolution the electric field determined by the one electron distribution just cancels the electric field due to the hypothetic positive charge [i.e., $0=\mathbf{E}+\nabla(m/e)\omega_r(\Omega-\omega_r)r^2/2$]. Taking the divergence yields the electron density inside the surface of revolution [i.e., $-4\pi n e=\nabla\cdot\mathbf{E}=-(m/e)\omega_r(\Omega-\omega_r)2$, or $\omega_p^2=2\omega_r(\Omega-\omega_r)$]. For the frequency ordering considered here (i.e., $\omega_p\ll\Omega$), the latter relation reduces to $\omega_r\simeq\omega_p^2/2\Omega=cE/Br$. For the case of strong correlation one expects this picture to be modified significantly only in the Debye sheath region near the edge of the plasma (i.e., near the surface of revolution).

In carrying out the averages, $\hat{\mathbf{r}}\cdot\delta\mathbf{v}_j$ should be associated with $\hat{\mathbf{r}}\cdot\mathbf{v}_j$ and $\hat{\boldsymbol{\theta}}\cdot\delta\mathbf{v}_j$ with $\hat{\boldsymbol{\theta}}\cdot(\mathbf{v}_j-r_j\omega_r\hat{\boldsymbol{\theta}}_j)$. Substituting Eq. (25) into Eq. (12) and averaging over the distribution yields the result

$$\frac{d\langle H\rangle}{dt}=-\left(\frac{d}{dt}+2\nu_\lambda\right)\frac{e^2}{4c^2}\frac{kT}{m}\times\int_{-\infty}^{+\infty}dz\int_0^\infty 2\pi r\,dr\,n(r,z)\times\frac{|(\hat{\mathbf{r}}-i\hat{\boldsymbol{\theta}})\cdot\mathbf{A}_\lambda(r,\theta,z)|^2}{[\omega_\lambda-\Omega(z)+\Delta_p(\lambda,z)]^2+[\nu_\lambda+\nu_p(\lambda,z)]^2}, \tag{27}$$

where $n(r,z)\equiv\int d\mathbf{v}_1\,d\mathbf{x}_2\,d\mathbf{v}_2\cdots d\mathbf{x}_N\,d\mathbf{v}_N NF$ is the electron density. Use has been made of the relation $d\langle G\rangle/dt=\langle dG/dt\rangle$, which follows from the fact that $d/dt(F d\mathbf{x}_1\,d\mathbf{v}_1\cdots d\mathbf{x}_N\,d\mathbf{v}_N)=0$. Here, the time derivatives are total derivatives, and G is an arbitrary function of $(t,\mathbf{x}_1,\mathbf{v}_1\cdots\mathbf{x}_N,\mathbf{v}_N)$. There is a subtle point buried here. The actual distribution, which is constant along phase trajectories, is very complicated. It is the coarse grain average of this distribution that is given by Eq. (26), with slowly varying T and ω_r. Also, it is interesting to note that correlation effects do not enter Eq. (27). This is because the velocities $\delta\mathbf{v}_j$ and $\delta\mathbf{v}_k$ are completely uncorrelated, even if the positions $\mathbf{x}_j$ and $\mathbf{x}_k$ are strongly correlated.

On the right-hand side of Eq. (27), d/dt can be neglected in comparison with ν_λ since $d/dt\sim 1/\tau_r$ and $1/\tau_r\ll\nu_\lambda$. Also, with the aid of Eq. (24), the integral over radius can be rewritten in terms of ν_p. The result is

$$\frac{d\langle H\rangle}{dt}=-\frac{kT}{\pi}\int d\Omega\frac{2\nu_\lambda\nu_p(\lambda,\Omega)}{[\omega_\lambda-\Omega+\Delta_p(\lambda,\Omega)]^2+[\nu_\lambda+\nu_p(\lambda,\Omega)]^2}. \tag{28}$$

Likewise, substituting Eq. (25) into Eq. (15) and averaging over the distribution yields

$$\frac{d\langle P_\theta\rangle}{dt}=-\frac{kT}{\pi}\frac{l}{\omega_\lambda}\times\int d\Omega\frac{2\nu_\lambda\nu_p(\lambda,\Omega)}{[\omega_\lambda-\Omega+\Delta_p(\lambda,\Omega)]^2+[\nu_\lambda+\nu_p(\lambda,\Omega)]^2}. \tag{29}$$

It is easy to understand these results in a particular limit. By comparing Eq. (28) with Eq. (12) [or Eq. (29) with Eq. (15)], one can see that the average energy in the mode is given by

$$2\omega_\lambda^2\langle|q_\lambda|^2\rangle=\frac{kT}{\pi}\int d\Omega\frac{\nu_p(\lambda,\Omega)}{[\omega_\lambda-\Omega+\Delta_p(\lambda,\Omega)]^2+[\nu_\lambda+\nu_p(\lambda,\Omega)]^2}. \tag{30}$$

In the limit where $\nu_p(\lambda,\Omega)$ is nearly constant over the width of the resonance, the integral is easily evaluated

$$2\omega_\lambda^2\langle|q_\lambda|^2\rangle=kT[\nu_p/(\nu_\lambda+\nu_p)]. \tag{31}$$

For $\nu_\lambda\ll\nu_p$, the mode should be in thermal equilibrium with the plasma, and, as expected, Eq. (31) reduces to $2\omega_\lambda^2\langle|q_\lambda|^2\rangle=kT$.

Next, $d\langle H\rangle/dt$ and $d\langle P_\theta\rangle/dt$ are related to dT/dt and $d\omega_r/dt$. Averaging $H=\sum_j mv_j^2/2+U(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ over the distribution yields

$$\langle H\rangle=\tfrac{3}{2}NkT+\langle U\rangle+\tfrac{1}{2}(m\omega_r^2)\int d^3\mathbf{x}\,n(\mathbf{x},t)r^2, \tag{32}$$

where $\langle U\rangle$ is the sum of a Vlasov contribution and a correlation contribution

$$\langle U\rangle=\langle U\rangle_v+\langle U\rangle_c, \tag{33}$$

$$\langle U\rangle_v=-e\int d^3\mathbf{x}_1\,n(\mathbf{x}_1,t)\left(\varphi(\mathbf{x}_1)+\tfrac{1}{2}\int d^3\mathbf{x}_2\,n(\mathbf{x}_2,t)\varphi(\mathbf{x}_1,\mathbf{x}_2)\right), \tag{34}$$

$$\langle U\rangle_c=-\frac{e}{2}\int d^3\mathbf{x}_1\int d^3\mathbf{x}_2\,n(\mathbf{x}_1,t)n(\mathbf{x}_2,t)g(\mathbf{x}_1,\mathbf{x}_2,t)\varphi(\mathbf{x}_1,\mathbf{x}_2). \tag{35}$$

Here, $-e\varphi(\mathbf{x})$ is the electrostatic energy of a single electron at position $\mathbf{x}$ in the cavity and $-e\varphi(\mathbf{x},\mathbf{x}')$ is the addition to the energy of that electron due to the existence of another electron at position $\mathbf{x}'$. The pair correlation function $g(\mathbf{x}_1,\mathbf{x}_2,t)$ is defined in the usual fashion

$$N^2\int d^3\mathbf{v}_1\,d^3\mathbf{v}_2\,d^3\mathbf{x}_3\,d^3\mathbf{v}_3\cdots d^3\mathbf{x}_N\,d^3\mathbf{v}_N F=n(\mathbf{x}_1,t)n(\mathbf{x}_2,t)[1+g(\mathbf{x}_1,\mathbf{x}_2,t)]. \tag{36}$$

In the limit of weak correlation, the correlation energy is given by $\langle U\rangle_c\simeq-(3NkT/2)[1/(12\pi n\lambda_D^3)]$.[14]

The quantity $d\langle U\rangle_v/dt$ is closely related to $d\langle P_\theta\rangle/dt$. From Eq. (34) and the continuity equation, $\partial n/\partial t+\nabla\cdot(n\mathbf{u})=0$, one finds that

$$\frac{d\langle U\rangle_v}{dt}=e\int d^3\mathbf{x}\,n(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t). \tag{37}$$

where

$$\mathbf{E}=-\nabla\varphi-\nabla\int d^3\mathbf{x}'\,n(\mathbf{x}',t)\varphi(\mathbf{x},\mathbf{x}').$$

As mentioned earlier, inside the plasma the electric field is given by $E=-\nabla(m/e)\omega_r\Omega r^2/2$; so Eq. (37) takes the form

$$\frac{d\langle U\rangle_v}{dt}\simeq -m\omega_r\Omega\int d^3\mathbf{x}\,n(\mathbf{x},t)u_r(\mathbf{x},t)r\,. \quad (38)$$

The average of $P_\theta=\sum_j mv_{\theta_j}r_j-m\Omega r_j^2/2$ over the distribution (26) is given by

$$\langle P_\theta\rangle=-m(\tfrac{1}{2}\Omega-\omega_r)\int d^3\mathbf{x}\,n(\mathbf{x},t)r^2$$
$$\langle P_\theta\rangle\simeq -m\Omega\int d^3\mathbf{x}\,\tfrac{1}{2}[n(\mathbf{x},t)r^2]\,. \quad (39)$$

Taking the derivative, using the continuity equation, and comparing to Eq. (38) yields the relation

$$\frac{d\langle U\rangle_v}{dt}\simeq\omega_r\frac{d\langle P_\theta\rangle}{dt}\,. \quad (40)$$

Thus, the time derivative of Eq. (32) can be written as

$$\frac{d\langle H\rangle}{dt}\simeq\frac{d}{dt}\left(\frac{3}{2}NT+\langle U\rangle_c\right)+\omega_r\frac{d\langle P_\theta\rangle}{dt}, \quad (41)$$

where the derivative of $\frac{1}{2}m\omega_r^2\int d^3\mathbf{x}\,nr^2$ has been neglected compared with $\omega_r d\langle P_\theta\rangle/dt$. In this regard, note that $(1/\omega_r)(d\omega_r/dt)\simeq(1/n)(dn/dt)$ is the same order as $(1/\langle P_\theta\rangle)(d\langle P_\theta\rangle/dt)$. From Eqs. (28) and (27) it follows that $d\langle P_\theta\rangle/dt=l/\omega_\lambda d\langle H\rangle/dt$. Since l is of order unity and $\omega_r\ll\omega_\lambda$, Eq. (40) reduces to

$$\frac{d\langle H\rangle}{dt}\simeq\frac{d}{dt}\left(\frac{3}{2}NT+\langle U\rangle_c\right). \quad (42)$$

In essence, a mode with $\omega_\lambda/l\gg\omega_r$ removes a large amount of energy and a relatively small amount of angular momentum. Because the electron canonical angular momentum is nearly conserved, the electrons cannot move out radially and the energy $\langle U\rangle_v$ cannot be liberated. The plasma cools and becomes more correlated without changing its overall shape. The density and $\omega_r\simeq\omega_p^2/2\Omega$ remain essentially constant. Since the density remains constant, Eq. (42) can be rewritten as an equation for the rate of change of the temperature

$$\frac{d\langle H\rangle}{dt}=\frac{dT}{dt}\frac{\partial}{\partial T}\left(\frac{3}{2}NT+\langle U\rangle_c\right). \quad (43)$$

IV. MAXIMUM ENERGY LOSS RATE

In this section, the maximum value of $d\langle H\rangle/dt$ is determined for given values of N and T. It is convenient to rewrite $\Delta_p(\lambda,\Omega)$ and $\nu_p(\lambda,\Omega)$ in the form [see Eqs. (23) and (24)]

$$\Delta_p(\lambda,\Omega)=-\beta^2\frac{1}{\pi}\int d\Omega'\frac{P}{\Omega'-\Omega}f(\Omega'), \quad (44)$$
$$\nu_p(\lambda,\Omega)=\beta^2 f(\Omega), \quad (45)$$

where

$$\beta^2=\frac{\pi e^2}{4mc^2}\int_{-\infty}^{+\infty}dz\int_0^\infty 2\pi r\,dr\,n(r,z)|(\hat{\mathbf{r}}-i\hat{\boldsymbol{\theta}})\cdot\mathbf{A}_\lambda(r,\theta,z)|^2, \quad (46)$$

$$f(\Omega)=\frac{\left|\frac{dz}{d\Omega}\right|\int_0^\infty 2\pi r\,dr\,n(r,z)|(\hat{\mathbf{r}}-i\hat{\boldsymbol{\theta}})\cdot\mathbf{A}_\lambda(r,\theta,z)|^2}{\int_{-\infty}^{+\infty}dz\int_0^\infty 2\pi r\,dr\,n(r,z)|(\hat{\mathbf{r}}-i\hat{\boldsymbol{\theta}})\cdot\mathbf{A}_\lambda(r,\theta,z)|^2}. \quad (47)$$

Here, z and Ω are related implicitly through $\Omega=\Omega(z)$. Part of the maximization condition turns out to be a requirement that the ν_p be finite; so that $d\Omega/dz\neq 0$ in the resonant region, and $\Omega=\Omega(z)$ is monotonic in that region. The function $f(\Omega)$ is the distribution of cyclotron frequencies [i.e., $(1/N)(dN/d\Omega)$] weighted by the relative strength of $|(\hat{\mathbf{r}}-i\hat{\boldsymbol{\theta}})\mathbf{A}_\lambda|^2$ at the appropriate z (or Ω). Also, $f(\Omega)$ is normalized to unity [i.e., $\int d\Omega f(\Omega)=1$]. In terms of β^2 and $f(\Omega)$, Eq. (28) takes the form

$$\frac{d\langle H\rangle}{dt}=-\frac{T}{\pi}\int d\Omega\frac{2\nu_\lambda\beta^2 f(\Omega)}{\left(\Omega-\omega_\lambda+\frac{\beta^2}{\pi}\int\frac{d\Omega' Pf(\Omega')}{\Omega'-\Omega}\right)^2+[\nu_\lambda+\beta^2 f(\Omega)]^2}. \quad (48)$$

Using the normalization condition $\int d^3\mathbf{x}|\mathbf{A}_\lambda|^2=4\pi c^2$ to estimate β^2 yields

$$\beta^2\simeq\pi^2e^2N/mV\,, \quad (49)$$

where V is the volume of cavity. Thus, maximizing $d\langle H\rangle/dt$ for a given T and N is equivalent to maximizing it for a given T and β. The quantities ω_λ, ν_λ, and $f(\Omega)$ may all be varied. Suppose that $f(\Omega)$ is characterized by a width $\Delta\Omega$ and a height $(\Delta\Omega)^{-1}$ [recall that $\int d\Omega f(\Omega)=1$]. By considering various limiting cases (e.g., $\Delta\Omega\ll\beta$, ν_λ, etc.), one can convince himself that the maximum occurs for $\nu_\lambda\sim\Delta\Omega\sim\beta$ and that the maximum value is of order

$$\frac{d\langle H\rangle}{dt}\simeq -kT\beta\,. \quad (50)$$

Essentially, this follows from dimensional analysis, since β is the only frequency being held fixed.

Note that the definition $\nu_p=\beta^2 f(\Omega)$ allows the maximization condition $\nu_\lambda\sim\Delta\Omega\sim\beta$ to be rewritten as $\nu_\lambda\sim\Delta\Omega\sim\nu_p$. The condition $\nu_\lambda\sim\nu_p$ is simply a requirement that the impedance of the cavity be matched to that of the plasma. Since ν_p is proportional to $1/\Delta\Omega$, the condition that $\Delta\Omega\sim\nu_\lambda$ is a requirement that ν_p be as small as possible; that is, that the Q of the resonance be as large as possible, subject to the constraint that all electrons participate in the resonance.

The Q of the resonance can be written as

$$Q=\Omega/\beta\simeq\Omega(mV/\pi^2e^2N)^{1/2}\,. \quad (51)$$

Since $(\pi^2e^2N/mV)\lesssim\omega_p^2$, the frequency ordering $\omega_p\ll\Omega$ corresponds to the case of a high-Q resonance.

It is interesting to compare the energy loss rate for single mode resonance to that for many mode resonance. One expects that an optically thin plasma in a cavity which is sufficiently large and lossy so that each electron can resonate with many modes will lose energy at the rate $d\langle H\rangle/dt=-N2e^2\Omega^2\langle v_\perp^2\rangle/3c^3$, where $\langle v_\perp^2\rangle=2kT/m$. For an optically thick plasma, a large amount of the radiated energy would be reabsorbed; so this expression represents the maximum energy loss rate for a plasma experiencing resonance with many modes. The ratio of the energy loss rate for the single mode resonance to that for the many mode is given by

$$\frac{(d\langle H\rangle/dt)_s}{(d\langle H\rangle/dt)_m}=\left(\frac{3\pi^2}{4}\frac{(c/\Omega)^3}{V}\right)Q\,. \tag{52}$$

For a cavity with the axial dimension comparable to the radial dimension and for resonance with the lowest frequency mode that fits in the cavity, the quantity in parentheses is of order unity and the ratio is of order Q. In principle, a high-Q resonance with a single low-order mode always leads to faster energy loss than resonance with many modes in a large low-Q cavity.

In practice, cavities with dimensions of order $2\pi(c/\Omega)$ would be inconveniently small for a large magnetic field [e.g., $2\pi(c/\Omega)\simeq 1$ mm for $B\simeq 100$ kG]. For such a large magnetic field, one would probably employ the many mode resonance. It is particularly effective at large magnetic fields, since $d\langle H\rangle/dt$ is proportional to Ω^2. The practical significance of the single mode resonance is to extend the range of cyclotron cooling to lower magnetic fields, where the many mode resonance is ineffective.

In this regard, note that the volume V in the expression $d\langle H\rangle/dt=-NkT(\pi^2e^2/mNV)^{1/2}$ need not increase indefinitely as the magnetic field decreases (i.e., as c/Ω increases). For cyclotron frequencies that are low enough so that circuits may be used, V may be thought of as the volume of a capacitor which is part of a resonant circuit. The capacitor is connected to an external inductor and a cooled resistor. An appropriate geometry for the capacitor is two coaxial cylinders with negatively biased end sections and an axial magnetic field. If the inner cylinder is biased negatively relative to the outer cylinder, thermal equilibrium solutions are possible in which the electrons are confined in the region between the two cylinders. Axial confinement results from the negatively biased end sections. The high frequency field can be written as $\mathbf{E}=(-1/c)\mathbf{A}_\lambda q_\lambda+\text{c.c.}$, where $\mathbf{A}_\lambda=\hat{r}(1/r)M$ and M is a normalization constant chosen so that $\int d^3\mathbf{x}|\mathbf{A}_\lambda|^2=4\pi c^2$. The Hamiltonian for the circuit (without a resistor) takes the same form as the Hamiltonian for the cavity modes [i.e., Eq. (5)] if one makes the identifications

$$\sqrt{L}\,\dot{Q}=P_\lambda,\quad \sqrt{L}\,Q=Q_\lambda,\quad \omega_\lambda^2=1/LC\,, \tag{53}$$

where Q is the charge on the capacitor, C is capacitance, and L is the inductance. To include a resistor, we identify ν_λ with $R/2L$. Thus, Eq. (9) describes the evolution of $q_\lambda=\sqrt{L}\,Q/2-\sqrt{L}\,\dot{Q}/2i\omega_\lambda$, and we may take over the results for the cavity modes and apply them to the case of a circuit.

Of course, there are technological advantages in operating at lower magnetic fields. Also, the lowest temperature that can be reached by cyclotron cooling (i.e., $kT\simeq\hbar\Omega$) is lower for lower magnetic fields.

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