A confinement theorem for nonneutral plasmas

T. M. O'Neil

Department of Physics, University of California, La Jolla, California 92093
(Received 27 December 1979; accepted 14 August 1980)

A plasma consisting solely of particles of a single species is initially in the shape of a long column. It is confined by an axial magnetic field in a region of space bounded by a perfectly conducting and perfectly absorbing cylindrical wall. Conservation of angular momentum and conservation of energy are used to place an upper bound on the fraction of electrons that can ever reach the wall.

Recent experiments have involved the confinement of a collection of electrons of sufficient density to be called a plasma, that is, a pure electron plasma.[1] The confinement geometry for these experiments consists of a conducting cylinder which is immersed in a uniform axial magnetic field and is divided into three sections, the two end sections being biased negatively relative to the central section. The electrons reside in the central section, with axial confinement provided by electrostatic fields and radial confinement provided by the axial magnetic field. In such a geometry one need not worry about the axial confinement, assuming that the bias on the end cylinders is sufficiently negative. It is the radial confinement (i.e., magnetic confinement) that is worrisome.

In this paper, an infinitely long version of the geometry is considered (i.e., an infinitely long central cylinder), and an upper bound is placed on the fraction of electrons that can ever reach the cylindrical wall. The wall is assumed to be perfectly conducting and perfectly absorbing, and the bound relies on conservation of angular momentum and conservation of energy for the electron and field system. Effects such as electron collisions with neutral atoms, which transfer angular momentum to the electrons, are assumed not to exist. The physics underlying the bound is similar to that involved in certain stability theorems for nonneutral plasmas.[2,3]

The z component of the total angular momentum is given by

$$L_z = \sum_j m\gamma_j v_{\theta j} r_j + \int d^3\mathbf{x}\, r\hat{\theta}\cdot(\mathbf{E}\times\mathbf{B})/4\pi c\,, \tag{1}$$

where $\mathbf{x}=(r,\theta,z)$ is a cylindrical coordinate system, with the z axis coincident with the axis of the cylinder, $\mathbf{v}_j$ is the velocity of the jth electron, $\gamma_j=(1-v_j^2/c^2)^{-1/2}$, and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, respectively. The magnetic field can be expressed as $\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1$, where $\mathbf{B}_0$ is the uniform axial magnetic field and $\mathbf{B}_1$ is the change in the field due to the electrons. With the aid of Poisson's equation, one can show that

$$\int d^3\mathbf{x}\; r\hat{\theta}\cdot(\mathbf{E}\times\hat{z}B_0)/4\pi c = \sum_j \Omega m(R^2/2 - r_j^2/2),$$

where R is the radius of the cylinder, $\Omega=eB_0/mc$ is the electron cyclotron frequency, and the convention $e=|e|$ is used. Note that the quantity $-m\Omega r_j^2/2$ is simply the vector potential contribution to the canonical angular momentum for the jth electron [i.e., $-(e/c)A_\theta(r_j)r_j$, where $A_\theta(r_j)=B_0 r_j/2$]. Conservation of angular momentum may be written as

$$\sum_j m\Omega r_j^2/2 = \sum_j m\Omega r_j^2(0)/2 + \sum_j m\gamma_j v_{\theta j} r_j - \sum_j m\gamma_j(0)v_{\theta j}(0)r_j(0) + \int d^3\mathbf{x}\, r\hat{\theta}\cdot(\mathbf{E}\times\mathbf{B}_1)/4\pi c - \int d^3\mathbf{x}\, r\hat{\theta}\cdot[\mathbf{E}(0)\times B_1(0)]/4\pi c\,, \tag{2}$$

where $r_j(0)$ refers to $t=0$ and r_j to any later time t. The same convention holds for the other variables. Conservation of energy may be written as

$$K = K(0) + \int d^3\mathbf{x}[E^2(0)+B_1^2(0)]/8\pi - \int d^3\mathbf{x}(E^2+B_1^2)/8\pi\,, \tag{3}$$

where $K=mc^2\sum_j(\gamma_j-1)$ is the electron kinetic energy, and the fact that a perfectly conducting wall passes no flux has been used to set

$$\int d^3\mathbf{x}\,\mathbf{B}_1(0)\cdot\mathbf{B}_0 = \int d^3\mathbf{x}\,\mathbf{B}_1\cdot\mathbf{B}_0\,.$$

Before proceeding, we must establish the conventions necessary to insure that Eqs. (2) and (3) are valid even though electrons hit the wall. Of course, the interaction of an electron with a real wall is quite complicated. Even the idealization of a perfectly conducting wall involves a complication. As an electron approaches a perfectly conducting wall, the potential energy of interaction between the electron and its image in the wall approaches minus infinity, and the electron kinetic energy approaches plus infinity. In order to avoid these infinities, we consider an imaginary surface located a distance d out from the wall, where d is chosen to satisfy the inequality $R \gg d \gg e^2N/W(0)$. Here, $W(0)/N$ is the initial energy per electron minus the self-energy of an electron in unbounded space [see Eq. (14)]. For example, if $W(0)/N$ were in the range of 100 eV, then d could be chosen to be a few angstroms. For any electron that hits the wall, we retain, in Eqs. (2) and (3), the values of r_j and $\mathbf{v}_j$ and the electrostatic and magnetostatic fields associated with the electron at the moment it crosses the imaginary surface. The electron is assumed to

 0031-9171/80/112216-03$00.90

have negligible interaction with the other electrons after it crosses the surface. In other words, the fields associated with the electron are assumed to be highly localized and to involve an interaction between the electron and its image but not the other electrons. The energy associated with the electron-image interaction is of order e^2/d, which by hypothesis is negligible compared with $W(0)/N$. We will make use of this point shortly. Also, it is assumed that no electron returns from the wall to the plasma.

A simple heuristic argument leading to a bound on the fraction of electrons that can reach the wall was given previously.[4] This argument assumes that all of the terms in Eq. (2) are negligible except $\sum_j m\Omega r_j^2/2$ and $\sum_j m\Omega r_j^2(0)/2$. In other words, the heuristic argument approximates the total angular momentum by the vector potential term in the electron canonical angular momentum, i.e.,

$$L_z \simeq \sum_j -(e/c)A_\theta(r_j)r_j = -\sum_j m\Omega r_j^2/2 .$$

If N is the number of electrons initially in the column, ΔN is the number that ultimately reach the wall, R is the radius of the wall, and a^2 is the mean-square initial radius of the column, i.e.,

$$\sum_j r_j^2(0) = Na^2,$$

then the heuristic argument yields the result

$$\Delta N R^2 \leq \sum_j r_j^2 = \sum_j r_j^2(0) = Na^2 .$$

This can be rewritten as the bound $\Delta N/N \leq (a/R)^2$.

The difficulty with this argument is that the terms that are assumed to be negligible might not be negligible. Even if the initial conditions are such that $\sum_j m\gamma_j v_{\theta j} r_j$ starts off being negligible compared with $\sum_j m\Omega r_j^2/2$, it may not remain negligible. As the electrons move radially outward, the large radial electric field (required by Gauss' law) does work on the electrons, and $m\gamma v_\theta r$ can become large for at least some electrons. This might involve some electrons becoming relativistic. Also, it is not obvious that the angular momentum in the field, $\int d^3\mathbf{x}\, r\hat{\theta}\cdot(\mathbf{E}\times\mathbf{B}_1)/4\pi c$, remains negligible. One might worry that the electrons reach the wall by simply radiating away their angular momentum. In what follows, conservation of energy is used to bound the terms that were previously neglected and a corrected bound is obtained for $\Delta N/N$.

First, consider the term

$$\left|\sum_j m\gamma_j v_{\theta j} r_j\right| \leq \sum_j m\gamma_j v_j r_j .$$

We want to choose the v_j's so that the right-hand side of this inequality is a maximum for given values of the r_j's and for a given value of K. In other words, the v_j's must be chosen so that

$$\delta\left(\sum_j m\gamma_j v_j r_j - \alpha K\right) = 0,$$

where δ indicates variation with respect to the v_j's and α is a Lagrange multiplier. This equation reduces to

$$\sum_j m(r_j - \alpha v_j)\gamma_j^3 \delta v_j = 0.$$

Since the δv_j's are to be treated as independent of one another, there is only one extremum and it occurs for $v_j = r_j/\alpha$. Also, one can check that the extremum is a maximum; so we obtain the bound

$$\sum_j m\gamma_j v_j r_j \leq \alpha \sum_j mc^2 (r_j^2/\alpha^2c^2)(1 - r_j^2/\alpha^2c^2)^{-1/2}, \quad (4)$$

where the parameter α is determined by the relation

$$K = mc^2 \sum_j \left[(1 - r_j^2/\alpha^2c^2)^{-1/2} - 1\right]. \quad (5)$$

With the aid of Taylor expansions, one can show that $x(1-x)^{-1/2} \leq 2[(1-x)^{-1/2} - 1]$ for $0 < x < 1$; so, inequality (4) can be rewritten as

$$\sum_j m\gamma_j v_j r_j \leq 2\alpha K. \quad (6)$$

Inverting Eq. (5) to find α as a function of K and the r_j's is difficult; but an upper bound may be obtained for α easily. First consider the case where α satisfies the inequality $R/\alpha c \leq \epsilon$. Here, ϵ is any number in the range $0 < \epsilon < 1$. By taking into account the fact that $r_j < R$ and by using Taylor expansions, one finds that

$$\left[(1 - r_j^2/\alpha^2c^2)^{-1/2} - 1\right] \leq (r_j^2/\alpha^2c^2)\epsilon^{-2}\left[(1-\epsilon^2)^{-1/2} - 1\right], \quad (7)$$

and by summing over j one obtains the bound

$$\alpha \leq \left[(\beta m/K)\sum_j r_j^2\right]^{1/2}, \quad (8)$$

where $\beta \equiv \epsilon^{-2}[(1-\epsilon^2)^{-1/2} - 1]$ has been introduced for brevity. For the opposite case (i.e., $R/\alpha c > \epsilon$), one trivially obtains the bound $\alpha < R/\epsilon c$. In general, α must be smaller than the sum of the bounds obtained separately for the two cases

$$\alpha \leq \left[(\beta m/K)\sum_j r_j^2\right]^{1/2} + R/\epsilon c . \quad (9)$$

At this point, ϵ should be chosen to minimize the right-hand side of this inequality. However, the minimization procedure is difficult analytically and little is gained over a judicious choice for ϵ. To understand this latter point, note that for $0 < \epsilon < 1$ the minimum value of $1/\epsilon$ is 1, obtained for $\epsilon = 1$, and the minimum value of $\beta = \epsilon^{-2}[(1-\epsilon^2)^{-1/2} - 1]$ is 1/2, obtained for $\epsilon = 0$; whereas, a choice such as $\epsilon = \frac{3}{4}$ yields $1/\epsilon \simeq 1.3$ and $\beta \simeq 0.9$, which are both reasonably close to the respective minima. For generality, we leave ϵ unspecified but bear in mind that $1/\epsilon$ and β are both close to unity. Combining inequality (9) with inequality (6) yields the result

$$\sum_j m\gamma_j v_j r_j \leq 2\left(\beta m K \sum_j r_j^2\right)^{1/2} + 2RK/(\epsilon c). \quad (10)$$

Conservation of energy can be used to bound K. If the electric field is expressed as the sum of a longitudinal and a transverse part (i.e., $\mathbf{E} = -\nabla\phi + \mathbf{E}_1$, where $\nabla^2\phi = -4\pi\rho$ and $\nabla\cdot\mathbf{E}_1 = 0$), the electric field energy can be written as

$$\int d^3\mathbf{x}\, E^2/8\pi = \frac{1}{2}\int d^3\mathbf{x}\,\rho(\mathbf{x})\phi(\mathbf{x}) + \int d^3\mathbf{x}\, E_1^2/8\pi\,, \quad (11)$$

where the boundary condition $\phi = 0$ on the conducting wall has been used. Thus, Eq. (3) yields the inequality

$$K \le K(0) + \int d^3\mathbf{x}[E^2(0) + B_1^2(0)]/8\pi - \frac{1}{2}\int d^3\mathbf{x}\rho(\mathbf{x})\phi(\mathbf{x})\,. \quad (12)$$

The last term can be expressed as

$$\frac{1}{2}\int d^3\mathbf{x}\rho(\mathbf{x})\phi(\mathbf{x}) = \sum_j \frac{1}{2}\int d^3\mathbf{x}\rho_j(\mathbf{x})\phi_j(\mathbf{x}) + \sum_{i\ne j}\frac{1}{2}\int d^3\mathbf{x}\rho_i(\mathbf{x})\phi_j(\mathbf{x})\,, \quad (13)$$

where $\rho_i(\mathbf{x})$ is the charge density of the ith electron and $\phi_i(\mathbf{x})$ is the instantaneous potential due to that electron. We have in mind here that the charge for each electron is distributed over a spherical region of radius comparable to the classical electron radius. The last term on the right-hand side of Eq. (13) represents the interaction energy between the various electrons. From the fact that $\phi = 0$ on the cylindrical wall and the fact that $\nabla^2\phi_j \ge 0$ everywhere inside the cylinder, one can see that $\phi_j \le 0$ everywhere inside the cylinder. Thus, the last term on the right-hand side of Eq. (13) is non-negative and may be neglected when Eq. (13) is substituted into inequality (12). The first term is the sum of the self-energies for the electrons. For an individual electron, the self-energy differs from the self-energy for an electron in unbounded space by the interaction energy between the electron and its image in the wall. By our choice of the distance between the wall and the imaginary surface where we declare an electron to be lost, the interaction energy is negligible [i.e., $e^2/d \ll W(0)/N$]. Thus, inequality (12) can be rewritten as

$$K \le K(0) + \int d^3\mathbf{x}[E^2(0) + B_1^2(0)]/8\pi - N\epsilon_s \equiv W(0)\,, \quad (14)$$

where ϵ_s is the self-energy of an electron in unbounded space.

From inequalities (10) and (14) we find the bound

$$\left|\sum_j m\gamma_j v_{\theta j} r_j\right| \le 2\left[\beta m W(0)\sum_j r_j^2\right]^{1/2} + 2RW(0)/\epsilon c\,. \quad (15)$$

The same bound holds for the term $\left|\sum_j m\gamma_j(0)v_{\theta j}(0)r_j(0)\right|$.

Next, we consider the term $\int d^3\mathbf{x}\, r\hat{\theta}\cdot(\mathbf{E}\times\mathbf{B}_1)/4\pi c$. The momentum density, $(\mathbf{E}\times\mathbf{B}_1/4\pi c$, is also $1/c^2$ times the energy flux (i.e., Poynting vector). The self-field for a particle travels with the particle and produces an energy flux $\mathbf{v}\epsilon_s$ and a momentum $\mathbf{v}m_s$, where $m_s = \epsilon_s/c^2$. This momentum is usually thought of as part of the mechanical momentum, which we have just bounded. The remaining field energy is bounded by $W(0)$ and can correspond to a momentum no larger than $c[W(0)/c^2]$ and to an angular momentum no larger than $(R/c)W(0)$. The same bound holds for $\left|-\int d^3\mathbf{x}\, r\hat{\theta}\cdot\mathbf{E}(0)\times\mathbf{B}_1(0)/4\pi c\right|$.

Using these bounds Eq. (2) can be rewritten as the inequality

$$\sum_j m\Omega r_j^2/2 \le \sum_j m\Omega r_j^2(0)/2 + 4\left[\beta m W(0)\sum_j r_j^2\right]^{1/2} + (2 + 4/\epsilon)W(0)R/c\,. \quad (16)$$

With the aid of the quadratic formula the inequality can be rewritten as

$$\left\{\sum_j r_j^2\left[\sum_j r_j^2(0)\right]^{-1}\right\}^{1/2} \le 2^{3/2}\beta^{1/2}A + \{1 + A^2[8\beta + (2 + 4/\epsilon)(R\Omega/c)]\}^{1/2}\,, \quad (17)$$

where $A^2 = W(0)/\sum_j m\Omega^2 r_j^2(0)/2$. If ΔN is the number of electrons that are lost (i.e., have $r_j = R$) and a^2 is the mean square initial radius of the plasma [i.e., $Na^2 = \sum_j r_j^2(0)$], then we find the bound

$$(\Delta N/N) \le (a/R)^2 D^2\,, \quad (18)$$

where D is the right-hand side of inequality (17). When A and $(R\Omega/c)A^2$ are both small compared with unity, the bound reduces to that obtained from the heuristic argument mentioned earlier [(i.e., $\Delta N/N \le (a/R)^2$]. Recall from the discussion preceding inequality (10) that $1/\epsilon$ and β may be chosen to be near unity. We note that the conditions for the experiment of Ref. 1 are such that A and $(R\Omega/c)A^2$ are both small compared with unity; so, the bound $(\Delta N/N) < (a/R)^2$ is appropriate for these experiments. However, the corrections obtained here may be important for future experiments.

Of course, in the experiments, essentially all the electrons ultimately reach the wall even though $(a/R)^2 \ll 1$. The utility of the present theorem is that in searching for the cause of the electron loss one need only consider those effects that can transfer angular momentum to the electron and field system, that is, effects such as electron collisions with neutral atoms, field errors that are not cylindrically symmetrical, and finite wall resistance.

ACKNOWLEDGMENT

This work was supported by National Science Foundation Grant PHY-77-20613.

[1]J. deGrassie and J. Malmberg, Phys. Rev. Lett. **39**, 1077 (1977).
[2]R. Davidson, *Theory of Nonneutral Plasmas* (Benjamin, New York, 1974), p. 155.
[3]H. V. Wong, M. L. Sloan, J. Thompson, and A. T. Dobrot, Phys. Fluids **16**, 903 (1973).
[4]J. Malmberg and T. O'Neil, Phys. Rev. Lett. **21**, 1333 (1977).