Equilibration rate of spin temperature in a strongly magnetized pure electron plasma

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The equilibration of spin temperature $T_{\rm spin}$ with kinetic temperature T is examined in a weakly correlated pure electron plasma in the strongly magnetized limit, where the distance of closest approach is large compared to the Larmor radius. In this limit, the spin precession frequency $\Omega_p = g\Omega_c/2$ is large so the component of spin along the magnetic field is an adiabatic invariant that is broken only by resonant magnetic fluctuations of frequency Ω_p. (Here Ω_c is the electron cyclotron frequency and $g \cong 2.002$.) In this case, the most important spin flip mechanism stems from electron–electron collisions in a spatially inhomogeneous magnetic field. Such collisions cause an exchange of spin and cyclotron quanta, and consequently the conventional many-electron adiabatic invariant (i.e., the total number of cyclotron quanta) is broken and is replaced by a new adiabatic invariant, equal to the sum of the spin and cyclotron actions. A quantum Boltzmann equation is derived to describe the equilibration of $T_{\rm spin}$ toward T.

I. INTRODUCTION

Recent experiments have confined and cooled a pure-electron plasma to cryogenic temperatures, $T \sim 1\text{--}10^2$ K, in a strong solenoidal magnetic field, $B \sim 10\text{--}60$ kG.[1] This range of temperatures and magnetic fields places the plasma in the novel regime of strong magnetization, in which the average distance of closest approach $\bar{b} \equiv 2e^2/kT_\parallel$ is large compared to the average Larmor radius $\bar{r}_L = \sqrt{kT_\perp/m}/\Omega_c$ (where e is the electron charge, $T_\parallel$ and $T_\perp$ are the kinetic temperatures associated with the distributions of velocities parallel and perpendicular to the magnetic field, m the electron mass, and $\Omega_c = eB/mc$ is the electron cyclotron frequency).

In this paper we consider a strongly magnetized pure electron plasma that initially has a temperature associated with the distribution of electron spins, T_s, which is different from the kinetic temperatures $T_\parallel$ and $T_\perp$. We calculate the rate at which T_s, $T_\parallel$, and $T_\perp$ should relax to a common value. We assume throughout that the plasma is weakly correlated (i.e., that $n\lambda_D^3 \gg 1$, where n is the density and $\lambda_D = \sqrt{kT/4\pi e^2 n}$ is the Debye length).

After examining several mechanisms that couple the spin and kinetic degrees of freedom, we conclude that the dominant spin-flip process is an electron–electron collision in a spatially inhomogeneous magnetic field. In the experiments, the confining magnetic field is inhomogeneous due, among other things, to the finite length of the solenoid. The degree of field nonuniformity can be controlled by confining the plasma at different distances from the end of the solenoid. The ability to control the rate of spin temperature relaxation may be useful in future experiments which rely on measurements of the degree of electron spin polarization. Two such experiments are briefly discussed in the conclusion of the paper.

In order to understand the spin-flip process intuitively, it is useful to consider a classical model of the spin dynamics in which the spin is regarded as a classical magnetic moment. (It is well known that this classical picture is rigorously correct if one considers the dynamics of the quantum expectation value of the spin operator.) The direction of the moment precesses around the magnetic field at the spin precession frequency $\Omega_p = g\Omega_c/2$, where g is the Landé g factor, equal to 2.002... for electrons. In the regime of strong magnetization Ω_p is large and hence the component of the magnetic moment along the magnetic field is an adiabatic invariant. In order to flip the spin this adiabatic invariant must be broken. If collisions with neutrals are neglected (and we will see that this effect is unimportant in Sec. II), the only way to break the invariant is through a resonant fluctuation in the magnetic field—that is, a fluctuation at frequency Ω_p in the electrons' rest frame. Electron cyclotron motion in a spatially nonuniform (but time-independent) magnetic field is almost of the right frequency to cause such a fluctuation in the electron's rest frame, since the electron g factor is nearly equal to 2. However, $\Omega_p - \Omega_c \simeq 0.001\Omega_c$ is still a large frequency, and so cyclotron motion by itself is not enough to break the invariant, and a perturbation of the cyclotron motion must occur which is of sufficiently high frequency to make up the difference and cause a resonance between the spin and orbital dynamics. In the regime of strong magnetization the only perturbation of such high frequency is an electron–electron collision, which induces orbital perturbations with frequencies of order $\bar{v}/b$, where $\bar{v}$ is the thermal speed.

In order to estimate the magnitude of the spin depolarization rate due to electron–electron collisions in a spatially nonuniform **B** field, consider a strong static magnetic field $B\hat{z}$ along with a small time varying magnetic field $\delta\mathbf{B}(t)$ in the electron's rest frame. This time-dependent field is due to electron motion through the spatially inhomogeneous but time-independent external magnetic field. We will estimate δB presently, but for now all we need to assume is that for a time $\Delta t \sim \bar{b}/\bar{v}$, the time scale of an electron–electron collision, $\delta\mathbf{B}(t)$ has a right-circularly po-

larized component rotating at frequency $\omega = \Omega_p$. This component resonates with the spin precession and drives a spin flip. The magnitude of this resonant component, δB_R, will be given approximately by a sum over all temporal Fourier components of the right-polarized part of $\delta B(t)$ with frequencies ω satisfying $|\omega - \Omega_p| \lesssim 2\pi/\Delta t$:

$$\delta B_R \sim \int_{|\omega - \Omega_p| < 2\pi/\Delta t} d\omega \, \frac{(\hat{x} + i\hat{y})}{\sqrt{2}} \cdot \delta\widetilde{B}(\omega),$$

where $\delta\widetilde{B}(\omega)$ is the Fourier transform of $\delta B(t)$. The probability amplitude ΔC of the spin flip is then given, in perturbation theory, by the angle through which the spin precesses in time Δt due to this resonant field:

$$|\Delta C| \sim \frac{eg}{2mc} |\delta B_R| \Delta t.$$

Now, $\delta B(t)$ can be estimated for an electron executing cyclotron motion in a slightly nonuniform magnetic field: $\delta \mathbf{B}(t) \sim \boldsymbol{\rho}(t) \cdot \nabla B$, where $\boldsymbol{\rho}$ is a vector describing the cyclotron motion: $\boldsymbol{\rho}(t) = \bar{r}_L[\cos(\Omega_c t + \theta)\hat{x} + \sin(\Omega_c t + \theta)\hat{y}]$, where θ is the constant gyrophase. If one further assumes that the electron suffers a collision for which the impact parameter is large compared to the Larmor radius, we will see in Sec. III that the most important effect is that θ becomes a function of time, adding Fourier components that bring $\delta B(t)$ into resonance with the spin precession: $\theta = \theta_0 + \delta\theta(t)$. Since $\delta\theta(t)$ is small in such a collision, one can expand to first order in $\delta\theta$ to find that the resonant magnetic field is approximately

$$\delta B_R \sim B \frac{\bar{r}_L}{L} \int_{|\omega - \Omega_p| < 2\pi/\Delta t} \frac{d\omega}{2\pi} \delta\widetilde{\theta}(\omega - \Omega_c),$$

where $\delta\widetilde{\theta}$ is the Fourier transform of $\delta\theta(t)$ and L is the scale length of variation of B. Since $\delta\theta(t)$ varies on a time scale of order $\Delta t = \bar{b}/\bar{v}$, if one assumes that $(\Omega_p - \Omega_c)\Delta t \lesssim 1$ this integral can be estimated as approximately

$$\delta B_R \sim B \frac{\bar{r}_L}{\Delta t \, L} \delta\widetilde{\theta}(\Omega_p - \Omega_c).$$

Furthermore, the magnitude of $\delta\widetilde{\theta}(\Omega_p - \Omega_c)$ can be estimated using dimensional analysis of the integral expression for the Fourier transform:

$$|\delta\widetilde{\theta}(\Omega_p - \Omega_c)| = \left| \int_{-\infty}^{\infty} dt \, \delta\theta(t) e^{-i(\Omega_p - \Omega_c)t} \right| \sim \frac{\Delta\theta}{(\Omega_p - \Omega_c)},$$

where $\Delta\theta$ is the total change in $\delta\theta(t)$ during the collision, and again we have assumed $(\Omega_p - \Omega_c)\Delta t \lesssim 1$. In this regime we show in Sec. III that $\Delta\theta$ is roughly on order of the small parameter $\bar{\epsilon} \equiv \bar{v}_z/\bar{b}\Omega_{c_0}$, where $\bar{v}_z$ is the parallel relative thermal speed. This parameter is the ratio of the frequency associated with a collision compared to the cyclotron frequency. When $\bar{\epsilon} \ll 1$ the plasma is in the strongly magnetized regime. Using this estimate for $\Delta\theta$, the spin-flip amplitude is approximately

$$|\Delta C| \sim \frac{\bar{r}_L}{L} \frac{\Delta\theta}{\Omega_p - \Omega_c} \frac{\Omega_c}{\Omega_p - \Omega_c} \sim \frac{\bar{\epsilon}\bar{r}_L}{L} \frac{\Omega_c}{\Omega_p - \Omega_c}. \quad (1)$$

Finally, over many uncorrelated collisions the spin direction gradually diffuses in a random walk and the rate of spin depolarization is given by $\nu_{\text{spin}} \sim \nu_c |\Delta C|^2$, where $\nu_c = \pi n \bar{v} \bar{b}^2$ is the electron–electron collision frequency.

This estimate for the depolarization rate gives the proper scaling of the spin depolarization rate, provided that $|\Omega_p - \Omega_c| \lesssim \bar{v}/\bar{b}$, or $\bar{\epsilon} \gtrsim g/2 - 1 = 0.001$. For $\bar{\epsilon} < 0.001$ we will see that ν_{spin} becomes exponentially small. This is because $(\Omega_p - \Omega_c)\Delta t$ becomes greater than unity in this regime, so that $\Delta\theta$ becomes exponentially small.

Although there has been considerable previous work, both theoretical and experimental, on the spin relaxation in neutral gases and solids, spin relaxation in plasmas has not received as much attention. However, the problem has been considered theoretically for plasma parameters of fusion interest. In this interesting work[2] it was noted that the fusion cross section for D–T reactions is enhanced when the reacting nuclei's spins are aligned, and so an increase of the fusion power output is achieved if the plasma ions are spin polarized. A calculation of the rate at which the nuclear spins are depolarized by various effects was then carried out.

It was found that, except for the effect of plasma waves, collisional depolarization in an inhomogeneous magnetic field is also the dominant depolarization effect in fusion plasmas. However, although collisions give rise to spin relaxation effects for both fusion plasmas and pure electron plasmas, the relaxation rates are quite different in the two cases. For collisions in a fusion plasma, the time scale on which the orbit changes, or the effective duration time of collision, is much shorter than the gyroperiod and so the detailed dynamics of an individual collision, which may be termed an "impulsive" random kick, is expected to be unimportant. In this case it suffices to take $\Delta\theta \sim 2\pi$ in Eq. (1), and then the relaxation rate given by Eq. (36) of Ref. 2 is recovered. On the other hand, during the effective duration time of collision in a strongly magnetized plasma, the electron gyrates over many cycles. In this case, there is only a small change of the gyrophase due to the Coulomb interaction. Evidently, the detailed collisional gyrodynamics is important for the determination of this change during a given collision.

The collisional process considered here causes an exchange of spin and cyclotron energy, and consequently the many electron adiabatic invariant of O'Neil and Hjorth,[3] equal to the sum of the perpendicular kinetic energies $\Sigma_i E_{\perp i}$, is broken. However, as we will see this adiabatic invariant is replaced by a new N-electron invariant equal to the sum of the spin and cyclotron actions:

$$\mu^{(N)} = \sum_i \left(s_{iz} + \frac{E_{\perp i}}{\Omega_c(\mathbf{x}_i)} \right) = \text{const}, \quad (2)$$

where s_{iz} is the component of the spin along the magnetic field for electron i and $E_{\perp i}/\Omega_c(\mathbf{x}_i)$ is the cyclotron action. The conservation of $\mu^{(N)}$ implies that this collisional process cannot by itself drive the system to complete thermal equilibrium and, in general, $T_s \neq T_\perp \neq T_\parallel$ will be the result. Rather, in Sec. V we obtain the relation

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$$\frac{1}{T_\perp} + \frac{(g/2-1)}{T_\parallel} - \frac{g/2}{T_s} = 0, \qquad (3)$$

which holds for the state of partial thermal equilibrium that is achieved after many collisions, which conserve $\mu^{(N)}$. Of course, since $\mu^{(N)}$ is not an exact invariant, electron–electron collisions occur, which cause exponentially small changes in its value. Because the spatial variation of the magnetic field is slow compared to the Larmor radius of the strongly magnetized electrons, almost all these collisions are of the type described by O'Neil and Hjorth in which the spin plays no role, and these collisions cause $T_\perp$ to approach $T_\parallel$ according to the equations described in Ref. 3. In turn, collisions considered in this paper that conserve $\mu^{(N)}$ cause T_s to approach the common value of $T_\perp$ and $T_\parallel$ [see Eq. (3) for $T_\perp = T_\parallel$], and hence a state of complete thermal equilibrium is achieved. This is the qualitative picture of spin relaxation that emerges from our analysis.

In Sec. II we make order-of-magnitude estimates of various spin-flip processes, including spin flip due to the mutually generated magnetic field, radiative transitions, and interactions with background waves, Thomas precession, electron–neutral collisions, and single particle electron motion through the inhomogeneous B field. We find that all processes except for electron–electron collisions in an inhomogeneous B field produce depolarization time scales that are longer than the plasma confinement time of approximately 10^5 sec, provided that neutrals with partially filled valence shells, such as N_2, are kept at pressures below $\sim 10^{-14}$ Torr (this is a reasonable upper bound in the cryogenic environment of the present experiments). In the regime $\hbar\Omega_c < kT_\perp$ and $\bar{\epsilon} > 0.001$, we find that spin depolarization rate is $\nu_{spin} = 1.5 \times 10^4 \, \nu_c \, (\bar{\epsilon}\bar{r}_L/L)^2$. For a typical plasma density of 10^8 cm^{-3} and $B = 10$ kG; this implies that the B-field inhomogeneity scale length L must satisfy L (cm) $< 7.15 T^{6/4}$ (K) in order for ν_{spin}^{-1} to be less than the plasma confinement time. In Sec. III we present a calculation of the spin-flip transition rate due to electron–electron collisions in a weakly inhomogeneous magnetic field, assuming that the orbital motion can be treated classically (that is, assuming that the electron's kinetic energy is large compared to $\hbar\Omega_c$). This calculation improves the estimate for ν_{spin} given by Eq. (1), extending it to cover the range $\bar{\epsilon} \lesssim 0.1$. In Sec. IV, the calculation is repeated using a quantum description of the orbital motion, since, in fact, $T_\perp$ and $T_\parallel$ can be of order $\hbar\Omega_c$ in the experiments. In the classical limit this rate agrees with that calculated in Sec. III. In Sec. V we present a derivation of a Boltzmann collision operator for spin relaxation that conserves $\mu^{(N)}$ and drives the system to a partial thermal equilibrium described by Eq. (3). We summarize our results in the conclusion and discuss two possible experiments that rely on measurements of spin polarization.

II. ORDER OF MAGNITUDE ESTIMATES FOR SPIN DEPOLARIZATION PROCESSES

As pointed out in our discussion, spin depolarization is caused by a resonant perturbing magnetic field of fre-
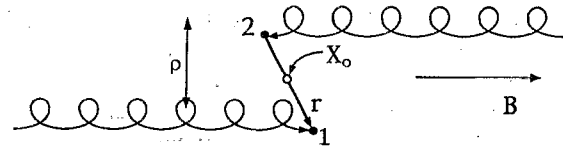

FIG. 1. Schematic picture for an electron–electron collision in a uniform magnetic field in the strongly magnetized limit.

quency Ω_p due to an electron–electron collision in a non-uniform magnetic field. Such resonant fields can also be induced by other mechanisms. We will consider four such processes, as well as a fifth process due to spin exchange in electron–neutral collisions. In order to simplify results we assume that $T_\perp$ and $T_\parallel$ are of the same order of magnitude.

A. Spin flip due to mutually generated magnetic field

Consider two electrons, 1 and 2, immersed in a uniform external field B, separated by relative distance r, and passing by one another with impact parameter ρ on the order of $\bar{b} \gg \bar{r}_L$ (see Fig. 1). Then electron 1 sees a time varying magnetic field induced by the relative motion of electron 2 as well as electron 2's intrinsic magnetic moment. In the former case the field is $\delta\mathbf{B} = (e/c)(\dot{\mathbf{r}} \times \mathbf{r})/r^3$, and the component of this field that is resonant with the spin precession is approximately $\delta B_R \sim (e/c)v_\perp z/r^3$, to lowest order in $\bar{r}_L/\bar{b}$. Taking v_1 equal to the thermal velocity $\bar{v} \equiv \sqrt{kT/m}$ and the effective interaction time of the electrons equal to $\bar{b}/\bar{v}$, we find the change in direction of spin is $\Delta C \sim eg\,\delta B_R/2mc \cdot \bar{b}/\bar{v} \sim (\bar{v}/c)^2 \sim 10^{-10}\bar{T}$, where $\bar{T}$ is the temperature in K. This gives rise to an extremely small depolarization rate $\nu_c(\Delta C)^2 \sim 10^{-12}T^{1/2}\bar{n}(\text{sec}^{-1})$, where $\nu_c \simeq \pi\bar{b}^2n\bar{v}$ is the electron–electron collision frequency and $\bar{n}$ is the electron density (n) in units of 10^7 cm^{-3}.

The intrinsic magnetic dipole moment of electron 2 also induces a time varying magnetic field at electron 1. However, this magnetic field is so weak that the spin depolarization effect is negligible, even compared to the above estimate.

B. Radiative transitions and interactions with background waves

As an electron's spin precesses, its related intrinsic magnetic moment will radiate spontaneously through magnetic dipole transition. The rate for the spontaneous radiation is[4]

$$\frac{2}{3}\left(\frac{g}{2}\right)^2 \frac{e^2\hbar}{m^2c^3}\,\Omega_c^3 \simeq 7 \times 10^{-11}\bar{B}^3(\text{sec}^{-1}),$$

where B is the magnitude of the magnetic field in Tesla.

In addition, as pointed out by, for example, Kulsrud et al.,[2] in a uniform magnetic field, the right-circularly polarized component of an electromagnetic wave with harmonics near Ω_p will cause an electron spin depolarization. It is easy to show that a thermal level of electromagnetic waves produces negligible depolarization, provided that the

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plasma is optically thin. When the plasma is optically thick, the problem is more complicated due to the dielectric behavior of the magnetized plasma, but we believe that for a thermal equilibrium plasma at cryogenic temperature, the electric current fluctuation is negligibly small and there is not appreciable excitation of magnetic fluctuations. However, waves that are unstable in the range of electron spin precession frequency could cause appreciable spin depolarization. Although electromagnetic instabilities could be driven when $T_\perp$ and $T_\parallel$ differ, if the temperature difference $|T_\perp - T_\parallel|$ is not too large and no external heating is assumed, then, unlike the spin polarized fusion plasma, the presence of strong cyclotron damping should make the existence of unstable waves at the spin precession frequency unlikely [since the electron spin precession frequency $\Omega_p = (g/2)\Omega_c$ is close to the cyclotron frequency Ω_c].

Another possible depolarization effect is due to the electron position shift driven by electrostatic waves at the spin precession frequency Ω_p. In a spatially nonuniform magnetic field the magnetic field seen in the electron's rest frame is perturbed at frequency Ω_p and the electron spin is flipped by the resonant magnetic field perturbation. However, one may show that compared with the collisional effect, this effect is also negligible for a thermal level of waves in the strongly magnetized cryogenic plasma. Physically this is due to the relatively few degrees of freedom involved in these collective electrostatic modes compared with the perturbing electrostatic field due to collisions.

C. Thomas precession

Due to this pure relativistic effect, the electron sees an additional perturbing magnetic field corresponding to a precession frequency, $\omega_T(t) \simeq \dot{\bar{\mathbf{v}}} \times \bar{\mathbf{v}}/2c^2$. The magnitude of this frequency does not equal Ω_p, except during an electron–electron collision. During the collision, a component of the Thomas precession frequency given by $\omega_T(t) \simeq \dot{\bar{\mathbf{v}}}_\perp \times \bar{v}_\parallel/2c^2$ varies at the resonant frequency and so leads to a spin direction change $\Delta C \sim (\Omega_c \bar{v}_\perp \bar{v}_\parallel/2c^2) \cdot (\bar{b}/\bar{v}_\parallel) \sim (\bar{b}/\bar{r}_L) \cdot (\bar{v}_\perp/c)^2$. Here, again, we have kept only the lowest-order component of ω_T in an expansion in $\bar{r}_L/\bar{b}$. The depolarization rate is then $\nu_c(\Delta C)^2 \sim 10^{-7}\bar{n}\bar{B}^2\sqrt{T_\perp}\,T_\parallel^{-3}$ (sec^{-1}). As before, $\bar{B}$ is in units of Tesla and $\bar{n}$, $T_\perp$, and $T_\parallel$ are again in units as defined in Sec. II A.

D. Electron–neutral collisions

In the cryogenic environment of the experiment it is likely that the residual neutrals are almost entirely helium since most of the neutral gas freezes on the wall. Nevertheless, there may be traces of other gases, and here we also consider collisions with nitrogen molecules as a representative example. To calculate the spin depolarization rate due to electron–neutral collision, we note that the spin-flip cross section due to spin exchange between the free electron and atomic electron is several orders of magnitude larger than that due to other effects,[5] such as the spin–orbit interaction. For electron–helium collisions the spin ex-

change is inhibited by the Pauli exclusive principle, and so the depolarization effect is effectively negligible for them. For an electron–nitrogen collision, the spin-flip cross section is $\sigma_{\text{spin flip}} \simeq \sigma_{\text{spin exchange}} \lesssim \sigma_{\text{kinetic}} \simeq 3$ Å^2, and thus the depolarization rate is approximately $\sigma_{\text{spin flip}} \cdot \bar{n}_{N_2} \bar{v}_e \lesssim 10^{-5}T^{-1/2}\bar{n}_{N_2}$ (sec^{-1}), where $\bar{n}_{N_2}$ is the density of nitrogen molecules in units of 10^4/cm^3 and T is again the temperature in units of Kelvin.

In addition, electron–neutral collisions change the electron's orbit randomly, resulting in a fluctuating magnetic field in the electron's rest frame due to the nonuniform external magnetic field. This perturbing field causes a spin flip at the rate[2] $\nu_{\text{spin}} \approx 9.57 \times 10^{-13}T^{3/2}B^{-2}L^{-2}\bar{n}_N$, where the scale length of magnetic field inhomogeneity L is in units of cm and $\bar{n}_N$ is the neutral density in units of 10^4/cm^3.

E. Single-particle motion

Single-particle motion consists of cyclotron motion together with a slow $\mathbf{E}\times\mathbf{B}$ rotation of the plasma column and parallel streaming along the slightly curved magnetic field lines. Neither of these drifts cause sufficiently high-frequency perturbations to the magnetic field observed in the electron rest frame, and so these effects cause negligible spin depolarization. However, as an electron approaches the end of the plasma along a field line, the electron feels an electric potential with a scale length of gradient of order of λ_D, the Debye length. Due to the electric potential, the electron gyro-orbit is disturbed, and thus, in the slightly nonuniform B field, as in an electron–electron collision, a secular spin depolarization results. However, since $\lambda_D \gg b$, the "collision" with the end of the plasma is much slower than an electron–electron collision, and the resonant field δB_R is much smaller. The size of this effect can be estimated by substitution of λ_D for b in Eq. (1) and use of the axial bounce frequency $\nu_b = \bar{v}/L_p$ rather than ν_c, where L_p is the length of the plasma. This implies a depolarization rate smaller than that given by Eq. (1) by the factor $(\nu_b/\nu_c) \cdot (\bar{b}/\lambda_D)^2 \sim 10^{-3}/T\bar{L}_p$, where $\bar{L}_p$ are lengths in units of centimeters. This result is further reduced if the electron mean-free path is less than L_p, and so should be regarded as an upper bound.

The depolarization rate ν_{spin} for various spin relaxation processes are plotted as a function of small parameter $\bar{\epsilon} = \bar{r}_L/\bar{b} = 10^{-3}\bar{T}^{3/2}/\bar{B}$ in Fig. 2, where the other parameters $\bar{B}$, $\bar{n}$, $\bar{n}_{N_2}$, and $\bar{n}_N$ are set to be unity and $\bar{L}$ is set to be 10 cm. The conclusion we draw from the figure is that the spin relaxation time ν_{spin}^{-1} due to all effects considered other than that of collisional depolarization in a nonuniform magnetic field is longer than the maximum plasma confinement time of approximately 10^5 sec, provided that $\bar{n}_{N_2} \lesssim 1$. Therefore, we conclude that the dominant depolarization effect is due to collisional depolarization in an inhomogeneous magnetic field.
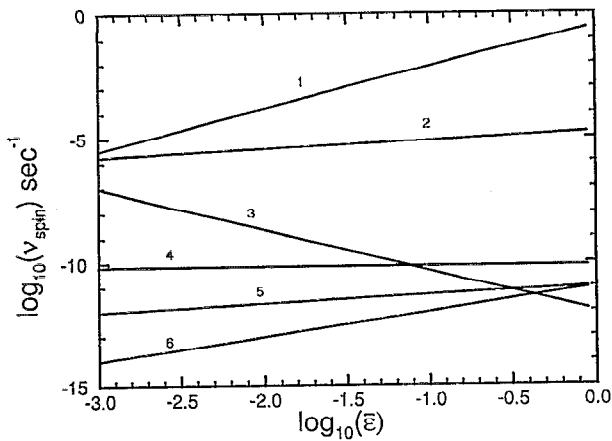
FIG. 2. Plot of the spin depolarization rate as a function of $\bar{\varepsilon} = \bar{r}_L/\bar{b} = 10^{-3}\bar{T}^{3/2}/\bar{B}$ for different processes. Curve 1: collisional depolarization in a nonuniform magnetic field. Curve 2: spin exchange effect during electron–neutral collisions. Curve 3: Thomas precession. Curve 4: spontaneous magnetic dipole radiation. Curve 5: spin flip due to mutually generated magnetic field. Curve 6: spin flip due to electron–neutral collision in a nonuniform magnetic field. The electron density n_e is assumed to be 10^7 cm^{-3}, the neutral density is taken to be 10^4 cm^{-3}, the magnetic field is 1 T and the scale length of magnetic field inhomogeneity is taken to be 10 cm.

III. COLLISIONAL SPIN DEPOLARIZATION IN AN INHOMOGENEOUS MAGNETIC FIELD

In this section we consider in detail the problem of spin depolarization due to electron–electron collisions in a weakly inhomogeneous magnetic field. The velocities of the colliding electrons are taken to be sufficiently large so that we can treat the orbital dynamics classically. We will eventually expand in the small parameters r_L/b and r_L/L, but in order to set up the problem we consider the spin dynamics of a spin-$\frac{1}{2}$ particle moving on a general classical trajectory through an inhomogeneous magnetic field.[2] In a fixed laboratory frame of reference the spin part of the wave function $|\psi\rangle$ evolves according to

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = \mathbf{s}\cdot\boldsymbol{\Omega}_p|\psi\rangle, \qquad (4)$$

where $\Omega_p(t) = (g/2)[eB(\mathbf{x}(t))/mc]$, $\mathbf{x}(t)$ is the position of the electron, $\mathbf{s} = \hbar/2\ (\sigma_x, \sigma_y, \sigma_z)$ is the spin operator for spin-$\frac{1}{2}$ particles, and σ_x, σ_y, and σ_z are the Pauli matrices, with respect to some fixed coordinate axes. The classical approximation employed throughout this section implies that $\mathbf{x}(t)$ is unaffected by the spin state and so is a given function of time.

Now, because the spin component along the field is an adiabatic invariant we consider the evolution of the spin in a noninertial frame of reference, which follows the electron and which keeps the z axis directed along the magnetic field. Since these coordinate axes rotate in time as the field varies in direction in the electrons' rest frame, the spin Hamiltonian $\mathbf{s}\cdot\boldsymbol{\Omega}_p$ transforms into the noninertial frame according to the usual relation $\hat{H}' = \mathbf{s}\cdot\boldsymbol{\Omega}_p - \mathbf{s}\cdot\boldsymbol{\omega}$, where $\boldsymbol{\omega} = \hat{b}\times d\hat{b}/dt - \omega_z\hat{b}$ is the rate of rotation of the coordinate

frame, ω_z represents an arbitrary rotation of the coordinates around $\mathbf{B}$, and $\hat{b} = \mathbf{B}/B$. Thus, in the rotating frame, Eq. (4) becomes[2]

$$i\hbar \frac{\partial|\psi\rangle}{dt} = (\boldsymbol{\Omega}_p - \boldsymbol{\omega})\cdot\mathbf{s}|\psi\rangle.$$

Writing $|\psi\rangle$ as $|\psi\rangle = C_+(t)|+\rangle + C_-(t)|-\rangle$, where $|+\rangle$ and $|-\rangle$ are states polarized parallel and antiparallel to $\hat{b}$ (i.e., they are eigenstates of σ_z in the coordinates moving with the electron), linearized solutions can be found for the transition amplitudes as a function of time assuming that at the initial time $t = t_1$ the spin is in either the $+$ or $-$ state only, so that $C_{+(-)}(t_1) = 1$. The probability amplitude of transition to the opposite state follows after some simple algebra:

$$C_{+(-)} \approx \frac{i}{2}\int_{t_1}^{t} dt'\, \omega_{+(-)}(t')$$

$$\times \exp\left(\mp i \int_{t_1}^{t}[\Omega_p(t'') - \omega_z(t'')]dt''\right),$$

where $\omega_+ = \omega_x \pm i\omega_y$. This expression clearly shows that $|C_+|^2 = |C_-|^2$, so we consider only C_+ from now on.

In order to make further progress we choose to set $\omega_z = 0$ and further specialize to the regime of strong magnetization in which one may write $\omega(t)$ in a guiding center expansion:

$$\omega(t) = \sum_n \omega^{(n)}(t)\exp\left(in\int_{t_1}^{t}\Omega_c(t')dt'\right), \qquad (5)$$

where the $\omega^{(n)}$'s are relatively slowly varying functions compared to the oscillatory factor; $\omega^{(0)}$ is the term stemming from guiding center motion, and the other terms in the series are associated with harmonics of the cyclotron motion. The largest terms are $\omega^{(0)}$ and $\omega^{(\pm 1)}$. These are of magnitude v/L, as can be seen from the expression $\omega = \hat{b} \times \mathbf{v}\cdot\nabla\hat{b}$. Before we evaluate the $\omega^{(n)}$'s explicitly in terms of the strongly magnetized electron trajectory, it proves useful to integrate by parts in order to separate out a small oscillatory contribution due to the limits of integration:

$$C_+(t) = \frac{i}{2}\sum_n \left\{ \frac{\omega_-^{(n)}(t')}{i(n+g/2)\Omega_c(t')} \right.$$

$$\left. \times \exp\left[i\int_{t_1}^{t'}\left(n+\frac{g}{2}\right)\Omega_c(t'')dt''\right]\Bigg|_{t'=t_1}^{t'=t} \right.$$

$$- \int_{t_1}^{t} dt' \frac{(d/dt')\omega_-^{(n)}(t')}{i(n+g/2)\Omega_c(t')}$$

$$\left. \times \exp\left[i\int_{t_1}^{t'}\left(n+\frac{g}{2}\right)\Omega_c(t'')dt''\right] \right\}. \qquad (6)$$

We neglect the first term because it is small and nonsecular. By this we mean that even after many collisions, the velocity of the electron remains on the order of the

thermal speed, and so $\omega^{(n)}/\Omega_c$ also remains small. Furthermore, although there is a nearly resonant denominator for the $n = -1$ term in the series, the term is still only of order $r_L/(g-2)L \ll 1$. It is also true that after any single collision the change of the second term of Eq. (6) is small [in fact, it is smaller than the first term by $O(\epsilon)$, as we will see]. However, over the course of *many* collisions this second term grows secularly in a random walk and hence dominates the expression for C_+ over time. Physically, the first term represents the effect of fast spin precession in a slowly varying magnetic field, which causes small oscillations in the z component of the spin as **B** changes direction in the electron rest frame. To make an analogy with the classical theory of adiabatic invariants, the exact adiabatic invariant is not s_z, but is instead an infinite asymptotic expansion with s_z as the lowest-order term. The small oscillations in s_z represented by the first term of Eq. (6) are due to higher-order nonsecular terms in the invariant, and are not important in determining the secular change of the invariant.

We further simplify the expression for C_+ by neglecting terms of order $(r_L/L)^2$ and higher. Since ω is already of $O(r_L/L)$ we can therefore neglect the magnetic field gradient in the dynamics of the electron orbits and evaluate the collisional dynamics in a constant field $\mathbf{B}_0 = \mathbf{B}(\mathbf{x}_0)$, where we choose $\mathbf{x}_0$ as the center of mass position at the instant of closest approach of the colliding electrons (see Fig. 1). Furthermore, to lowest order in r_L/L, ω itself can be written as

$$\omega = \hat{b}_0 \times \mathbf{v} \cdot (\nabla \hat{b})_0,$$

where $\hat{b}_0 = \hat{b}(\mathbf{x}_0)$ and $(\nabla \hat{b})_0 = \nabla \hat{b}(\mathbf{x}_0)$ are constant, and the velocity **v** has a guiding center expansion of the same form as Eq. (6). Then, keeping only the near-resonant $n = -1$ term in the series over n in Eq. (6), the expression becomes

$$C_+(t) = \frac{-1}{(g-2)\Omega_{c_0}} \int_{t_1}^{t} \left(\hat{b}_0 \times \frac{d}{dt'} \mathbf{v}^{(-1)} \cdot (\nabla \hat{b})_0 \right)$$
$$\times e^{i(t'-t_1)(g/2-1)\Omega_{c_0}} dt' + O\left[\left(\frac{r_L}{L} \right)^2 \right], \quad (7)$$

where $\Omega_{c_0} = \Omega_c(\mathbf{x}_0)$. All other terms in the series give contributions that are exponentially small because of the fast variation of the phase factor in the integrand.

Finally, $(d\mathbf{v}^{(-1)}/dt)(t)$ is evaluated in a guiding center expansion in the small parameter $\epsilon \equiv v_z(t_1)/b\Omega_0$, where $b = 2e^2/\mu v_z^2(t_1)$ is the distance of closest approach, $v_z(t_1)$ is the initial relative parallel velocity, and $\mu = m/2$ is the reduced mass. Again we consider two electrons, labeled 1 and 2, colliding in a uniform magnetic field **B**. In the strongly magnetized regime the collision may be pictured schematically, as shown in Fig. 1. The electrons spiral in tight Larmor orbits toward one another along the magnetic field lines, and their mutual Coulomb repulsion perturbs the orbits. This perturbation shifts the cyclotron frequency, bringing it into resonance with the spin precession, and inducing a spin-flip transition. We will determine the trajectories of the electrons and use them to calculate C_+ for electron 1. The equations of motion for two electrons are

$$m\ddot{\mathbf{x}}_{1 \atop (2)} = e \frac{\partial}{\partial \mathbf{x}_{1 \atop (2)}} \phi(\mathbf{x}_1 - \mathbf{x}_2) - \frac{e}{c} \dot{\mathbf{x}}_{1 \atop (2)} \times \mathbf{B}_0,$$

where $\phi = -e/|\mathbf{x}_1 - \mathbf{x}_2|$ is the interaction potential. The center of mass motion can be separated out by transforming to center of mass coordinates through $\mathbf{R} = (\mathbf{x}_1 + \mathbf{x}_2)/2$, $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, leading to

$$m\ddot{\mathbf{R}} = -\frac{e}{c} \dot{\mathbf{R}} \times \mathbf{B}_0, \quad (8)$$

$$m\ddot{\mathbf{r}} = 2e \frac{\partial}{\partial \mathbf{r}} \phi(r) - \frac{e}{c} \dot{\mathbf{r}} \times \mathbf{B}_0. \quad (9)$$

Equation (8) describes center of mass motion, which is just a combination of constant amplitude Larmor gyrations and parallel streaming. Since $d\mathbf{v}^{(-1)}/dt$ is zero for this motion, the center of mass motion makes no contribution to C_+.

Turning to the equation for relative motion, we solve for $\ddot{\mathbf{r}}$ by expanding in ϵ using standard asymptotic techniques. To $O(\epsilon)$ the result is[6]

$$\dot{\mathbf{r}} = \mathbf{v}_g(t) + v_{\perp 0} \, \mathrm{Re}[e^{i[\Omega_{c_0}(t-t_1)+\theta_0+\delta\theta(t)]}(\hat{x}-i\hat{y})] + O(\epsilon^2), \quad (10)$$

where θ_0 is the initial relative gyrophase, $\mathbf{v}_g(t)$ is the slowly varying guiding center relative velocity, $v_{\perp 0} = v_\perp(t_1)$ is the initial perpendicular relative velocity, and $\delta\theta$ is a $O(\epsilon)$ slow variation of the relative gyrophase, given by[6]

$$\delta\theta(t) = \frac{e^2}{\mu\Omega_{c_0}} \int_{t_1}^{t} dt' \frac{z^2(t') - \frac{1}{2}\rho_0^2}{[z^2(t')+\rho_0^2]^{5/2}}, \quad (11)$$

where the function $z(t)$ is the lowest-order z position of the guiding center, determined by the solution of the equation

$$\ddot{z} = \frac{e^2}{\mu} \frac{z}{(z^2+\rho_0^2)^{3/2}}, \quad (12)$$

and $\rho_0 = \sqrt{x(t_1)^2 + y(t_1)^2}$ is the initial impact parameter. The time t_1 is chosen so that the electrons are initially far apart, i.e., $|z_0(t=0)| \gg \rho_0$.

A further simplification can be made by noting that $d\mathbf{v}^{(-1)}/dt$ appears in Eq. (7) only in the combination

$$\frac{d\omega_-^{(-1)}}{dt} = \left(\hat{b}_0 \times \frac{d\mathbf{v}^{(-1)}}{dt} \cdot \nabla \hat{b}_0 \right)_-.$$

Using the fact that $\nabla \cdot \mathbf{B} = \nabla \times \mathbf{B} = 0$, this expression can be rewritten as

$$\frac{d\omega_-^{(-1)}}{dt} = \frac{i}{2B_0} \left[\frac{\partial B_z}{\partial z_0} \frac{dv_-^{(-1)}}{dt} + \left(\frac{\partial B_y}{\partial y_0} - \frac{\partial B_x}{\partial x_0} + 2i \frac{\partial B_x}{\partial y_0} \right) \frac{dv_+^{(-1)}}{dt} \right.$$
$$\left. - 2 \left(\frac{\partial B_z}{\partial x_0} - i \frac{\partial B_z}{\partial y_0} \right) \frac{dv_z^{(-1)}}{dt} \right],$$

where $v_\pm = v_x \pm i v_y$. However, Eq. (10) implies that only the term involving $dv_-^{(-1)}/dt$ provides a resonant contribution at $O(\epsilon)$, so to this order we find

$$\frac{d\omega^{(-1)}_-}{dt} = \frac{1}{4B_0}\frac{\partial B_z}{\partial z_0}\cdot v_{\perp\,0}\,\frac{d\delta\theta}{dt}e^{i\theta_0}+O(\epsilon^2).$$

Here an extra factor of $\frac{1}{2}$ appears because electron 1's velocity equals $\dot{\mathbf{R}}+\frac{1}{2}\dot{\mathbf{r}}$. Thus, to $O(\epsilon)$ only the slow time variation of the gyrophase contributes to C_+.

Substitution of this expression into Eq. (7), together with Eq. (11), leads to a simple form for the secular change in C_+ during a single collision,

$$\Delta C_+ = \frac{1}{4(g-2)}\frac{r_L}{B_0}\frac{\partial B_z}{\partial z_0}\frac{e^{-i\theta'}}{\Omega_{c_0}}\frac{e^2}{\mu}\int_{-\infty}^{\infty}\frac{z^2(t')-\frac{1}{2}\rho_0^2}{[z^2(t')+\rho_0^2]^{5/2}}$$

$$\times e^{i\Omega_{c_0}(g/2-1)t'}\,dt'+O(\epsilon^2)+O\left[\left(\frac{r_L}{L}\right)^2\right],\quad(13)$$

where $\theta' = \theta_0 + \Omega_{c_0}(g/2 - 1)t_1$ and $r_L=v_{\perp\,0}/\Omega_{c_0}$ is the initial relative Larmor radius. Here we have taken the limits of integration to $\pm\infty$ in order to determine the total change in C_+ after a single collision. Of course, this assumes that the plasma is weakly correlated so that two particle collisions are well separated in time.

It is also useful to work with dimensionless distances and times, defining $\bar{t}=tv_z(t_1)/b$, $\bar{\rho}=\rho_0/b$, and $\bar{z}=z_0/b$. Then Eq. (13) becomes, after some simple algebra,

$$\Delta C_+ = \frac{1}{8(g-2)}\frac{r_L}{B_0}\frac{\partial B_z}{\partial z_0}e^{-i\theta'}\epsilon I\left(\frac{g/2-1}{\epsilon},\bar{\rho}\right),\quad(14)$$

where the function $I(x,\bar{\rho})$ is defined by

$$I(x,\bar{\rho}) = \int_{-\infty}^{\infty}d\bar{t}\,e^{-ix\bar{t}}\frac{\bar{z}^2-\bar{\rho}^2/2}{(\bar{\rho}^2+\bar{z}^2)^{5/2}},\quad(15)$$

and $\bar{z}(t)$ satisfies the differential equation [see Eq. (12)],

$$\dot{\bar{z}}^2+\frac{1}{\sqrt{\bar{\rho}^2+\bar{z}^2}}=1,\quad(16)$$

with initial conditions $z(t=-\infty)=-\infty$, $\dot{\bar{z}}(t=-\infty)=1$. We note that Eq. (16) can be analytically integrated and $\bar{t}$ can be expressed in terms of $\bar{z}$ through elliptic integrals (see Appendix B).

In a few special cases analytical forms for I can be obtained: For example,

$$I(x,\bar{\rho}) = -x^2K_0(x\bar{\rho}),\quad\text{for }\bar{\rho}\gg1,$$
$$I(x,\bar{\rho}) = h(x,\bar{\rho})e^{-xg(\bar{\rho})},\quad\text{for }x\gg1,\quad(17)$$

where

$$g(\bar{\rho}) \equiv \left|\int_{\bar{\rho}}^{1}d\bar{x}\frac{\bar{x}^{3/2}}{\sqrt{\bar{x}^2-\bar{\rho}^2}\sqrt{1-\bar{x}}}\right|,$$

and $h(x,\bar{\rho})$ is a function that is neither exponentially small nor exponentially large. For head-on collisions,

$$I(x,0) \simeq \frac{8\pi}{9}xe^{-(\pi/2)x},\quad\text{for }x\gg1,\quad(18)$$

$$I(x,0) = \tfrac{8}{3}+x^2\ln x+O(x^2),\quad\text{for }x\ll1.\quad(19)$$

However, for general values of x and $\bar{\rho}$, $I(x,\bar{\rho})$ must be determined numerically. The integral over $\bar{t}$ in the defini-
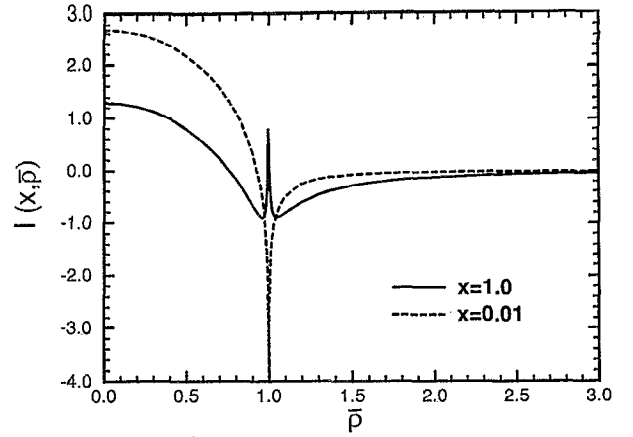

FIG. 3. Plot of $I(x,\bar{\rho})$ as a function of $\bar{\rho}$ for different values of x. The behavior of $I(x,\bar{\rho})$ becomes singular at $\bar{\rho}=1$, the separatrix point between passage and reflection of the two electrons. For $x=0.01$, $I(x,0)\simeq\frac{8}{3}$ coincides with Eq. (19).

tion of I was performed by transformation of the integration variable from $\bar{t}$ to $\bar{z}$ via elliptic integral expressions of the guiding center orbit $\bar{t}(\bar{z})$ derived in Appendix B, and then the $\bar{z}$ integral was calculated using the SLATEC[7] subroutine DQAGSE. The function $I(x,\bar{\rho})$ is plotted in Fig. 3 as a function of $\bar{\rho}$ for $x=0.01$ and $x=1$. The singular behavior at $\bar{\rho}=1$ is due to the effectively infinite collision time at the separatrix for the electrons to pass by or reflect from each other. The behavior of $I(x,\bar{\rho})$ for large x is also plotted as a function of x for different impact parameters $\bar{\rho}$ in Fig. 4, where the numerical results are compared to the analytic expressions.

Equation (14) gives the probability amplitude for spin flip due to the classical electrostatic collision of two strongly magnetized electrons in a weakly inhomogeneous magnetic field. By averaging over a Maxwellian distribu-
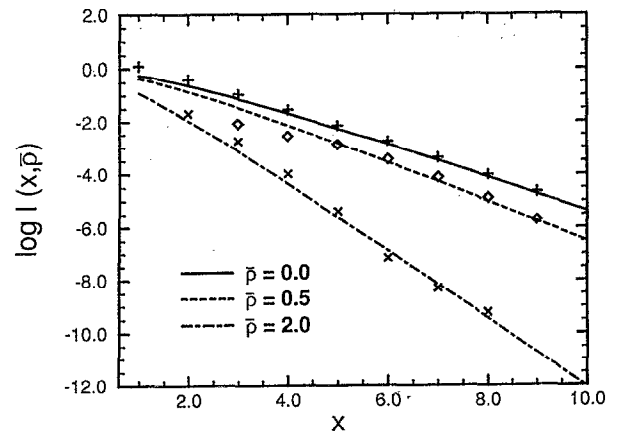

FIG. 4. Numerical test of the asymptotic form of $|I(x,\bar{\rho})|$ for large x. The curves represent asymptotic values given by Eq. (17) and Eq. (18), where the function $h(x,\bar{\rho})$ is approximated by $h(x,0)=(8\pi/9)x^3$. This form works reasonably well, even for values of ρ where I is negative. This is because the exponential dependence of I on x dominates the behavior for large x. The numerical results are denoted by $+$: $\bar{\rho}=0$; $\diamondsuit$: $\bar{\rho}=0.5$ and $\times$: $\bar{\rho}=2$.

tion of electrons the average rate of spin flip can be obtained. This calculation is carried out in Sec. V.

IV. QUANTUM ANALYSIS

In this section, the previous assumption of classical orbital motion is relaxed. For the strong magnetic fields and low temperatures of the experiments on cryogenic electron plasmas, the perpendicular mean thermal energy $kT_\perp$ can be as low as the spacing of the Landau levels $\hbar\Omega_c$, so quantum mechanics is necessary to describe the orbital motion. Moreover, since $kT_\perp$ is then also comparable with the energy difference $\hbar\Omega_p$ between spin up and down, a spin flip changes the orbital state of the electron appreciably. This spin-orbit energy exchange process is important for the plasma thermal equilibration, as will be seen in Sec. V. Since the electron thermal de Broglie wavelength is small compared to the classical distance of closest approach, the antisymmetry of the two-electron wave function will be ignored, as this approximation will only cause an exponentially small relative correction.

As in Sec. III, we calculate the probability amplitude of a spin-flip transition during the collision of two electrons in a spatially inhomogeneous magnetic field. The collision is described by the two-electron Hamiltonian,

$$\hat{H} = \frac{[\mathbf{p}_1 + (e/c)\mathbf{A}_1]^2}{2m} + \mathbf{s}_1 \cdot \mathbf{\Omega}_p(\mathbf{x}_1) + \frac{[\mathbf{p}_2 + (e/c)\mathbf{A}_2]^2}{2m} + \mathbf{s}_2 \cdot \mathbf{\Omega}_p(\mathbf{x}_2) + \frac{e^2}{|\mathbf{x}_1 - \mathbf{x}_2|}.$$

We also follow Sec. III in assuming that $\mathbf{B}(\mathbf{x})$ varies slowly compared to the scale lengths associated with the electron–electron collision, and so we expand $\mathbf{B}$ to linear order about an arbitrary point: $\mathbf{B} = B_0\hat{z} + \mathbf{x} \cdot \nabla\mathbf{B}$, where $\mathbf{x}$ is measured with respect to this point. Although the eigenfunctions of $\hat{H}$ are not localized, through a judicious choice of the initial states of the colliding electrons, this arbitrary point will become the collision center $\mathbf{x}_0$ in the classical limit, so this expansion is justified on physical grounds. We will see that the expansion is justified mathematically by the convergence of the overlap integrals that couple the initial and final states through the magnetic perturbation.

In terms of the center of mass position $\mathbf{R} = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$ and the relative position $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, to the first order in ∇B, the Hamiltonian expands out to the form $\hat{H} = \hat{H}_{cm} + \hat{H}_{rel} + \hat{H}_{spin} + \delta\hat{H}_{orbit} + \hat{H}_{sf}$, where

$$\hat{H}_{cm} = \frac{1}{2m}\left(\mathbf{P}_R - \frac{e}{c}\mathbf{R} \times \mathbf{B}_0\right)^2,$$

$$\hat{H}_{rel} = \frac{1}{2\mu}\left(\mathbf{p} - \frac{e}{c}\mathbf{r} \times \mathbf{B}_0\right)^2 + \frac{e^2}{r},$$

$$\hat{H}_{spin} = (\mathbf{s}_1 + \mathbf{s}_2) \cdot \mathbf{\Omega}_{p_0},$$

$$\delta\hat{H}_{orbit} = \frac{e}{mc}\delta\mathbf{A}(\mathbf{x}_1) \cdot \left(\mathbf{p}_1 - \frac{e}{2c}\mathbf{x}_1 \times \mathbf{B}_0\right) + \frac{e}{mc}\delta\mathbf{A}(\mathbf{x}_2) \cdot \left(\mathbf{p}_2 - \frac{e}{2c}\mathbf{x}_2 \times \mathbf{B}_0\right),$$

and

$$\hat{H}_{sf} = (\mathbf{s}_1 + \mathbf{s}_2) \cdot (\mathbf{R} \cdot \nabla)\mathbf{\Omega}_{p_0} + \frac{1}{2}(\mathbf{s}_1 - \mathbf{s}_2) \cdot (\mathbf{r} \cdot \nabla)\mathbf{\Omega}_{p_0}.$$

Here, $\mathbf{\Omega}_{p_0} \equiv (g/2)\mathbf{\Omega}_{c_0} = (g/2)(eB_0/mc)$, $\mathbf{P}_R = -i\hbar(\partial/\partial\mathbf{R})$, and $\mathbf{p} = -i\hbar(\partial/\partial\mathbf{r})$ are the momentum operators of the center of mass and relative motion, respectively. The function $\delta\mathbf{A}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) - \frac{1}{2}\mathbf{B}_0 \times \mathbf{x}$ is the correction to the vector potential due to the spatial variation in $\mathbf{B}(\mathbf{x})$.

Since the spin and orbital dynamics decouple in $\hat{H}_{cm}$, $\hat{H}_{rel}$, $\hat{H}_{spin}$, and $\delta\hat{H}_{orbit}$, these Hamiltonians are not responsible for the spin-flip transition. The spin-flip transition is due only to $\hat{H}_{sf}$. According to Fermi's "golden rule", the probability per unit time of a transition from state $|i\rangle$ to state $|f\rangle$ is given by

$$a_i^f = \frac{2\pi}{\hbar}\rho_f|\langle f|H_{sf}|i\rangle|^2. \tag{20}$$

Here, $|i\rangle$ and $|f\rangle$ are the eigenstates of $\hat{H}_{cm} + \hat{H}_{rel} + \hat{H}_{spin} + \delta\hat{H}_{orbit}$, ρ_f is the density of the final states, and the transition conserves the total spin and orbital energy.

Before beginning the calculation of the transition rate, we note that the spin-flip Hamiltonian $\hat{H}_{sf}$ can be rewritten as

$$\hat{H}_{sf} = X\,\partial_x\mathbf{\Omega}_{p_0} \cdot (\mathbf{s}_1 + \mathbf{s}_2) + Y\,\partial_y\mathbf{\Omega}_{p_0} \cdot (\mathbf{s}_1 + \mathbf{s}_2) + Z\,\partial_z\mathbf{\Omega}_{p_0} \cdot (\mathbf{s}_1 + \mathbf{s}_2) + \frac{x}{2}\,\partial_x\mathbf{\Omega}_{p_0} \cdot (\mathbf{s}_1 - \mathbf{s}_2) + \frac{y}{2}\,\partial_y\mathbf{\Omega}_{p_0} \cdot (\mathbf{s}_1 - \mathbf{s}_2) + \frac{z}{2}\partial_z\mathbf{\Omega}_{p_0} \cdot (\mathbf{s}_1 - \mathbf{s}_2),$$

where (X,Y,Z) and (x,y,z) are center of mass and relative coordinates, respectively. To calculate the transition rate to the leading order of ∇B, we use for the states $|i\rangle$ and $|f\rangle$ the states of colliding electrons in a uniform field B_0, i.e., the eigenstates of $\hat{H}_{cm} + \hat{H}_{rel} + \hat{H}_{spin}$ (in the absence of $\delta\hat{H}_{orbit}$), since $\delta\hat{H}_{orbit}$ is of order ∇B.

Several simplifications of H_{sf} can now be made. First, we note that the operators $\hat{s}_1$ and $\hat{s}_2$ are linearly combined in $\hat{H}_{sf}$, so only one spin can be flipped in the transition. This implies that a spin-flip transition always involves a spin energy change of magnitude $\hbar\Omega_{p_0}$. Now, the first two terms of $\hat{H}_{sf}$ couple the spin and the center of mass motion. However, this motion is described by $\hat{H}_{cm}$, which has well-known harmonic-oscillator eigenstates with energies separated by $\hbar\Omega_{c_0}$ for the X–Y motion, and free streaming for the Z motion. Since $\hbar\Omega_{p_0} \neq \hbar\Omega_{c_0}$ and the parallel electron states are unchanged, energy conservation forbids a spin-flip transition so the first two terms of $\hat{H}_{sf}$ may be neglected.

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The third and sixth terms in $\hat{H}_{\rm sf}$ can be neglected for a similar reason. These terms couple the spin and parallel dynamics, so during a spin-flip transition energy conservation requires a parallel energy change of magnitude $\hbar\Omega_{p_0}$. However, in the strongly magnetized regime this is a large change; the initial and final parallel states would have extremely disparate wave numbers, leading to an exponentially small contribution to the overlap integrals appearing in the golden rule, Eq. (20). There then remains only the resonant interaction between the relative (x,y) dynamics and the spin, which involves the fourth and fifth terms in $\hat{H}_{\rm sf}$:

$$\left(\frac{x}{2}\partial_x\Omega_{p_0}+\frac{y}{2}\partial_y\Omega_{p_0}\right)\cdot(\mathbf{s}_1-\mathbf{s}_2)$$

$$=\frac{eg}{2mc}\frac{\rho}{8}\left\{\left[-\frac{\partial B_z}{\partial z_0}e^{-i\theta}+\left(\frac{\partial B_x}{\partial x_0}-\frac{\partial B_y}{\partial y_0}-2i\frac{\partial B_y}{\partial x_0}\right)\cdot e^{i\theta}\right]\right.$$

$$\left.\times\hat{s}^{(+)}+2\left(\frac{\partial B_z}{\partial x_0}-i\frac{\partial B_z}{\partial y_0}\right)(\hat{s}_{1z}-\hat{s}_{2z})\right\}$$

$$+\text{(Hermitian conjugate)}.$$

On the right-hand side we have written (x,y) in terms of polar coordinates (ρ,θ) and we have introduced the spin creation and annihilation operators $\hat{s}^{+}$ and $\hat{s}^{-}$, where $s^{(\pm)}=\hat{s}_{1x}-\hat{s}_{2x}\pm i(\hat{s}_{1y}-\hat{s}_{2y})$. The term involving $\hat{s}_{1z}-\hat{s}_{2z}$ cannot induce transitions between different spin states, and so can be neglected. Thus, the effective spin-flip Hamiltonian is

$$\hat{H}_{\rm sf}=\frac{eg}{2mc}\frac{\rho}{8}\left[-\frac{\partial B_z}{\partial z_0}e^{-i\theta}+\left(\frac{\partial B_x}{\partial x_0}-\frac{\partial B_y}{\partial y_0}-2i\frac{\partial B_y}{\partial x_0}\right)e^{i\theta}\right]\hat{s}^{(+)}$$

$$+\text{Hermitian conjugate.} \qquad (21)$$

It can easily be verified that the term proportional to $\partial B_z/\partial z_0$ commutes with the operator $\hat{s}_z-i\hbar\partial/\partial\theta$, so this term conserves total angular momentum in the z direction. However, the other term is proportional to field gradients expressing the cylindrical asymmetry of the external field, and so it is not surprising that this term does not conserve total angular momentum. This difference will have important ramifications when we employ Eq. (21) in the calculation of Fermi's "golden rule," Eq. (20).

However, before we can apply the golden rule to Eq. (21) we will require expressions for the initial and final states that are the eigenstates of $\hat{H}_{\rm cm}+\hat{H}_{\rm rel}+\hat{H}_{\rm spin}$. These states can be expressed as the product $\psi_{\rm cm}(\mathbf{R})\times\psi_{\rm rel}(\mathbf{r})|s_{1z},s_{2z}\rangle$, where $\psi_{\rm cm}(\mathbf{R})$, $\psi_{\rm rel}(\mathbf{r})$, and $|s_{1z},s_{2z}\rangle$ are eigenstates of $\hat{H}_{\rm cm}$, $\hat{H}_{\rm rel}$, and $\hat{H}_{\rm spin}$, respectively. Since the center of mass dynamics does not appear and the spin eigenstates are trivial, it remains only to find $\psi_{\rm rel}$. We therefore calculate $\psi_{\rm rel}$ for two colliding electrons using a quantum version of the classical guiding center expansion.[3] The expansion is most easily derived by first expressing the relative Hamiltonian in terms of cylindrical coordinates:

$$\hat{H}_{\rm rel}=\frac{-\hbar^2}{2\mu}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right]+\frac{\hbar\Omega_{c_0}}{2i}\frac{\partial}{\partial\theta}$$

$$+\frac{\mu\Omega_{c_0}}{8}\rho^2+\frac{e^2}{\sqrt{\rho^2+z^2}}.$$

Since $\hat{H}_{\rm rel}$ is θ independent, the z component of the angular momentum $L_z=l\hbar$ is conserved, and we look for eigenstates of the form $(e^{il\theta}/\sqrt{2\pi})\psi_{l\alpha}(\rho,z)/\sqrt{\rho}$, where α denotes the two quantum numbers associated with dynamics in ρ and z. Replacing $\partial/\partial\theta$ by il in $\hat{H}_{\rm rel}$ yields the reduced Hamiltonian for $\psi_{l\alpha}(\rho,z)$:

$$\hat{H}_{\rm rel}=\frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial z^2}-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial\rho^2}+\frac{\mu\Omega_{c_0}^2}{8\rho^2}(\rho^2-\rho_l^2)^2-\frac{\hbar^2}{8\mu}\frac{1}{\rho^2}$$

$$+\frac{e^2}{\sqrt{\rho^2+z^2}},$$

where $\rho_l^2=-2\hbar l/\mu\Omega_{c_0}$.

We will see that the main contribution to the integral expression for the spin-flip rate comes from wave functions with ρ_l, such that $r_{q\rm L}\ll\rho_l\lesssim b$, where $r_{q\rm L}\equiv\sqrt{2\nu\hbar/\mu\Omega_{c_0}}$ is the quantum Larmor radius, ν is the quantum number of cyclotron motion, and b is the classical distance of closest approach. Physically, $\rho=\rho_l$ corresponds to the impact parameter of the guiding center of a reduced mass electron incident on the force center. The wave function $\psi_{l\alpha}(\rho,z)$ is peaked near $\rho\simeq\rho_l$, at the minimum point of the centrifugal potential of $\hat{H}_{\rm rel,b}$, and $\psi_{l\alpha}(\rho,z)$ falls off rapidly in a distance of order the cyclotron radius $r_{q\rm L}\ll\rho_l$. It is therefore useful to introduce the variable $x_l\equiv\rho-\rho_l$ in the relative Hamiltonian. Expansion of $\hat{H}_{\rm rel}$ to second order in $r_{q\rm L}/\rho_l$ then yields

$$\hat{H}_{\rm rel}=H_l^{(0)}+\hat{H}^{(1)}+\hat{H}^{(2)}+O\left(\frac{r_{q\rm L}}{\rho_l}\right)^3, \qquad (22)$$

where

$$\hat{H}_l^{(0)}(x_l,z)=\frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial z^2}+\frac{e^2}{\sqrt{\rho_l^2+z^2}}-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x_l^2}+\frac{1}{2}\mu\Omega_{c_0}^2x_l^2,$$

$$\hat{H}^{(1)}(x_l,z)=-\frac{1}{2}\mu\Omega_{c_0}^2\frac{x_l^3}{\rho_l}+g(z)x_l,$$

$$\hat{H}^{(2)}(x_l,z)=\frac{5}{8}\mu\Omega_{c_0}^2x_l^2\left(\frac{x_l}{\rho_l}\right)^2+f(z)x_l^2-\frac{\hbar^2}{8\mu\rho_l^2},$$

and the functions f and g arise from Taylor expansion of the Coulomb potential, and are defined by

$$f(z)\equiv\frac{e^2(\rho_l^2-z^2/2)}{(\rho_l^2+z^2)^{5/2}}$$

and

$$g(z)\equiv-\frac{e^2\rho_l}{(\rho_l^2+z^2)^{3/2}}.$$

Each term $\hat{H}^{(n)}$ in this expansion has magnitude of order $\hbar\Omega_{c_0}(r_{q\rm L}/\rho_l)^n$, since x_l is of order $r_{q\rm L}$. Furthermore,

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it is clear that eigenfunctions of $\hat{H}_{\text{rel}}$ are, in the position representation, functions of ρ through the variable x_l:

$$\psi_{l\alpha}(\rho,z) = \bar{\psi}_{l\alpha}(x_l,z), \tag{23}$$

where $\bar{\psi}_{l\alpha}$ is the eigenfunction of the Hamiltonian of Eq. (22), and α denotes the two quantum numbers, which, along with l, parametrize the state. In this form, $\hat{H}_{\text{rel}}$ is a perturbed harmonic oscillator Hamiltonian in the variable x_l, so $\bar{\psi}_{l\alpha}(x,z)$ is highly peaked around $x=0$.

The unperturbed Hamiltonian $H_l^{(0)}(x_l,z)$ has eigenstates $|l,\alpha\rangle^{(0)}$, which we write in the position representation as $|l,\alpha\rangle^{(0)} = G_\nu(x_l)F_{lk}(z)$. Here, $G_\nu(x)$ is a harmonic oscillator eigenfunction with eigenenergy $(\nu+\tfrac{1}{2})\hbar\Omega_{c_0}$, and $F_{lk}(z)$ is the eigenfunction of the parallel dynamics, with energy κ. Thus, α can be represented by the values of ν and κ. The total energy of an eigenstate of $H_l^{(0)}$ is denoted by E_α and is given by $E_\alpha = (\nu+\tfrac{1}{2})\hbar\Omega_{c_0} + \kappa$. (Although E_α is also a function of l through the dependence of κ on l, we drop this subscript in order to save space.)

Taking $|l,\alpha\rangle^{(0)}$ as the base vector and using second-order perturbation theory, we obtain a perturbation expansion for $|l,\alpha\rangle$:

$$|l,\alpha\rangle = \left(1 - \frac{d_\alpha}{2}\right)|l,\alpha\rangle^{(0)} + \sum_{\alpha'}(a_{\alpha\alpha'} + b_{\alpha\alpha'} + c_{\alpha\alpha'})$$

$$\times |l,\alpha'\rangle^{(0)} + O\left(\frac{r_{qL}}{\rho_l}\right)^3, \tag{24}$$

where

$$a_{\alpha\alpha'} = H_{\alpha\alpha'}^{(1)}/(E_\alpha - E_{\alpha'}),$$

$$b_{\alpha\alpha'} = H_{\alpha\alpha'}^{(2)}/(E_\alpha - E_{\alpha'}),$$

$$c_{\alpha\alpha'} = \sum_{\alpha_1} H_{\alpha'\alpha_1}^{(1)} H_{\alpha_1\alpha}^{(1)}/(E_\alpha - E_{\alpha'})(E_\alpha - E_{\alpha_1}),$$

$$d_\alpha = \sum_{\alpha_1} \frac{|H_{\alpha\alpha_1}^{(1)}|^2}{(E_\alpha - E_{\alpha_1})},$$

and where we employ the notation $H_{\alpha\alpha'}$ to denote the matrix element $^{(0)}\langle l\alpha | H | l\alpha'\rangle^{(0)}$.

However, to calculate the transition matrix element of Eq. (20), we will also require an expression for $|l\pm 1,\alpha\rangle$. Although this expression can, in principle, be obtained from Eq. (23) by substitution of $l\pm 1$ for l, it is more convenient to determine $|l\pm 1,\alpha\rangle$ in terms of $|l,\alpha\rangle^{(0)}$ rather than $|l\pm 1,\alpha\rangle^{(0)}$. The ket $|l\pm 1,\alpha\rangle^{(0)}$ is the eigenstate of $\hat{H}_{l\pm 1}^{(0)}$, which is related to $\hat{H}_l^{(0)}$ through a Taylor expansion of ρ_l:

$$\hat{H}_{l\pm 1}^{(0)}(x,z) = \hat{H}_l^{(0)}(x,z) \mp \frac{\hbar}{\mu\Omega_{c_0}} \frac{e^2}{(\rho_l^2 + z^2)^{3/2}} + O\left(\frac{r_{qL}}{\rho_l}\right)^3.$$

Taking the term $\mp (\hbar/\mu\Omega_{c_0})[e^2/(\rho_l^2 + z^2)^{3/2}]$ as the perturbation, we find that in the position representation the kets are related by

$$\bar{\psi}_{l\pm 1,\alpha}^{(0)}(x,z) = \bar{\psi}_{l\alpha}^{(0)}(x,z) \mp \sum_{\alpha'} h_{\alpha\alpha'} \bar{\psi}_{l\alpha'}^{(0)}(x,z), \tag{25}$$

where

$$h_{\alpha\alpha'} = \frac{\{(-\hbar/\mu\Omega_{c_0})[e^2/(\rho_l^2 + z^2)^{3/2}]\}_{\alpha\alpha'}}{(E_\alpha - E_{\alpha'})}. \tag{26}$$

Substituting Eq. (25) in Eq. (24) with l replaced by $l\pm 1$ yields

$$\bar{\psi}_{l\pm 1,\alpha}(x,z) = \bar{\psi}_{l\alpha}(x,z) \mp \sum_{\alpha'} h_{\alpha\alpha'} \bar{\psi}_{l,\alpha'}^{(0)}(x,z) + O\left(\frac{r_{qL}}{\rho_l}\right)^3. \tag{27}$$

We now evaluate the transition matrix element $\langle f | H_{\text{sf}} | i\rangle$. Without loss of generality, we take the initial state to be

$$|i\rangle = \frac{e^{il\theta}}{\sqrt{2\pi}} \frac{1}{\sqrt{\rho}} \bar{\psi}_{l\alpha}(x_l,z)\left|-\frac{\hbar}{2},\frac{\hbar}{2}\right\rangle, \tag{28}$$

and the final state to be

$$|f\rangle = \frac{e^{il_f\theta}}{\sqrt{2\pi}} \frac{1}{\sqrt{\rho}} \bar{\psi}_{l_f\alpha_f}(x_{l_f},z)\left|\frac{\hbar}{2},\frac{\hbar}{2}\right\rangle, \tag{29}$$

so that during the transition spin 1 is flipped from down to up. Energy conservation at zero order in r_{qL}/ρ_l requires that

$$\left(\nu+\frac{1}{2}\right)\hbar\Omega_{c_0} - \frac{\hbar}{2}\Omega_{p_0} + \kappa = \left(\nu_f + \frac{1}{2}\right)\hbar\Omega_{c_0} + \frac{\hbar}{2}\Omega_{p_0} + \kappa_f.$$

Then the zeroth-order parallel energy change is $\kappa_f - \kappa = (\nu - \nu_f)\hbar\Omega_{c_0} - \hbar\Omega_{p_0} = (\nu - \nu_f - g/2)\hbar\Omega_{c_0}$. As discussed at the beginning of this section, since the z motion is very slow compared with spin precession at frequency Ω_{p_0}, by far the largest contribution to the transition comes from $\nu_f = \nu - 1$; then $|(\kappa_f - \kappa)/\hbar| = \Omega_{p_0} - \Omega_{c_0} = (g/2 - 1)\Omega_{c_0} \ll \Omega_{p_0}$. That is, while the spin is excited from down to up, the orbital perpendicular motion provides one quantum of energy $\hbar\Omega_{c_0}$ to the spin, and since $\hbar\Omega_{p_0} \neq \hbar\Omega_{c_0}$ the spin also absorbs energy $\hbar(\Omega_{p_0} - \Omega_{c_0})$ from the parallel motion. Thus, the orbital state jumps to a lower energy state with new quantum numbers (ν_f,κ_f,l_f). Since a spin flip from down to up is induced by the spin creation operator $\hat{s}^{(+)}$ in $\hat{H}_{\text{sf}}$, from Eq. (21) we have

$$\langle f | \hat{H}_{\text{sf}} | i\rangle$$

$$= \frac{\hbar}{8} \frac{eg}{2mc}\left[-\frac{\partial B_z}{\partial z_0}\int_{-\infty}^{+\infty} dz \int_0^\infty d\rho\right.$$

$$\times \psi_{l-1,\alpha_f}^*(\rho,z)\rho\psi_{l\alpha}(\rho,z) + \left(\frac{\partial B_x}{\partial x_0} - \frac{\partial B_y}{\partial y_0} - 2i\frac{\partial B_y}{\partial y_0}\right)$$

$$\times \left.\int_{-\infty}^{+\infty} dz \int_0^\infty d\rho\psi_{l+1,\alpha_f}^*(\rho,z)\rho\psi_{l\alpha}(\rho,z)\right], \tag{30}$$

where $l_f = l\pm 1$ is a result of the integral over θ and the matrix element $\langle \hbar/2,\hbar/2 | \hat{s}^{(+)} | -\hbar/2,\hbar/2\rangle = \hbar$ has been used.

The inner products appearing here can be evaluated in the quantum guiding center approximation by changing integration variables from ρ to x_l. The required integrals are

$$M_\pm = \int_{-\infty}^{+\infty} dz \int_0^\infty d\rho \, \psi_{l\pm 1,\alpha_f}^*(\rho,z)\, \rho\, \psi_{l\alpha}(\rho,z)$$

$$= \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} dx_l \, \bar\psi_{l\pm 1,\alpha_f}^*(x_{l\pm 1},z)(\rho_l + x_l)\bar\psi_{l\alpha}(x_l,z),$$

where the integration range in x_l is extended to $\pm\infty$ because $\bar\psi_{l\alpha}(x,z)$ is highly peaked around $x = 0$. The first argument of the barred wave functions appearing in $M_\pm$ are evaluated at different positions, $x_{l\pm 1}$ and x_l. However, these positions are related through the equation

$$x_{l\pm 1} - x_l = \rho_l - \rho_{l\pm 1} = \pm \frac{\hbar}{\mu\Omega_{c_0}\rho_l}.$$

In order to simplify the evaluation of the integrals, we then Taylor expand $\psi_{l\pm 1,\alpha_f}(x_{l\pm 1},z)$ around x_l:

$$\bar\psi_{l\pm 1,\alpha_f}(x_{l\pm 1},z) = \left[1 \pm \frac{i\hat P_l}{\mu\Omega_{c_0}\rho_l} - \frac{\hat P_l^2}{2(\mu\Omega_{c_0}\rho_l)^2} \right.$$
$$\left. + O\left(\frac{r_{q\mathrm{L}}}{\rho_l}\right)^3 \right] \bar\psi_{l\pm 1,\alpha_f}(x_l,z),$$

where $\hat P_l = -i\hbar\partial/\partial x_l$ is the momentum operator. Then, to second order in $r_{q\mathrm{L}}/\rho_l$, $M_\pm$ is given by

$$M_\pm = \rho_l \langle l\pm 1,\alpha_f | 1 - \frac{\hat P_l^2}{2(\mu\Omega_{c_0}\rho_l)^2} | l\alpha\rangle$$
$$+ \langle l\pm 1,\alpha_f | x_l \frac{i\hat P_l}{\mu\Omega_{c_0}} | l\alpha\rangle + O\left(\frac{r_{q\mathrm{L}}}{\rho_l}\right)^3, \qquad (31)$$

where the inner products denote integrals with respect to z and x_l of barred wave functions evaluated at the same point; for example,

$$\langle l'\alpha' | l\alpha\rangle \equiv \int_{-\infty}^\infty dz\, dx\, \bar\psi_{l'\alpha'}(x,z)\bar\psi_{l\alpha}(x,z).$$

Equation (31) can be further simplified since some of the terms are negligible. For example,

$$\langle l-1,\alpha_f | l\alpha\rangle = \langle l\alpha_f | l\alpha\rangle + \sum_{\alpha'} h_{\alpha_f\alpha'}\langle l\alpha' | l\alpha\rangle + O\left(\frac{r_{q\mathrm{L}}}{\rho_l}\right)^3$$

$$= h_{\alpha_f\alpha} + O\left(\frac{r_{q\mathrm{L}}}{\rho_l}\right)^3,$$

where Eq. (25) has been employed, and in the second line we have used the orthogonality of $|l\alpha\rangle$ and $|l\alpha'\rangle$ together with the selection rule $\nu_f = \nu - 1$. However, Eq. (26) implies that $h_{\alpha\alpha'}$ is proportional to $\delta_{\nu\nu'}$ so $\langle l-1,\alpha_f | l\alpha\rangle \sim O(r_{q\mathrm{L}}/\rho_l)^3$ and may be neglected. Similarly, one can also show that

$$\langle l-1,\alpha_f | \frac{\hat P_l^2}{2(\mu\Omega_{c_0}\rho_l)^2} | l\alpha\rangle \sim O\left(\frac{r_{q\mathrm{L}}}{\rho_l}\right)^3,$$

so we neglect this term's contribution to Eq. (31) as well. Combining Eq. (30) and Eq. (31) then yields a simple result for the spin-flip transition matrix element:

$$\langle f | H_{\mathrm{sf}} | i\rangle = -\frac{\hbar r_{q\mathrm{L}}}{8\sqrt{\nu}}\frac{eg}{2mc}
\begin{cases}
\dfrac{\partial B_z}{\partial z_0}\langle l_f\alpha_f | \hat a | l\alpha\rangle, & l_f = l-1, \\[2ex]
\left(\dfrac{\partial B_x}{\partial x_0} - \dfrac{\partial B_y}{\partial y_0} - 2i\dfrac{\partial B_y}{\partial x_0}\right) \\[1ex]
\quad \times \langle l_f\alpha_f | \hat a^+ | l\alpha\rangle, & l_f = l+1,
\end{cases}$$
$$(32)$$

where $\hat a^+$ and $\hat a$ are the creation and annihilation operators for cyclotron quanta: $\hat a \equiv \sqrt{\mu\Omega_{c_0}/2\hbar}(x_l + i\hat P_l/\mu\Omega_{c_0})$. This form for $\langle f | H_{\mathrm{sf}} | i\rangle$ has a clear physical interpretation. The case $l_f = l-1$ corresponds to the transition dynamics we have already described. As one of the electron spin flips from down to up, a relative cyclotron quantum is annihilated by $\hat a$, and the z component of the relative orbital angular momentum, $\hbar l$, is reduced by one unit, conserving the sum of the spin and orbital angular momentum. However, in the second case, $l_f = l+1$, and the sum of the spin and orbital angular momentum is increased by two units because the transition occurs in a nonuniform external magnetic field with a cylindrical asymmetry described by the combination of gradients preceding the matrix element. In this case a quantum of cyclotron action is created by $\hat a^+$, but does not go into cyclotron dynamics, since ν_f must equal $\nu - 1$ in order to conserve energy. Instead, two cyclotron quanta are distributed into energy and canonical angular momentum associated with a change in the relative radial guiding center position, so that the final state still has one fewer quantum in the cyclotron motion. This interpretation follows from the fact that the radial guiding center position, i.e., the position of the peak of $\psi_{l\alpha}$ in ρ, is characterized by the combination $\nu - l$,[8] so the guiding centers end up farther apart by a distance of order $r_{q\mathrm{L}}$. However, since the guiding center motion is relatively slowly varying compared to the cyclotron dynamics, we would expect that such a process is off resonance, and so it should give a negligible contribution to the transition probability.

The guiding center expansion for $\bar\psi_{l\alpha}(x,z)$, Eq. (24), can now be employed in order to explicitly calculate the matrix elements up to $O(r_{q\mathrm{L}}/\rho_l)^2$. This lengthy algebraic exercise is left to Appendix A. We find that the case $l_f = l+1$ does not contribute, as expected. The other case, $l_f = l-1$, is given by Eq. (A3), and leads to a relatively simple form for the transition matrix element:

$$\langle f | H_f | i\rangle = \frac{\partial B_z}{\partial z_0}\frac{e^3\hbar r_{q\mathrm{L}}}{8\mu^2 c\Omega_{c_0}^2(g-2)}$$
$$\times \left(\int dz \frac{F_{\kappa_f}^{(l)*}(z)(z^2 - \rho_l^2/2)F_\kappa^{(l)}(z)}{(\rho_l^2 + z^2)^{5/2}} \right). \qquad (33)$$

As discussed in Appendix A, this expression neglects terms of order $(r_{q\mathrm{L}}/\rho_l)^3$ and higher.

Equation (33) is the transition matrix element for the spin of electron 1 to flip from down to up, which upon substitution into Eq. (20) yields the transition probability

per unit time a_i^f. However, in the Boltzmann analysis of the next section, rather than a_i^f we need P_i^f, the transition probability per collision given by $a_i^f J_\kappa^{-1}$, where J_κ is the incident flux associated with the initial relative wave function of parallel energy κ. To calculate J_κ and the density of final states ρ_f of Eq. (20), we impose periodic boundary conditions at $z = \pm L$ ($L \gg \rho_l$).[3] One finds that $\rho_f = L/\pi \hbar v_z(\kappa_f)$ and $J_\kappa = v_z(\kappa)/2L$, where $(\mu/2)v_z^2(\kappa) \equiv \kappa$, and the incident (initial) state and outgoing (final) state are taken to be $|i\rangle$ of Eq. (28) and $|f\rangle$ of Eq. (29). Finally, we have $P_i^f = |\Delta C_+|^2$, where

$$|\Delta C_+| = \frac{|\partial B_z/\partial z_0|}{\Omega_{c_0}^2} \frac{e^3 r_{qL}}{8\mu^2 c(g-2)} \frac{2L}{\sqrt{v_z(\kappa)v_z(\kappa_f)}}$$

$$\times \left| \int dz \frac{F_{\kappa_f}^{(l)*}(z)(z^2-\rho_l^2/2)F_\kappa^{(l)}(z)}{(\rho_l^2+z^2)^{5/2}} \right|. \quad (34)$$

Since the parallel thermal de Broglie wavelength is much smaller than the distance of closest approach a WKB solution for $F_\kappa^{(l)}(z)$ is valid. Then, if we further assume that $\kappa \sim kT_\parallel \gg (g/2-1)\hbar\Omega_{c_0} = 10^{-3}\hbar\Omega_{c_0}$, a quasiclassical expansion of the WKB wavefunction can be carried out, and the z integral can be transformed into a time-history integral over the classical orbit:[3]

$$\frac{2L}{\sqrt{v_z(\kappa)v_z(\kappa_f)}} \int_{-L}^{L} dz \frac{F_{\kappa_f}^{(l)*}(z)(z^2-\rho_l^2/2)F_\kappa^{(f)}(z)}{(\rho_l^2+z^2)^{5/2}}$$

$$= \int dt \frac{z^2-\rho_l^2/2}{(\rho_l^2+z^2)^{5/2}} e^{-i(g/2-1)\Omega_{c_0}t}. \quad (35)$$

In Eq. (35) the limit $\pm L$ has been extended to $\pm \infty$ since $L \gg \rho_l$, and $z(t)$ is given by Eq. (16).

Substitution of Eq. (35) into Eq. (34) then yields the final form for the transition amplitude in the quantum regime:

$$|\Delta C_+| = \frac{|\partial B_z/\partial z_0|}{\Omega_{c_0}^2} \frac{e^3 r_{qL}}{8\mu^2 c(g-2)}$$

$$\times \left| \int dt\, e^{-i(g/2-1)\Omega_{c_0}t} \frac{z^2-\rho_l^2/2}{(\rho_l^2+z^2)^{5/2}} \right|. \quad (36)$$

For large quantum number ν, Eq. (36) returns to the classical result of Eq. (13) because r_{qL} approaches the classical Larmor radius r_L, as may be seen by the energy correspondence

$$\tfrac{1}{2}\mu r_L^2 \Omega_{c_0}^2 = (\nu + \tfrac{1}{2})\hbar\Omega_{c_0} \simeq \nu\hbar\Omega_{c_0}.$$

V. BOLTZMANN ANALYSIS FOR THE SPIN TEMPERATURE EQUILIBRATION RATE

In this section a collision operator is derived for spin relaxation due to electron–electron collisions in an inhomogeneous magnetic field. The plasma is assumed to be weakly correlated and the effective spin-flip interaction only occurs over a short range of order b, so only two-particle interactions are important, and these collisions can be regarded as point collisions. We therefore use the Boltz-

mann equation to describe the spin relaxation process. Since the electron de Broglie wavelength is small compared to the average interparticle distance, classical Boltzmann statistics rather than the quantum Fermi statistics will be used throughout the calculation.

We first focus on the spin temperature relaxation problem for the classical electron motion discussed in Sec. III. In this case, the kinetic temperatures $T_\perp$ and $T_\parallel$ are large compared to $\hbar\Omega_p$, and so the kinetic energy of the electrons behaves like an infinite temperature heat reservoir supplying energy to excite the spin motion. For this classical case the orbital state of the electron is not affected by the spin flip though the spin-flip probability is determined by the orbital motion, so the spin-flip transitions from $|+\rangle$ to $|-\rangle$ and from $|-\rangle$ to $|+\rangle$ have equal probability. Therefore, we may immediately write down the time rate of change of the spin population due to collisions:

$$\left(\frac{d}{dt}x_-\right)_{\text{coll}} = \nu_{\text{spin}}(\mathbf{x})(x_+ - x_-), \quad (37)$$

where $x_\pm$ is the concentration of electrons with spin state $|+\rangle$ or $|-\rangle$ in a volume element at position $\mathbf{x}$, where the size of the mathematically infinitesimal volume is physically large compared with the average interparticle distance, but small compared with the scale length of the magnetic field inhomogeneity. The spin depolarization rate is given by

$$\nu_{\text{spin}} = \int d^3\mathbf{v}\, f(v_\perp, v_z) \int 2\pi\rho_0\, d\rho_0\, n|v_z|\,|\Delta C|^2. \quad (38)$$

Here $|\Delta C|$ is given by Eq. (14) and $f(v_\perp, v_z)$ is the two-temperature Maxwellian distribution function. A two-temperature Maxwellian distribution is employed since the perpendicular kinetic energy is an adiabatic invariant, and so electron–electron collisions drive the velocity distribution to the two-temperature Maxwellian form on a fast time scale on the order of the electron–electron collision frequency.[6]

Directly substituting Eq. (14) for $|\Delta C|$ in Eq. (38) and performing the integrals over $\mathbf{v}_\perp$, one obtains

$$\nu_{\text{spin}}(\mathbf{x}) = \left(\frac{\bar{r}_L(\mathbf{x})}{8(g-2)L(\mathbf{x})}\right)^2 n\sqrt{\frac{\mu}{2\pi kT_\parallel}}$$

$$\cdot \int dv_z \exp\left(\frac{-\mu v_z^2}{2kT_\parallel}\right)|v_z|$$

$$\cdot \epsilon^2 \cdot \int 2\pi\rho_0\, d\rho_0\, \left|I\left[\left(\frac{g}{2}-1\right)\epsilon^{-1}, \bar{\rho}\right]\right|^2$$

where I is the integral given by Eq. (15), and where $\bar{r}_L(\mathbf{x}) = \sqrt{2kT_\perp/\mu}/\Omega_{c_0}(\mathbf{x})$ is the Larmor radius and $L(\mathbf{x}) = [(1/B_0)(\partial B_z/\partial z_0)]^{-1}$ is the scale length of the magnetic field inhomogeneity.

Furthermore, Eq. (37) implies a simple form for the time evolution equation for the local spin temperature $T_s(\mathbf{x})$, which is defined by $1/T_s = (\partial S/\partial E)_N = (1/\hbar\Omega_{p_0}) \times \ln(x_-/x_+)$, where S, E are the entropy and energy of the spin system:

$$\frac{\dot{T}_s(\mathbf{x})}{T_s} = \frac{2kT_s}{\hbar\Omega_{p_0}}\sinh\left(\frac{\hbar\Omega_{p_0}}{kT_s}\right)\nu_{\text{spin}}, \qquad (39)$$

where

$$\nu_{\text{spin}} = 2.5\times10^2\nu_c\left(\frac{\bar{\epsilon}r_{\text{L}}}{L}\right)^2\eta(\bar{\epsilon}). \qquad (40)$$

Here $\nu_c = \pi\bar{b}^2 n\bar{v}_z$ is the electron–electron collision frequency, $(g-2)$ is taken to be approximately 0.0023, and $\eta(\bar{\epsilon})$ is given by

$$\eta(\bar{\epsilon}) = \int_0^\infty du\, u^{1/3}\exp\left(-\frac{1}{2}u^{2/3}\right)$$
$$\times \int_0^\infty d\bar{\rho}\, 2\pi\bar{\rho}\left|I\left[\left(\frac{g}{2}-1\right)(u\epsilon)^{-1},\bar{\rho}\right]\right|^2, \qquad (41)$$

where we have transformed the integral over velocities by introducing the parameter $u\equiv\epsilon/\bar{\epsilon}=(v_z/\bar{v}_z)^3$, where $\bar{\epsilon}=\bar{v}_z/\bar{b}\Omega_{c_0}$ is the mean adiabaticity parameter, $\bar{v}_z = \sqrt{kT_{\parallel}/\mu}$ is the relative thermal speed, and $\bar{b}=e^2/\frac{1}{2}\mu\bar{v}_z^2=2e^2/kT_{\parallel}$ is the mean distance of closest approach.

To evaluate the numerical value of $\eta(\bar{\epsilon})$, two integrals over $\bar{\rho}$ and u, respectively, were performed after the numerical integration of I. The $\bar{\rho}$ integral was calculated numerically using the IMSL[9] subroutine DQDAGP with the upper integration limit cut off at $\bar{\rho}=8$, which introduces an error of less than $\pm0.6\%$. For the u integral, the integrand is a smoothly varying function of u, and so a cubic spline interpolation method was then applied by using subroutines SPLINE and SPLINT in Ref. 10 to obtain the interpolated integrand. Finally, the u integration was completed by IMSL subroutine DQAGS. A careful estimate of the errors involved in the cubic spline interpolation along with the cutoff in the $\bar{\rho}$ integral implies an error of less than $\pm2\%$ for the value of $\eta(\bar{\epsilon})$.

It is useful to note that for $\bar{\epsilon}\gg(g/2-1)\simeq0.001$, $I[(g/2-1)/\epsilon u,\bar{\rho}]$ can be approximated by $I(0,\bar{\rho})$ since the distribution $u^{1/3}\exp(-\frac{1}{2}u^{2/3})$ is peaked near $u=1$. In this case a numerical integration yields

$$\eta(\bar{\epsilon}) \approx \int_0^\infty du\, u^{1/3}\exp\left(-\frac{1}{2}u^{2/3}\right)$$
$$\times \int_0^\infty d\bar{\rho}\, 2\pi\bar{\rho}|I(0,\bar{\rho})|^2\approx61, \quad \bar{\epsilon}\gg0.001. \qquad (42)$$

We then recover the simple scaling of Eq. (1); the numerical coefficients of the two results are within an order of magnitude of one another. The function $\eta(\bar{\epsilon})$ is plotted in Fig. 5.

The spin depolarization effect is appreciable in a large variety of parameter regimes. As an example, we take $T_\perp \sim T_{\parallel} \sim 20$ K, $\bar{n}\sim10^2$, $B\sim10$ kG, and $L\sim10$ cm. In this case $\bar{\epsilon}\sim8.4\times10^{-2}$ and $\nu_{\text{spin}}\sim9.1\times10^{-2}$ (sec^{-1}), corresponding relaxation time $\nu_{\text{spin}}^{-1}\sim11$ sec. However, if B is sufficiently uniform or strong so that spins are tightly bound to the magnetic field line the depolarization effect is negligible.
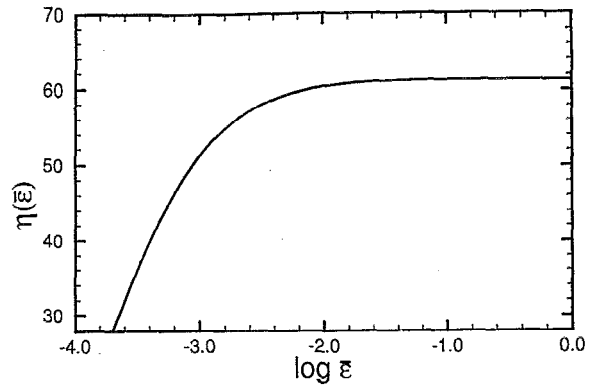

FIG. 5. Plot of $\eta(\epsilon)$. For $\bar{\epsilon}\gtrsim0.01$, $\eta(\bar{\epsilon})$ is almost a constant [see Eq. (42)]. For $\bar{\epsilon}\lesssim0.01$, $\eta(\bar{\epsilon})$ decreases exponentially since spin and orbital motion are out of resonance as $\bar{\epsilon}$ decreases. This $\eta(\bar{\epsilon})$ curve is valid only for $\bar{\epsilon}\lesssim0.1$ due to our assumption of guiding center dynamics during the electron–electron collision.

Now, the spin temperature equilibration determined by Eq. (39) implies that a thermal equilibrium state is reached only when $n_+ = n_-$, i.e., $T_s\to\infty$. Physically, this conclusion is the direct result of the assumption of classical orbital motion. The kinetic energy of the orbital dynamics is assumed large compared to $\hbar\Omega_p$, and serves as an infinite heat reservoir for the spin motion. In order to observe true thermal equilibrium one must therefore treat the orbital motion quantum mechanically.

Denote the occupation number of state $|s,\Gamma\rangle$ in a volume element at position $\mathbf{x}$ by $\tilde{f}(s,\Gamma,\mathbf{x}),=x_s(\mathbf{x})f(\Gamma,\mathbf{x})$, where s represents the spin state and Γ stands for the local single-particle orbital state with respect to the local magnetic field $\mathbf{B}(\mathbf{x})$, which is virtually constant inside the volume element. The orbital distribution function $f(\Gamma)$ is normalized by $\Sigma_\Gamma f(\Gamma) = N$, where N is the total number of electrons in the volume element. Obviously, x_s, the concentration of electrons with spin state $s(=\pm)$ in the volume element, is normalized by $\Sigma_s x_s = x_+ + x_- = 1$.

The rate of change of $\tilde{f}$ due to collisions is governed by the following master equation:

$$\left(\frac{d}{dt}\tilde{f}_i\right)_{\text{coll}} = \sum_{jkl}(a_{kl}^{ij}\tilde{f}_k\tilde{f}_l - a_{ij}^{kl}\tilde{f}_i\tilde{f}_j), \qquad (43)$$

where $\tilde{f}_i\equiv\tilde{f}(s_i,\Gamma_i,\mathbf{x})$, etc., and a_{kl}^{ij} is the transition rate for electron 1 scattered from state $|s_k\Gamma_k\rangle$ to state $|s_i\Gamma_i\rangle$ and electron 2 scattered from $|s_l\Gamma_l\rangle$ to $|s_j\Gamma_j\rangle$. In Eq. (43), the time derivative is a partial derivative at a fixed position $\mathbf{x}$; it denotes the rate of change of the distribution function due to collisions.

Making use of the normalization condition $\Sigma_{\Gamma_i}\tilde{f}_i = Nx_i$ together with the "detailed balance" symmetry relation[11] $\Sigma_k a_{kl}^{ij}=\Sigma_k a_{ij}^{kl}$ in Eq. (43), we find a general expression for the rate of change of the spin distribution due to collisions:

$$\left(\frac{d}{dt}x_i\right)_{\text{coll}}=\frac{1}{N}\sum_{\Gamma_i}\frac{d}{dt}\tilde{f}_i$$

$$=\frac{1}{N}\sum_{\Gamma_i}\left(\sum_{jkl}a_{kl}^{ij}(\tilde{f}_k\tilde{f}_l-\tilde{f}_i\tilde{f}_j)\right)$$

$$=\frac{1}{N}\sum_{\Gamma_i}\left(\sum_{jkl}a_{kl}^{ij}[x_kx_lf(\Gamma_k)f(\Gamma_l)\right.$$

$$\left.-x_ix_jf(\Gamma_i)f(\Gamma_j)]\right).$$

We now assume that $s_i=+$ and consider the form of this rate equation when the golden rule, Eq. (20), is used to determine the a's. As noted previously, the form of H_{sf} implies that in any given two-particle interaction, at most, one spin can be flipped, so transition rates like $a_{-\Gamma_{k'},+\Gamma_l}^{+\Gamma_{i'}-\Gamma_j}$ vanish. Furthermore, the form of $\hat{H}_{\text{sf}}$ also implies that the transition rate for electron 1 is independent of the spin state of electron 2. Also, if neither spin is flipped in the interaction, another detailed balance symmetry relation holds for transitions involving only orbital changes: $\Sigma_{\Gamma_k,\Gamma_l}a_{s_i\Gamma_{k'},s_j\Gamma_l}^{s_i\Gamma_{i'},s_j\Gamma_j}=\Sigma_{\Gamma_k,\Gamma_l}a_{s_j\Gamma_{i'},s_j\Gamma_l}^{s_i\Gamma_{k'},s_j\Gamma_l}$. This follows from the fact that the wave functions of the initial and final states separate into a product of a spin wave function and an orbital wave function, both of which are members of complete sets over the spin and orbital vector spaces.

Using these relations in the rate equation, several cancellations occur, and we are left with

$$\left(\frac{d}{dt}x_+\right)_{\text{coll}}=\frac{1}{N}\sum_{\Gamma_i\Gamma_j\Gamma_k\Gamma_l}a_{-\Gamma_{k'}\Gamma_l}^{+\Gamma_{i'}\Gamma_j}$$

$$\times[x_-f(\Gamma_k)f(\Gamma_l)-x_+f(\Gamma_i)f(\Gamma_j)],$$

(44)

where $a_{-\Gamma_{k'}\Gamma_l}^{+\Gamma_{i'}\Gamma_j}\equiv a_{-\Gamma_{k'}+\Gamma_l}^{+\Gamma_{i'}+\Gamma_j}=a_{-\Gamma_{k'}-\Gamma_l}^{+\Gamma_{i'}-\Gamma_j}$.

If, as before, we assume that $f(\Gamma)$ is an anisotropic Maxwellian distribution function of the form $\exp\{[-E_\perp(\Gamma)/T_\perp]-[E_\parallel(\Gamma)/T_\parallel]\}$, then we may rewrite the two-particle distribution function $f(\Gamma_k)f(\Gamma_l)$ as the product of center of mass (C) and relative (R) distribution function $f_c(\Gamma_{kl}^C)f_R(\Gamma_{kl}^R)$ with normalization condition $\Sigma_{\Gamma_{kl}^C}f_C(\Gamma_{kl}^C)=N$ and $\Sigma_{\Gamma_{kl}^R}f_R(\Gamma_{kl}^R)=N$. As we discussed in Sec. IV, the center of mass variables do not participate in the spin-flip transition. In other words, the transition rate is only a function of Γ_{ij}^R and Γ_{kl}^R. Then summing over the CM states in Eq. (44) and applying the normalization condition, we have

$$\left(\frac{d}{dt}x_+\right)_{\text{coll}}=\sum_{\Gamma_{kl}^R\Gamma_{ji}^R}a_{-\Gamma_{kl}^R}^{+\Gamma_{ij}^R}[x_-f_R(\Gamma_{kl}^R)-x_+f_R(\Gamma_{ij}^R)].$$

Further, taking Γ_{kl}^R to be the quantum numbers (l,ν,κ) associated with state $|i\rangle$ of Eq. (28) and Γ_{ij}^R to be the quantum numbers of state $|f\rangle$ of Eq. (29) with values $(l_f,\nu_f,\kappa_f)=[l-1,\nu-1,\kappa-\hbar(g/2-1)\Omega_{c_0}]$, we obtain

$$\frac{d}{dt}x_+=\sum_{\nu\kappa l}a_{-l\nu\kappa}^{+l-1,\nu-1,\kappa-\hbar(g/2-1)\Omega_{c_0}}\{x_-f_R(l,\nu,\kappa)$$

$$-x_+f_R[l-1,\nu-1,\kappa-\hbar(g/2-1)\Omega_{c_0}]\}$$

$$=\frac{1}{2L}\cdot\sum_{l=-\infty}^{0}\sum_{\nu=1}^{\infty}\sum_{\kappa=\hbar(g/2-1)\Omega_{c_0}}^{\infty}|\Delta C|^2v_z(\kappa).$$

$$\times\{x_-f_R(l,\nu,\kappa)-x_+f_R[l-1,\nu-1,$$

$$\kappa-\hbar(g/2-1)\Omega_{c_0}]\},$$

(45)

where the equation $a_{-l\nu\kappa}^{+l-1,\nu-1,\kappa-\hbar(g/2-1)\Omega_{c_0}}=|\Delta C|^2[v_z(\kappa)/2L]$ has been used [see Eq. (34)]. The sum over ν begins at one rather than zero because $|\Delta C|^2=0$ for $\nu=0$. Furthermore, the sum over κ begins at $\hbar(g/2-1)\Omega_{c_0}$ rather than zero, because, in a transition from $-$ to $+$, this is the minimum relative parallel energy required to conserve energy in the transition. Finally, the sum over l is cut off at 0 rather than ν because we consider only guiding center dynamics for which $\rho_l\gg r_{qL}$. This introduces a negligible relative error of order $(r_{qL}/\bar{b})^2$ to the total transition probability.

The sums can be performed when the explicit form for the relative Maxwellian distribution is employed:

$$f_R(l,\nu,\kappa)=A\exp\left(\frac{-(\nu+\frac{1}{2})\hbar\Omega_{c_0}}{kT_\perp}-\frac{\kappa}{kT_\parallel}\right),$$

where the constant A is determined by normalization condition $\Sigma_{l=-\infty}^0\Sigma_{\nu=0}^\infty\Sigma_{\kappa=0}^\infty f_R(l,\nu,\kappa)=N$. Making the substitutions

$$\sum_l\rightarrow\frac{\int_0^\infty 2\pi\rho_l\,d\rho_l}{\hbar/\mu\Omega_{c_0}};\quad\sum_\kappa\rightarrow\int_0^\infty\rho(\kappa)d\kappa,$$

where $\rho(\kappa)=L/\pi\hbar v_z(\kappa)$ is the density of states and $\kappa=\frac{1}{2}\mu v_z^2(\kappa)$, we find that

$$A=\frac{2Nh^2\sinh(\hbar\Omega_{c_0}/2kT_\perp)}{\mu\Omega_{c_0}V\sqrt{2\pi\mu kT_\parallel}},$$

where $V\equiv L\int 2\pi\rho_l\,d\rho_l$ is the volume of the volume element. Substituting this expression for $f_R(l,\nu,\kappa)$ in Eq. (45) yields

$$\left(\frac{d}{dt}x_+\right)_{\text{coll}}=\sum_{\nu=1}^\infty\int_0^\infty 2\pi\rho_l\,d\rho_l\int_{\kappa>\hbar(g/2-1)\Omega_{c_0}}dv_z(\kappa)$$

$$\times|\Delta C|^2\cdot\frac{\mu^2\Omega_{c_0}f_R(l,\nu,\kappa)}{h^2}$$

$$\times(x_--x_+e^{\hbar\Omega_{c_0}/kT_\perp+\hbar(g/2-1)\Omega_{c_0}/kT_\parallel}),$$

(46)

where $n\equiv N/V$ is the electron number density. Finally, the sum over ν can also be performed, and with the aid of Eq. (36) for $|\Delta C|$, Eq. (46) can be rewritten in terms of the spin temperature as

$$\frac{1}{kT_s}\frac{d}{dt}kT_s(\mathbf{x})=\frac{kT_s}{\hbar\Omega_{p_0}}(1+e^{\hbar\Omega_{p_0}/kT_s})\,(1$$

$$-e^{\hbar\Omega_{c_0}/kT_\perp\,+\hbar(g/2-1)\Omega_{c_0}/kT_\parallel\,-\hbar\Omega_{p_0}/kT_s})\,\nu_{\rm spin}^{(Q)},$$

$$(47)$$

where the quantum spin depolarization rate $\nu_{\rm spin}^{(Q)}$ is given by

$$\nu_{\rm spin}^{(Q)}=\left(\frac{\sqrt{2\hbar/\mu\Omega_{c_0}}}{8(g-2)L}\right)^2 n\sqrt{\frac{\mu}{2\pi kT_\parallel}}\,\frac{1}{2\sinh(\hbar\Omega_{c_0}/2kT_\perp)}$$

$$\cdot\int_{(1/2)\mu v_z^2>\hbar(g/2-1)\Omega_{c_0}}dv_z|v_z|\cdot\exp\left(\frac{-\mu v_z^2}{2kT_\parallel}\right)$$

$$\cdot\epsilon^2\int 2\pi\rho_l\,d\rho_l\left|I\left[\left(\frac{g}{2}-1\right)\epsilon^{-1},\bar\rho\right]\right|^2.\qquad(48)$$

When we again normalize the integrals as in Eq. (40), we obtain

$$\nu_{\rm spin}^{(Q)}=2.5\times10^2\nu_c\left(\frac{\overline{\epsilon}r_\perp}{L}\right)^2\cdot\left(\frac{\hbar\Omega_{c_0}/2kT_\perp}{\sinh(\hbar\Omega_{c_0}/2kT_\perp)}\right)$$

$$\cdot\eta^{(Q)}(\overline{\epsilon},\Omega_{c_0}),\qquad(49)$$

where $\eta^{(Q)}(\overline{\epsilon},\Omega_{c_0})$ is

$$\eta^{(Q)}(\overline{\epsilon},\Omega_{c_0})=\int_{u_m}^\infty du\,u^{1/3}\exp\left(-\frac{1}{2}u^{2/3}\right)$$

$$\times\int_0^\infty 2\pi\bar\rho\,d\bar\rho\left|I\left[\left(\frac{g}{2}-1\right)(u\overline{\epsilon})^{-1},\bar\rho\right]\right|^2,$$

and the lower cutoff u_m is

$$u_m=(g/2-1)^{3/2}(r_{qL}\Omega_c/\overline{v}_z)^3.$$

Note that $u_m\!\ll\!1$, provided that $kT_\parallel\!\gg\!(g/2-1)\hbar\Omega_c$, a condition well satisfied in the experiments. In this case, $\eta^{(Q)}(\overline{\epsilon},\Omega_{c_0})$ approaches the classical result $\eta(\overline{\epsilon})$. However, even when $u_m\!\ll\!1$, Eq. (49) implies that the spin relaxation rate is notably suppressed by quantum effects when $kT_\perp\!\ll\!\hbar\Omega_c$. This is because almost all the electrons will stay at the ground Landau level in this case, and they are forbidden to further give up energies to excite the spin flip. Aside from the quantum suppression factor in Eq. (49), the equilibration rate is also strongly modified by the factor

$$(1-e^{\hbar\Omega_{c_0}/kT_\perp\,+\hbar(g/2-1)\Omega_{c_0}/kT_\parallel\,-\hbar\Omega_{p_0}/kT_s})$$

in Eq. (47), which arises from the self-consistent consideration for the energy transfer between spin and kinetic degrees of freedom. However, if $kT_\perp\!\gg\!\hbar\Omega_c$ and $kT_\parallel\!\gg\!(g/2-1)\hbar\Omega_c$, one may verify that the spin temperature equilibration equation (47) returns to the form of the classical equation, Eq. (39).

As discussed in connection with Eq. (3), we see from Eq. (47) that the spin-flip collisions just calculated cannot drive T_s, $T_\perp$ and $T_\parallel$ toward a common equilibrium tem-

perature. Instead, they can only drive the plasma to a partial equilibrium between T_s, $T_\perp$ and $T_\parallel$ such that

$$\frac{\hbar\Omega_{c_0}}{T_\perp}+\frac{\hbar(g/2-1)\Omega_{c_0}}{T_\parallel}-\frac{\hbar\Omega_{p_0}}{T_s}=0,\qquad(50)$$

from which Eq. (3) immediately follows. This is a consequence of the fact that these collisions conserve an N-particle adiabatic invariant, which equals the sum of the cyclotron action and the spin component along the magnetic field for each particle. For each binary collision, this invariant reduces to the two-particle invariant, $\mu^{(2)}=s_{1z}+s_{2z}+E_\perp^R/\Omega_c+E_\perp^c/\Omega_c$, where $E_\perp^R$ and $E_\perp^c$ are the relative and center of mass perpendicular (cyclotron) energies. The invariance of $\mu^{(2)}$ is evident because for the spin-flip collisions discussed in this paper, $E_\perp^c$ and one of the two spins, say, s_{2z}, are not changed before and after collision, and the remaining part in the invariant, $s_{1z}+E_\perp^R/\Omega_c=s_{1z}+(\nu+\frac{1}{2})\hbar$, is also conserved, since $\Delta\nu=-\Delta s_{1z}/\hbar$. For a weakly correlated plasma in which the collisions are predominantly binary, one may generalize $\mu^{(2)}$ to a many-electron adiabatic invariant $\mu^{(N)}$:

$$\mu^{(N)}=\sum_i\left(s_{iz}+\frac{E_{i\perp}}{\Omega_c}\right),\qquad(51)$$

where the sum is over all the particles. This expression is an extension of the many-electron adiabatic invariant $\Sigma_i E_{i\perp}/\Omega_c$ derived previously for a system in which the spin orbital dynamics is decoupled.[3] In such a system the spin and cyclotron actions are conserved separately. However, an inhomogeneous magnetic field couples the spin and cyclotron dynamics causing an exchange of spin and cyclotron quanta, which leads to the generalized many-electron invariant of Eq. (51).

Equation (3) follows directly from the statistical mechanics of $\mu^{(N)}$-conserving collisions. As a consequence of the invariance of μ, the equilibrium distribution has the form $\rho=Z^{-1}\exp(-\beta H+\alpha\mu^{(N)})$, where $H=\Sigma_i(s_{iz}\Omega_p+E_{i\perp}+E_{i\parallel})$ is the total energy and Z, α, β are constants. By rearranging terms, ρ can be put in the form

$$\rho=Z^{-1}\exp\sum_i\left(-\frac{s_{iz}\Omega_p}{kT_s}-\frac{E_{i\perp}}{kT_\perp}-\frac{E_{i\parallel}}{kT_\parallel}\right),$$

where $T_\parallel$, $T_\perp$, and T_s are related to α and β through the equations $\beta=1/kT_\parallel$; $\beta-\alpha/\Omega_p=1/kT_s$; $\beta-\alpha/\Omega_c=1/kT_\perp$. These relations are equivalent to Eq. (3).

Equation (3) leads us to conclude that T_s will approach $T_\perp$ in this partial equilibrium if $T_\parallel\!\gg\!(g/2-1)T_\perp\approx10^{-3}T_\perp$. The fact that Eq. (3) does not result in the thermal equilibrium condition $T_\perp=T_\parallel=T_s$ implies that we cannot rely on these spin-flip collisions to drive the system to complete thermal equilibrium. Complete thermal equilibrium requires that action invariants such as $\mu^{(N)}$ must be broken. One of the most important $\mu^{(N)}$-breaking collisions is that involving collisional perpendicular and parallel energy exchange without spin flip, which has been discussed by another paper.[3] For a weakly inhomogeneous field, this kind of $\mu^{(N)}$-breaking

collision is the dominant mechanism and these $\mu^{(N)}$-breaking collisions cause equilibration between $T_\perp$ and $T_\parallel$ on a relatively fast time scale. If one assumes that $T_\perp = T_\parallel$ during the spin-kinetic temperature equilibration process the condition $T_\perp = T_\parallel = T_s$ follows directly from Eq. (50).

VI. DISCUSSION

We have seen that in a cryogenic strongly magnetized pure electron plasma the equilibration rate between the spin temperature and the kinetic temperature is dominated by a single process—electron–electron collisions in a non-uniform magnetic field. We have calculated this rate for the case of a weakly correlated plasma in which the collisions are uncorrelated binary events, taking into account the possibility that the cyclotron motion may be quantized. Although many other processes can also cause spin-flip transitions, we have estimated the rates for these processes to be longer than the typical loss rate of the plasma, which is on the order of $10^{-5}\ \text{sec}^{-1}$.

We find that the equilibration rate is proportional to L^{-2}, where L is the scale length of the magnetic field inhomogeneity. In the experiments the uniformity of the magnetic field can be varied over several orders of magnitude simply by confining the plasma at different distances from the end of the solenoid, which produces the magnetic field. Inhomogeneity scale lengths from $L \sim 10$ cm to $L \sim 10^3$ cm can easily be achieved through this technique. This suggests that the rate at which the electron spin temperature approaches the kinetic temperature can be relatively easily controlled. If this rate is reasonably fast, it might be possible to use a measurement of the plasma spin polarization as a thermometer for the kinetic degrees of freedom. Since the electron spin distribution becomes polarized as kT_s falls below $\hbar\Omega_c$, measurement of the degree of polarization of the electron spins could indirectly provide the kinetic temperature in a range of temperatures on the order of $\hbar\Omega_c/k$. For $B \sim 10\text{–}60$ kG, this temperature is on the order of 1 K, which is over an order of magnitude below the minimum temperatures which have been measured using current techniques.[12]

On the other hand, if the plasma is confined in the central region of the solenoid where the field is very uniform, the electron spin distribution is effectively time independent. This suggests a second experiment, in which one uses the spin of an electron as a tag in order to perform various test-particle measurements. For example, one might place a small subpopulation of the plasma in the opposite spin state from the bulk of the plasma, and follow this population's subsequent dynamics in order to evaluate test-particle spatial and velocity diffusion coefficients.

Of course, both of these experiments rely on some scheme for detection of the polarization state of the electrons, and in the test-particle experiment a technique to set up an initial spin distribution is also required. Fortunately, several methods for manipulation and measurement of electron spins have been perfected. For example, the phenomenon known as Mott scattering[13] has been employed for many years in order to both produce polarized elec-

trons and accurately measure their spin state. A novel technique has also recently been proposed[12] in order to produce large quantities of cryogenic spin polarized electrons by using the magnetic inhomogeneity due to finite solenoid length in a trap of the type discussed in this paper. The proposed technique makes use of the idea that the spin Hamiltonian $\mathbf{s}\cdot\boldsymbol{\Omega}_p(\mathbf{x})$ acts as an effective potential in the orbital energy, and this potential is of opposite sign for electrons of opposite spin. As the spatial distribution of electrons thermalizes along each magnetic field line, the $-$spins collect in regions of large $\Omega_p(\mathbf{x})$ and $+$spins collect in the regions of low $\Omega_p(\mathbf{x})$, provided that the parallel kinetic temperature $kT_\parallel$ is less than $\hbar|\Delta\Omega_p|$, where $\Delta\Omega_p$ is the difference between the spin precession frequency in the strong field and weak field regions. This proposed technique could be used to provide copious quantities of cryogenic spin polarized electrons for the spin tagging experiment, as well as other experiments involving polarized electrons.

Finally, we briefly discuss the effect of plasma rotation on the spin depolarization rate. The plasma is confined against radial expansion by the $\mathbf{v}\times\mathbf{B}$ force induced by rotation through the strong applied magnetic field. Throughout the paper we have assumed that the plasma rotation frequency ω_r is small compared to $\Omega_p - \Omega_c$, so that we may neglect the effect of rotation on the dynamics. This is the usual operating regime for the experiments, which generally involve low-density plasmas. For a uniform density plasma column the density is related to the rotation frequency through the expression $\omega_p^2 = 2\omega_r(\Omega_c - \omega_r)$.[14] However, the rotation frequency can at least theoretically be as large as Ω_c (although this can be difficult to achieve in practice), so it is useful to consider this situation.

In a frame rotating with the plasma the Coriolis force, which acts like a magnetic field, shifts the cyclotron frequency to the vortex frequency $\Omega_c - 2\omega_r$.[15] Furthermore, the spin precession frequency is Doppler shifted to $\Omega_p - \omega_r$. Thus, if ω_r is not too close to Ω_p or to $\Omega_c/2$, our results remain valid, provided that one substitutes for $\Omega_p - \Omega_c$ the expression $\Omega_p - \Omega_c + \omega_r$, and substitutes for $\bar{r}_L$ the effective Larmor radius in the rotating frame, $\bar{r}_L\Omega_c/(\Omega_c - 2\omega_r)$. For ω_r near $\Omega_c/2$ the guiding center approximation for the orbital dynamics breaks down, although s_z remains an adiabatic invariant. For ω_r near Ω_p the spin precession frequency is no longer large and s_z is no longer an adiabatic invariant. This introduces a rather novel density dependence in the spin depolarization rate, which can be summarized as follows. Starting at low densities, as the density increases the collision frequency increases and the rate of spin relaxation increases linearly with density. As density increases further, ω_r increases to $O(\Omega_p - \Omega_c)$ and the electron spin precession (as seen in the rotating frame) goes out of resonance with the cyclotron motion, exponentially *reducing* the rate of spin relaxation. However, as ω_r approaches Ω_c, the effective spin precession frequency in the rotating frame, $\Omega_p - \omega_r$, can become as small as $(g/2 - 1)\Omega_c$. Thus, for a narrow range of rotation frequencies near Ω_c the rate of spin relaxation should *increase* dramatically due to resonances between the spin precession

and any orbital motions having frequencies on the order of $(g/2-1)\Omega_c$, such as collisional dynamics parallel to **B**.

ACKNOWLEDGMENTS

APPENDIX A: CALCULATION OF THE TRANSITION MATRIX ELEMENTS

In this appendix we calculate the transition matrix elements in Eq. (32) for a spin flip from down to up. We will evaluate $\langle l-1,\alpha_f|\hat{a}|l\alpha\rangle$ first. The initial value of α is defined by quantum numbers (ν,κ) describing the cyclotron quantum state and the parallel energy, respectively. The final value $\alpha_f = [\nu - 1, \kappa - \hbar(g/2 - 1)]$ is in accordance with energy conservation in a resonant transition from spin down to spin up. According to Eqs. (23), (24), and (27), we have, to the second order of r_L/ρ_l,

$$\langle l-1,\alpha_f|\hat{a}|l\alpha\rangle = M_1+M_2+M_3+M_4, \qquad (A1)$$

where

$$M_1 = {}^{(0)}\langle l\alpha_f|\hat{a}|l\alpha\rangle^{(0)}(1-d_\alpha-d_\alpha^*),$$

$$M_2 = \sum_{\alpha'} (a_{\alpha_f\alpha'}^* + b_{\alpha_f\alpha'}^* + c_{\alpha_f\alpha'}^* + h_{\alpha_f\alpha'}^*)^{(0)}\langle l\alpha'|\hat{a}|l\alpha\rangle^{(0)},$$

$$M_3 = \sum_{\alpha'} (a_{\alpha\alpha'} + b_{\alpha\alpha'} + c_{\alpha\alpha'})^{(0)}\langle l\alpha_f|\hat{a}|l\alpha'\rangle^{(0)},$$

$$M_4 = \sum_{\alpha'\alpha''} a_{\alpha_f\alpha'}^* a_{\alpha\alpha''}{}^{(0)}\langle l\alpha'|\hat{a}|l\alpha''\rangle^{(0)}.$$

We now compute M_1, M_2, M_3, and M_4. Since $\kappa_f\neq\kappa$, the orthogonality of kets $|l\alpha\rangle^{(0)}$ and $|l\alpha_f\rangle^{(0)}$ implies that $M_1=0$. In order to calculate M_2, we note that ${}^{(0)}\langle l\alpha'|\hat{a}|l\alpha\rangle^{(0)} = \sqrt{\nu}\delta_{\nu',\nu-1}\delta_{\kappa'\kappa}$ for $\alpha'=(\nu'\kappa')$, and therefore we only need to calculate the perturbation coefficients $a_{\alpha_f\alpha'}$, $b_{\alpha_f\alpha'}$, $c_{\alpha_f\alpha'}$, $h_{\alpha_f\alpha'}$, for $\alpha'=(\nu-1,\kappa)$. However, Eqs. (22) and (23) imply that, in this case $a_{\alpha_f\alpha'} = 0$ because $\langle\nu_f|x^3|\nu-1\rangle = \langle\nu_f|x|\nu-1\rangle = 0$, for $\nu_f=\nu-1$. Furthermore,

$$b_{\alpha_f\alpha'} = \frac{5}{8}\frac{\mu\Omega_{c_0}^2}{\rho_l^2}\frac{{}^{(0)}\langle\nu_f\kappa_f|x^4|\nu-1,\kappa\rangle^{(0)}}{\kappa_f-\kappa}$$

$$+ \frac{{}^{(0)}\langle\kappa_f|f(z)|\kappa\rangle^{(0)(0)}\langle\nu_f|x^2|\nu-1\rangle^{(0)}}{\kappa_f-\kappa}$$

$$= \left(2-\frac{1}{\nu}\right)\left(\frac{r_{qL}}{2}\right)^2 f_{\kappa_f\kappa}/(\kappa_f-\kappa),$$

where $f_{\kappa_f\kappa} = {}^{(0)}\langle\kappa_f|f(z)|\kappa\rangle^{(0)}$, and we have made use of the matrix elements ${}^{(0)}\langle\nu_f|x^2|\nu-1\rangle^{(0)} = (2 - 1/\nu)r_{qL}^2/4$ and ${}^{(0)}\langle\nu_f\kappa_f|x^4|\nu - 1,\kappa\rangle^{(0)} = 0$ since $\kappa_f\neq\kappa$. Continuing next term in M_2, we have

$$c_{\alpha_f\alpha'} \equiv \sum_{\alpha_1} \frac{\langle\alpha_f|(-\mu\Omega_{c_0}^2/2\rho_l)x^3+g(z)x|\alpha_1\rangle\langle\alpha_1|(-\mu\Omega_{c_0}^2/2\rho_l)x^3+g(z)x|\alpha'\rangle}{(\kappa_f-\kappa)[(\nu_f-\nu_1)\hbar\Omega_{c_0}+\kappa_f-\kappa_1]}.$$

We observe that the numerator of each term vanishes unless $\nu_1=\nu_f\pm1$ or $\nu_f\pm3$ for $\alpha_1=(\nu_1,\kappa_1)$. But $\nu_1=\nu_f\pm3$ can be excluded, since then the numerator equals

$$\langle\nu_f\kappa_f|\frac{-\mu\Omega_{c_0}^2}{2\rho_l}x^3|\nu_f\pm3,\kappa_1\rangle\langle\nu_f\pm3,\kappa_1|\frac{-\mu\Omega_{c_0}^2}{2\rho_l}x^3|\nu_f,\kappa\rangle,$$

which is zero for $\kappa_f\neq\kappa$. Then, using the matrix elements $\langle\nu|x^3|\nu-1\rangle = \frac{3}{8}r_{qL}^3$ and $\langle\nu|x|\nu-1\rangle = \frac{1}{2}r_{qL}$, we find that the numerator equals

$$\frac{-3\mu}{32\rho_l}(\nu\Omega_{c_0}r_{qL}^2)^2 g_{\kappa_f\kappa}(\delta_{\kappa\kappa_1}+\delta_{\kappa_f\kappa_1}) + \frac{1}{4}r_{qL}^2 g_{\kappa_f\kappa_1}g_{\kappa_1\kappa}$$

for $\nu_1=\nu_f+1$. The result is identical for $\nu_1=\nu_f-1$, except that ν is replaced by $\nu-1$. Adding the expressions for $\nu=\nu_f\pm1$ together, we obtain

$$c_{\alpha_f\alpha'} = \frac{-3\mu(\Omega_{c_0}r_{qL}^2)^2 g_{\kappa_f\kappa}}{32\rho_l(\kappa_f-\kappa)\nu^2}\left(\frac{\nu^2}{-\hbar\Omega_{c_0}+\kappa_f-\kappa} - \frac{\nu^2}{\hbar\Omega_{c_0}}\right.$$

$$\left. + \frac{(\nu-1)^2}{\hbar\Omega_{c_0}+\kappa_f-\kappa} + \frac{(\nu-1)^2}{\hbar\Omega_{c_0}}\right) + \frac{r_{qL}^2}{4(\kappa_f-\kappa)\nu}\sum_{\kappa_1}g_{\kappa_f\kappa_1}g_{\kappa_1\kappa}$$

$$\times\left(\frac{\nu}{-\hbar\Omega_{c_0}+\kappa_f-\kappa_1} + \frac{\nu-1}{\hbar\Omega_{c_0}+\kappa_f-\kappa_1}\right).$$

Finally, $h_{\alpha_f\alpha'} = r_{qL}^2 g_{\kappa_f\kappa}/2\rho_l(\kappa_f - \kappa)\nu$ by definition. Now, combining the above results, we obtain the following expression for M_2:

$$M_2 = \frac{r_{qL}^2}{4(\kappa_f-\kappa)\sqrt{\nu}}(2\nu-1)f_{\kappa_f\kappa} + \frac{2g_{\kappa_f\kappa}}{\rho_l}\left[1 - \frac{3}{8\nu}\mu(\Omega_{c_0}r_{qL})^2\right.$$

$$\times\left(\frac{(\nu-1)^2}{\hbar\Omega_{c_0}} - \frac{\nu^2}{\hbar\Omega_{c_0}} + \frac{\nu^2}{-\hbar\Omega_{c_0}+\kappa_f-\kappa}\right.$$

$$+\frac{(\nu-1)^2}{\hbar\Omega_{c_0}+\kappa_f-\kappa}\Bigg)\Bigg]+4\sum_{\kappa_1}g_{\kappa_f\kappa_1}g_{\kappa_1\kappa}$$

$$\times\left(\frac{\nu}{-\hbar\Omega_{c_0}+\kappa_f-\kappa_1}+\frac{\nu-1}{\hbar\Omega_{c_0}+\kappa_f-\kappa_1}\right),$$

where $\kappa_f-\kappa=-\hbar(\Omega_p-\Omega_c)$. Turning to M_3, a similar calculation yields

$$M_3=\frac{\sqrt{\nu}r_{qL}^2}{4(\kappa-\kappa_f)\sqrt{\nu}}\Bigg[(2\nu+1)f_{\kappa_f\kappa}-\frac{3}{8}\frac{g_{\kappa_f\kappa}}{\rho_l\nu}\mu(\Omega_{c_0}r_{qL})^2$$

$$\times\left(\frac{(\nu+1)^2}{-\hbar\Omega_{c_0}+\kappa-\kappa_f}-\frac{(\nu+1)^2}{\hbar\Omega_{c_0}}+\frac{\nu^2}{\hbar\Omega_{c_0}+\kappa-\kappa_f}+\frac{\nu^2}{\hbar\Omega_{c_0}}\right)$$

$$+4\sum_{\kappa_1}g_{\kappa\kappa_1}g_{\kappa_1\kappa_f}\left(\frac{\nu+1}{-\hbar\Omega_{c_0}+\kappa-\kappa_1}+\frac{\nu}{\hbar\Omega_{c_0}+\kappa-\kappa_1}\right)\Bigg].$$

Turning to M_4, we notice that $\langle l\alpha'|\hat{a}|l\alpha''\rangle=\sqrt{\nu'+1}\delta_{\nu'',(\nu'+1)}\delta_{\kappa''\kappa'}$. Then

$$M_4=\sum_{\alpha'}\sqrt{\nu'+1}a^*_{\alpha_f\alpha'}a_{\alpha,(\nu'+1)\kappa'}$$

$$=\sum_{\alpha'}\sqrt{\nu'+1}$$

$$\times\frac{\langle\alpha'|H^{(1)}|\alpha_f\rangle\langle\alpha|H^{(1)}|\nu'+1,\kappa'\rangle}{[(\nu_f-\nu')\hbar\Omega_{c_0}+\kappa_f-\kappa'][(\nu-\nu'-1)\hbar\Omega_{c_0}+\kappa-\kappa']}.$$

From Eq. (22) the numerator is nonzero only when $\nu'-\nu_f=\pm1$ or ±3. In these cases, the denominator is of order $O(\hbar\Omega_{c_0})^2$ because $|\kappa_f-\kappa'|$ and $|\kappa-\kappa'|$ must be much smaller than $\hbar\Omega_{c_0}$; otherwise, the inner products involving the dynamics in z would result in exponentially small results. Recalling that $H^{(1)}$ is of order $O(r_{qL}\rho_l)$, we see that M_4 is of order $O(r_{qL}/\rho_l)^4$, which is negligible compared with M_2 and M_3.

Finally, combining these results, we have

$$\langle l-1,\alpha_f|\hat{a}|l\alpha\rangle=M_2+M_3+O\left(\frac{r_{qL}}{\rho_l}\right)^3$$

$$=\frac{r_{qL}^2}{4(\kappa_f-\kappa)\sqrt{\nu}}\Bigg[-2f_{\kappa_f\kappa}+\frac{2g_{\kappa_f\kappa}}{\rho_l}-\frac{3g_{\kappa_f\kappa}}{2\rho_l}$$

$$\times\left(1+\frac{\hbar\Omega_{c_0}-(2\nu^2+1)(\kappa_f-\kappa)}{\hbar\Omega_{c_0}-(\kappa_f-\kappa)^2/\hbar\Omega_{c_0}}\right)+4\sum_{\kappa_1}g_{\kappa_f\kappa_1}g_{\kappa_1\kappa}$$

$$\times\left(\frac{\hbar\Omega_{c_0}+(2\nu-1)(\kappa_f-\kappa_1)}{(\hbar\Omega_{c_0})^2-\kappa_f-\kappa_1}\right)$$

$$-\frac{\hbar\Omega_{c_0}+(2\nu+1)(\kappa-\kappa_1)}{(\hbar\Omega_{c_0})^2-(\kappa-\kappa_1)^2}\right)\Bigg]. \tag{A2}$$

This expression may be further simplified as follows. As shown in the Appendix of Ref. 3, the matrix element $g_{\kappa_f\kappa}$ can be evaluated to the lowest order in $(\kappa_f-\kappa)/\kappa=(g/2-1)\hbar\Omega_{c_0}\kappa$ by integration along the classical $\bar{z}(t)$

orbit: $g_{\kappa_f\kappa}=\int dt\, g[\bar{z}(t)]e^{-i(\kappa_f-\kappa)t/\hbar}$ [see Eq. (35)]. In order to avoid an exponentially small result, we require that $\hbar/(\kappa_f-\kappa)\gtrsim O(\rho_l/v)$, where ρ_l/v is the time scale during which the function $g[\bar{z}(t)]$ changes and $v=\sqrt{2\kappa/\mu}$. Then we obtain the ordering $(\kappa_f-\kappa)/\hbar\Omega_{c_0}\lesssim O(r_{qL}/\rho_l)\times\sqrt{\kappa/v\hbar\Omega_{c_0}}$, and therefore

$$\frac{\hbar\Omega_{c_0}-(2\nu^2+1)(\kappa_f-\kappa)}{\hbar\Omega_{c_0}-(\kappa_f-\kappa)^2/\hbar\Omega_{c_0}}=1+O\left(\frac{r_{qL}}{\rho_l}\right).$$

Using the same argument, we find that

$$\frac{\hbar\Omega_{c_0}+(2\nu-1)(\kappa_f-\kappa_1)}{(\hbar\Omega_{c_0})^2-(\kappa_f-\kappa_1)^2}=\frac{1}{\hbar\Omega_{c_0}}\left[1+O\left(\frac{r_{qL}}{\rho_l}\right)\right]$$

and

$$\frac{\hbar\Omega_{c_0}+(2\nu+1)(\kappa-\kappa_1)}{(\hbar\Omega_{c_0})^2-(\kappa-\kappa_1)^2}=\frac{1}{\hbar\Omega_{c_0}}\left[1+O\left(\frac{r_{qL}}{\rho_l}\right)\right],$$

so the difference between these two expressions is of order $(1/\hbar\Omega_{c_0})O(r_{qL}/\rho_l)$. This implies that the term in Eq. (A2) involving the sum over κ_1 is approximately equal to $\Sigma_{\kappa_1}g_{\kappa_f\kappa}(g_{\kappa_1\kappa}/\hbar\Omega_{c_0})\cdot O(r_{qL}/\rho_l)$, which is higher order in r_{qL}/ρ_l than the other terms in Eq. (A2). Combining the above results yields

$$\langle l-1,\alpha_f|\hat{a}|l,\alpha\rangle=\frac{r_{qL}^2}{4(\kappa_f-\kappa)\sqrt{\nu}}\left(-2f_{\kappa_f\kappa}-\frac{g_{\kappa_f\kappa}}{\rho_l}\right)+O\left(\frac{r_{qL}}{\rho_l}\right)^3$$

$$=\frac{r_{qL}e^2}{4\hbar(\Omega_p-\Omega_c)\sqrt{\nu}}\left(\frac{\rho_l^2-2z^2}{(\rho_l^2+z^2)^{5/2}}\right)_{\kappa_f\kappa}. \tag{A3}$$

We may easily calculate the other matrix element $\langle l+1,\alpha_f|\hat{a}^+|l,\alpha\rangle$ following the same procedure as for $\langle l-1,\alpha_f|\hat{a}|l,\alpha\rangle$. This matrix element can be written as $M_1'+M_2'+M_3'+M_4'+O(r_{qL}/\rho_l)^3$, where M_1' to M_4' have the same form as M_1 to M_4, except that $\hat{a}$ is changed to $\hat{a}^+$ and $h^*_{\alpha_f\alpha'}$ is changed to $-h^*_{\alpha_f\alpha'}$. We determine the order of magnitude of the matrix elements M_1' to M_4' in order to show that they are negligible. First, $M_1'=0$ since $\kappa_f\neq\kappa$. For M_2', the term $\langle l\alpha'|\hat{a}^+|l\alpha\rangle$ yields the selection rule: $\nu'=\nu+1$, $\kappa'=\kappa$. Then $a_{\alpha_f\alpha'}=0$ since $\langle\nu_f|x^3|\nu+1\rangle=\langle\nu_f|x|\nu+1\rangle=0$ for $\nu_f=\nu-1$, and

$$b_{\alpha_f\alpha'}=\frac{5}{8}\frac{\mu\Omega_{c_0}^2}{\rho_l^2}\frac{\langle\nu_f\kappa_f|x^4|\nu+1,\kappa\rangle}{\kappa_f-\kappa-2\hbar\Omega_{c_0}}+\frac{f_{\kappa_f\kappa}\langle\nu_f|x^2|\nu+1\rangle}{\kappa_f-\kappa-2\hbar\Omega_{c_0}}$$

$$\sim O\left(\frac{r_{qL}}{\rho_l}\right)^3,$$

since $\langle\nu_f|x^2|\nu+1\rangle\sim O(r_{qL}^2)$. Similarly, we have $c_{\alpha_f\alpha'}\sim h_{\alpha_f\alpha'}\sim O(r_{qL}/\rho_l)^3$, and therefore $M_2'\sim O(r_{qL}/\rho_l)^3$. One may also check that $M_3'\sim O(r_{qL}/\rho_l)^3$. Turning to M_4', the term $\langle l\alpha'|\hat{a}^+|l\alpha''\rangle$ implies the selection rule $\nu'=\nu''+1$, $\kappa'=\kappa''$, and therefore in analogy to M_4 we have $M_4'\sim O(r_{qL}/\rho_l)^4$. In conclusion, we

find that the matrix element $\langle l+1,\alpha_f|\hat{a}^+|l,\alpha\rangle$ is of $O(r_L/\rho_l)^3$, which is negligible compared to $\langle l-1,\alpha_f|\hat{a}|l,\alpha\rangle$.

APPENDIX B: EVALUATION OF THE ELLIPTIC INTEGRAL EXPRESSION FOR $t(z,\rho)$

In this appendix we obtain a closed-form analytic expression in terms of elliptic integrals for the parallel guiding center motion $t(z,\rho)$ given by Eq. (16). This simplifies numerical evaluation of the function $I(x,\bar{\rho})$. Although alternative expressions for $t(z,\rho)$ are possible, the one derived in this appendix has the advantage that it avoids (removable) singularities and is then useful for numerical calculations. From Eq. (16), $\bar{t}$ can be expressed as

$$\bar{t}(\bar{z},\bar{\rho})=\int_{\bar{z}_m}^{\bar{z}}\frac{d\bar{z}}{\sqrt{1-(1/\sqrt{\bar{z}^2+\bar{\rho}^2})}},\qquad (B1)$$

where $\bar{z}_m$ is the $\bar{z}$ value at the distance of closest approach:

$$\bar{z}_m=\begin{cases}0, & \text{for }\bar{\rho}>1,\\ \sqrt{1-\bar{\rho}^2}, & \text{for }\bar{\rho}<1.\end{cases}$$

For $\bar{\rho}=0$, the integral in Eq. (B1) can be easily calculated, and the result is that $\bar{t}(\bar{z},0)=\sqrt{\bar{z}^2-\bar{z}}+\ln(\sqrt{\bar{z}-1}+\sqrt{\bar{z}})$.

For $\bar{\rho}\neq0$, we introduce the new variable $u=\frac{1}{2}\arctan(\bar{z}/\bar{\rho})$; then Eq. (B1) becomes

$$\bar{t}(u,\bar{\rho})=2\bar{\rho}\int_{u_m}^{u}\frac{du}{(1-2\sin^2 u)^2\sqrt{(1-1/\bar{\rho})+(2/\bar{\rho})\sin^2 u}},\qquad (B2)$$

where

$$u_m\equiv\tfrac{1}{2}\arctan(\bar{z}_m/\bar{\rho}).$$

We now proceed to evaluate $\bar{t}(u)$ separately for the $\rho\geqslant1$ and $\rho<1$ cases.

1. $\rho>1$ case

We rewrite $\bar{t}(u)$ as

$$\frac{\bar{\rho}\sqrt{\bar{\rho}}}{2\sqrt{\bar{\rho}-1}}[P_{-2}(u,\bar{\rho})-P_{-2}(u_m,\bar{\rho})],$$

where we define

$$P_n(u,\bar{\rho})\equiv\int_0^u\frac{du(-1/2+\sin^2 u)^n}{\Delta_1},$$

$$\Delta_1\equiv\sqrt{1+p^2\sin^2 u},$$

$$p\equiv2/(\bar{\rho}-1).$$

Notice that P_{-2} can be expressed in terms of $P_{\pm1}$ through the identity[16]

$$-p^2P_1=-\Delta_1\tan 2u+\frac{p^2}{4}P_{-1}+\frac{1}{2}\left(1+\frac{p^2}{2}\right)P_{-2}.\qquad (B3)$$

Now we relate $P_{\pm1}$ to elliptic functions. First, we note that

$$P_1=\int\frac{-1/2+\sin^2 u}{\Delta_1}du$$

$$=-\left(\frac{1}{2}+\frac{1}{p^2}\right)\int\frac{du}{\Delta_1}+\frac{1}{p^2}\int\Delta_1\,du$$

$$=-\left(\frac{1}{2}+\frac{1}{p^2}\right)\frac{1}{\sqrt{1+p^2}}F_1$$

$$+\frac{1}{p^2}\left(\frac{\sqrt{1+p^2}E_1-p^2\sin u\cos u}{\Delta_1}\right),$$

where E_1, F_1 denote the first and second kind elliptic integrals: $E_1\equiv E(\beta,k_1)$; $F_1\equiv F(\beta,k_1)$ and

$$\beta\equiv\arcsin\left(\frac{\sqrt{1+p^2}\sin u}{\Delta_1}\right),$$

$$k_1\equiv p/\sqrt{1+p^2}.$$

For P_{-1} we use the identity

$$\frac{1}{(-1/2+\sin^2 u)\Delta_1}=\frac{-4}{p^2+2}\frac{\Delta_1}{1-2\sin^2 u}-\frac{2p^2}{(p^2+2)\Delta_1},$$

and obtain

$$P_{-1}=\frac{-4}{p^2+2}\frac{1}{\sqrt{1+p^2}}\Pi_1+\frac{p^2}{p^2+2}\frac{1}{\sqrt{1+p^2}}F_1,$$

where Π_1 denotes the third kind elliptic integral:

$$\Pi_1\equiv\Pi\left(\beta,\frac{2+p^2}{1+p^2},k_1\right).$$

Finally, substituting the elliptic integral expression for $P_{\pm1}$ in Eq. (B3) yields

$$P_{-2}(u,\bar{\rho})=\frac{2\bar{z}(u)}{\bar{\rho}\Delta_1}+\left(1+\frac{p^4}{(2+p^2)^2}\right)\frac{2}{\sqrt{1+p^2}}F_1$$

$$-\frac{4\sqrt{1+p^2}}{2+p^2}E_1+\frac{4p^2}{(2+p^2)^2}\frac{1}{\sqrt{1+p^2}}\Pi_1,\qquad (B4)$$

where $\bar{z}(u)=\bar{\rho}\tan 2u$.

2. $\rho<1$ case

We rewrite Eq. (B2) as

$$\bar{t}(v,\bar{\rho})=\frac{\bar{\rho}\sqrt{\bar{\rho}}}{2\sqrt{\bar{\rho}+1}}[Q_{-2}(v_m,\bar{\rho})-Q_{-2}(v,\bar{\rho})],$$

where

$$Q_n(v,\bar{\rho})\equiv\int_0^v\frac{dv(-1/2+\sin^2 v)^n}{\Delta_2},$$

$$\Delta_2\equiv\sqrt{1-k_2^2\sin^2 v},$$

$$k_2\equiv\sqrt{\frac{2}{1+\bar{\rho}}},$$

$$v\equiv\pi/2-u,$$

$$v_m \equiv \frac{\pi}{2} - u_m = \frac{\pi}{2} - \frac{1}{2}\arctan\sqrt{\frac{1}{\bar{\rho}^2} - 1}.$$

In analogy to Eq. (B3), we have, for Q_n,

$$k_2^2 Q_1 = -\Delta_2 \tan 2v - \frac{k_2^2}{4}Q_{-1} + \frac{1}{2}\left(1 - \frac{k_2^2}{2}\right)Q_{-2}, \quad (\text{B5})$$

where

$$Q_1 = \int \frac{-\frac{1}{2} + \sin^2 v}{\Delta_2}dv = \frac{-1}{k_2^2}E_2 + \left(\frac{1}{k_2^2} - \frac{1}{2}\right)F_2.$$

Here E_2 and F_2 denote elliptical integrals of the first and second kind: $E_2 \equiv E(v,k_2)$; $F_2 \equiv F(v,k_2)$. For

$$Q_{-1} = -2 \int \frac{dv}{(1 - 2\sin^2 v)\Delta_2} = -2\Pi(v,2,k_2),$$

we have[17]

$$Q_{-1} = 2\Pi_2 - 2F_2 - \frac{1}{p_1}\ln\left|\frac{p_1 \tan v + \Delta_2}{p_1 \tan v - \Delta_2}\right|,$$

where Π_2 denotes the elliptic integral of the third kind:

$$\Pi_2 \equiv \Pi\left(v, \frac{1}{1 + \bar{\rho}}, k_2\right),$$

and

$$p_1 \equiv \sqrt{\frac{\bar{\rho}}{\bar{\rho} + 1}}.$$

Finally, substituting the elliptic integral expression of $Q_{\pm 1}$ in Eq. (B5) yields

$$Q_{-2}(v,\bar{\rho}) = 2\left(1 - \frac{1}{\bar{\rho}}\right)F_2 - 2\left(1 + \frac{1}{\bar{\rho}}\right)E_2 + \frac{1}{\bar{\rho}}\Pi_2$$

$$-2\left(1 + \frac{1}{\bar{\rho}}\right)\frac{\bar{z}}{\bar{\rho}}\Delta_2 - \frac{1}{\bar{\rho}p_1}\ln\left|\frac{p_1 \tan v + \Delta_2}{p_1 \tan v - \Delta_2}\right|, \quad (\text{B6})$$

where $\bar{z}(v) = -\bar{\rho} \tan 2v$.

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