LETTERS

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Effects of electrostatic confinement fields and finite gyroradius on an instability of hollow electron columns

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(Received 15 July 1991; accepted 16 October 1991)

Exponentially growing diocotron instabilities with azimuthal mode number $l=1$ arise in hollow, cylindrical electron columns when the continuity equation includes small terms corresponding to finite gyroradius or axial variation of the radial electric field. The modes are similar in spatial form to the algebraically growing perturbations of the theory without these effects.

Diocotron instabilities are a well-studied class of shear-flow instabilities in non-neutral plasmas confined by transverse electrostatic and magnetostatic fields.[1] In this Letter we consider such instabilities in cylindrical plasma columns bounded by a cylindrical conductor at $r=R$ and confined by a strong, uniform magnetic field $\mathbf{B}=B\hat{z}$; we employ cylindrical coordinates (r,θ,z). The plasma is assumed to be nonrelativistic and of moderate density ($\omega_p \ll \Omega$ where ω_p and Ω are plasma and cyclotron frequencies, respectively). Recent experiments by Driscoll[2] on hollow, cylindrical electron columns exhibit such an instability with azimuthal mode number $l=1$. No such instability is predicted by the usual treatments based on spectral analysis of the cylindrical Rayleigh equation

$$\left(\frac{\partial}{\partial t}+il\omega_E(r)\right)\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{l^2}{r^2}\right)\phi-i\frac{4\pi ec}{B}\frac{dn_0}{dr}\frac{l}{r}\phi=0, \tag{1}$$

where ω_E is the local $\mathbf{E}\times\mathbf{B}$ rotation rate and the electrostatic potential perturbation is the real part of $\phi(r,t)\exp il\theta$ with $\phi=0$ at $r=R$ and regular at $r=0$. However, Rosenbluth discovered by initial-value treatment of Eq. (1) that there are algebraically growing disturbances with $l=1$ whenever the rotation frequency is nonmonotonic [i.e., there is a radius r_c where $\omega_E'(r_c)=0$ with $0<r_c<R$ and $\omega_E''(r_c)\neq 0$]. The details of this instability were presented in Ref. 3. The asymptotic form of these perturbations in linear theory is

$$\phi_{\text{asy}}\propto\begin{cases}t^{1/2}r[\omega_c-\omega_E(r)], & \text{for } r<r_c\\ 0, & \text{for } r>r_c,\end{cases} \tag{2}$$

where $\omega_c=\omega_E(r_c)$. In the experiments, on the other hand, exponential growth was observed. In the following analysis it is shown that there are exponential instabilities similar in spatial form to Eq. (2) when the continuity equation is modified to account for either finite gyroradius or the externally imposed electrostatic confinement field. We consider only small modifications, so the modes are still essentially waves superposed on a sheared $\mathbf{E}\times\mathbf{B}$ flow.

In all cases we take as the leading-order description of the equilibrium a density $n_0(r,z)$ with $\partial n_0/\partial z\sim 0$ except in boundary layers near the ends of the plasma columns $[n(r,z)\approx N_0(r)]$, with a single maximum in the rotation frequency profile $\omega_E(r)$ at $r_c>0$. The magnetic field is sufficiently strong that $\omega_E\ll\bar{v}/L$ although $\lambda_D\ll R\ll L$, where $\bar{v}$ is a thermal velocity, λ_D is the Debye length, and L is the length of the column. The representative profile

$$N_0(r)=\left[1-\left(\frac{r}{r_p}\right)^2\right]\left[1+9\left(\frac{r}{r_p}\right)^2-10\left(\frac{r}{r_p}\right)^4\right]H(r_p-r),$$

where $H(x)$ is the Heaviside step function and $r_p=0.5R$, shown in Fig. 1, was used for the numerical results shown below.

First let us consider the role of the electrostatic confinement field, which is imposed by setting boundary potentials in the experiments. Because of z dependence, the radial electric field (and hence the instantaneous rotation frequency) differs slightly from the field computed by radial integration of the line-averaged density. In the limit of large B, the $\mathbf{E}\times\mathbf{B}$ motion is slow compared to bouncing along $\mathbf{B}$, so the principal effect can be expressed as an increment in the bounce-averaged rotation frequency. (This assumes that terms arising from the z dependence of

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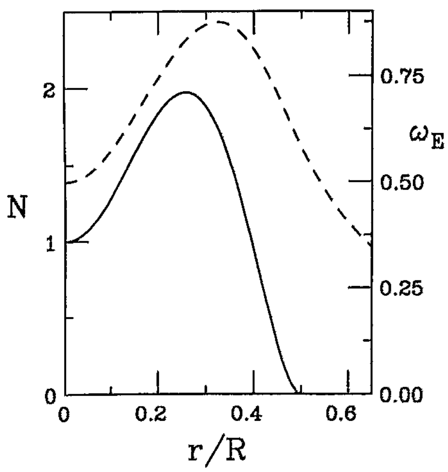

FIG. 1. Radial equilibrium profiles for a hollow cylindrical electron plasma: Solid line, left scale—line-integrated density normalized to central value; dashed line, right scale—rotation frequency normalized to central ω_p^2/Ω. A conducting cylinder is located at $r=R$.

the perturbation potential are negligible; this is plausible for small λ_D/R, but should be checked against a more complete analysis.) To avoid excessive detail depending on the confinement geometry, an illustrative form for this increment is considered, specifically

$$\overline{\omega_E}(r)=\frac{4\pi ec}{Br^2}\int dr'\,r'N_0(r')+Ar^2, \qquad (3)$$

where the overbar indicates a bounce average and the integral is the usual two-dimensional (2-D) form. A constant frequency offset can be eliminated by working in a rotating frame; the r^2 term is the simplest nontrivial one. The phenomenological parameter A represents the effective strength of the radial component of the confinement field; it presumably increases for shorter plasmas. For $l=1$, it is convenient to work with a radial displacement field $\xi(r)$, defined by $\phi(r,t)=r[\omega-\omega_E(r)]\xi(r)\exp(-i\omega t)$. The associated density perturbation is $\delta n=-\xi(r)(dn_0/dr)\times\exp(il\theta-i\omega t)$. The boundary conditions are $\xi(R)=0$ and $d\xi/dr=0$ at the origin. The ideal asymptotic displacement field is proportional to the step function $H(r_c-r)$. With the model of Eq. (3), the displacement field satisfies

$$\frac{d}{dr}r^3[\omega-\omega_E(r)]^2\frac{d}{dr}\xi-r^3[\omega-\omega_E(r)]8A\xi=0. \qquad (4)$$

Equation (4) is a somewhat unusual boundary-layer problem, so a summary of its asymptotic behavior may be of interest.[4] Near the peak in rotation rate we have $\omega_E\sim\omega_c+\frac{1}{2}\omega_c''(r-r_c)^2$. Define a real boundary-layer scale δ and phase μ by $\omega=\omega_c-\delta^2e^{2i\mu}\omega_c''$. We are interested in the scaling as $\delta\to 0$, so assume that $\delta\ll r_\omega(r)$ in the boundary layer, where $r_\omega(r)=\omega_E''/\omega_E'''$. The outer solutions obtained by integrating from $r=0$ and $r=R$ are

$$\xi_-\sim 1+8A\int_0^r\frac{dr'}{r'^3[\omega_c-\omega_E(r')]^2}$$

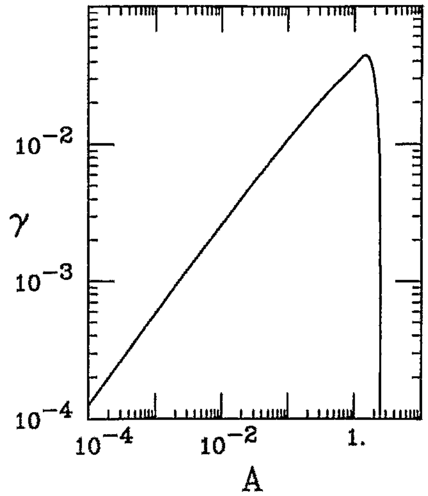

FIG. 2. Dependence of growth rate for $l=1$ diocotron instability on A (bounce-averaged frequency decrement—see text) for the equilibrium shown in Fig. 1. A is expressed in units of $\omega_p^2/R^2\Omega$ on axis. The real part of the mode frequency is close to $\omega_c\approx 0.88$ in these units.

$$\times\int_0^{r'}dr''\,r''^3[\omega_c-\omega_E(r'')] \qquad (5)$$

for $(r_c-r)\gg\delta$, and

$$\xi_+\sim b\int_r^R\frac{dr'}{r'^3[\omega_c-\omega_E(r')]^2} \qquad (6)$$

for $(r-r_c)\gg\delta$, where b is a constant determined by matching. In the matching regions where $r_\omega\gg|r-r_c|\gg\delta$, the outer solutions are approximately $\xi_-\sim 1-Ab_0+Ab_1(r-r_c)^{-3}$ and $\xi_+\sim b_2+b_3(r-r_c)^{-3}$; the constants b_2 and b_3 depend on A but b_0 and b_1 do not. With a boundary-layer coordinate $x=(r-r_c)/\delta$ the leading-order inner solution $y_0(x)\sim\xi(r)$ satisfies

$$(e^{2i\mu}+x^2)\frac{d^2y_0}{dx^2}+4x\frac{dy_0}{dx}=0, \qquad (7)$$

so we must match the outer solutions to

$$y_0(x)=c_0\left[\frac{xe^{i\mu}}{e^{2i\mu}+x^2}+\frac{i}{2}\ln\left(\frac{ie^{i\mu}+x}{ie^{i\mu}-x}\right)\right]+c_1, \qquad (8)$$

with the branch cut going through ∞ so that y_0 is regular for finite real x. We can choose the constants c_0 and c_1 so that

$$y_0\sim-\frac{2}{3\pi}e^{3i\mu}x^{-3}+\begin{cases}1, & \text{for } r<r_c\\ 0, & \text{for } r>r_c.\end{cases} \qquad (9)$$

Hence $\omega\sim\omega_c+c_3(-A)^{2/3}$. Only the branch with positive growth rate contributes to the long-time behavior of typical perturbations, since the continuous spectrum leads to algebraically decaying transients. The purely real frequency shift for $A<0$ would place the branch points on the real r axis, violating the assumptions of the asymptotic analysis; no smooth eigenfunctions were found numerically for $A<0$.

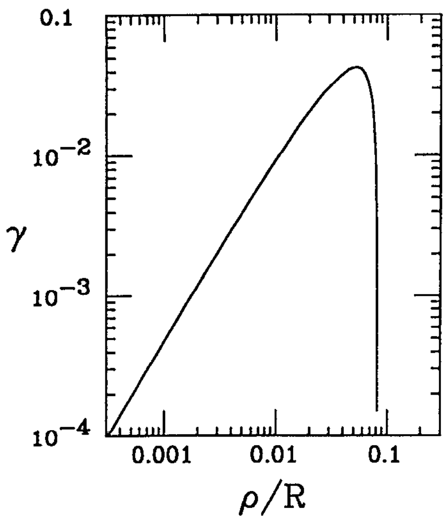

FIG. 3. Dependence of growth rate for $l=1$ diocotron instability on gyroradius for the equilibrium shown in Fig. 1.

Numerical solution of Eq. (4) yields growth rates as shown in Fig. 2; the eigenfunctions are slightly rounded approximations to the step function.[5] The asymptotic 2/3 power law only applies for very small A. To achieve γ/ω_c of a few percent (as observed in the experiments), it is only necessary to postulate relative increments in rotation frequency of about 10^{-2} over the 2-D value, which is reasonable for the experimental geometry.

As the relative strength of the confinement field increases, the growth rate turns over and eventually vanishes. Without the delicate cancellations which led to algebraic growth in Ref. 3, the profiles should be stable to $l=1$ modes for moderately large A. Similarly, moderate values of A can annihilate the growth rates of diocotron instabilities with larger l. This behavior may account for the experimental observation that hollow electron columns confined by sufficiently short cylindrical electrodes are stable to diocotron instabilities, even when positive growth rates are predicted by the Rayleigh equation.[6]

Next let us consider finite gyroradius ρ. For simplicity we neglect all variations along z in this section. A standard finite-gyroradius expansion will suffice; the terms $\rho^2\nabla^2$ which determine the eigenmode structure are everywhere small. The linearized continuity equation, neglecting terms of order $(\omega_p/\Omega)^2$, is

$$\left((\omega - l\omega_E) + \rho^2 \frac{4\pi ec}{B}\frac{l}{r}\frac{dN_0}{dr}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{l}{r^2}\right)\phi + \frac{4\pi ec}{B}\frac{l}{r}\frac{dN_0}{dr}\phi = 0\,. \quad (10)$$

Asymptotic analysis of Eq. (10) is similar to the previous model (but more cumbersome) and leads to $\omega - \omega_c \propto (i\rho)^{4/3}$. Numerical evaluation of the eigenvalue ω yields the dependence of the growth rate on ρ/R shown in Fig. 3.[5] For values characteristic of the experiments in Ref. 2 ($\rho/R \approx 10^{-4}$) the finite-gyroradius corrections alone yield a negligible growth rate for $l=1$ and have a negligible effect on higher l modes; the situation might differ for an ion plasma or weaker magnetic fields.

ACKNOWLEDGMENTS

The author gratefully acknowledges useful conversations with T. M. O'Neil and M. N. Rosenbluth.

This research was supported by National Science Foundation Grant No. PHY87-06538 and Office of Naval Research Contract No. N00014-82-K-0621.

[1] R. C. Davidson, *Theory of Nonneutral Plasmas* (Benjamin, Reading, MA, 1974).
[2] C. F. Driscoll, Phys. Rev. Lett. **64**, 645 (1990).
[3] R. A. Smith and M. N. Rosenbluth, Phys. Rev. Lett. **64**, 649 (1990).
[4] My objective here is to find a consistent asymptotic description when there is exponential growth; I leave aside the delicate question of perturbing the full initial-value problem.
[5] A relaxation method based on a fixed radial mesh furnishes estimates for ω. Accurate eigenvalues were then obtained using a shooting algorithm, which requires a good initial guess, especially for small growth rates.
[6] Short, hollow, stable electron columns were used in experiments described by C. F. Driscoll, J. H. Malmberg, and K. S. Fine, Phys. Rev. Lett. **60**, 1290 (1988).