Physics of Plasmas

Rotational pumping revisited

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Thomas M. O'Neil[a)]

AFFILIATIONS

Department of Physics, University of California at San Diego, La Jolla, California 92093-0319, USA

[a)]Author to whom correspondence should be addressed: toneil@ucsd.edu

ABSTRACT

This paper considers a pure electron plasma, with a small admixture of negative ions, confined in a Penning–Malmberg trap. When a diocotron mode is excited on the plasma, the end sheaths of the plasma are azimuthally distorted. During reflection at a distorted end sheath, an ion steps off of the surface characterizing the drift motion in the plasma interior, and this step produces transport. The diocotron mode transfers canonical angular momentum to the ions, and in response damps. These transport mechanism and associated damping are called rotational pumping. It is particularly strong when the axial bounce motion and the rotational drift motion, in the rotating frame of the mode, satisfy a resonance condition. This paper calculates the transport flux of ions and the associated damping rate of the mode in the resonant regime. Previous papers have discussed the theory and the experimental observation of rotational pumping for the special case of a diocotron mode with azimuthal wave number $l = 1$, and this paper extends the theory to modes with $l \neq 1$, which may sound like a trivial extension, but in fact is not.

I. INTRODUCTION

Non-neutral plasmas are often confined in a Penning–Malmberg trap.[1,2] The confinement region is bounded by a conducting cylinder, which is divided axially into three sections. For a plasma consisting of negatively charged species, the central section is held at ground potential, and the two end sections are held at a negative potential. The axial confinement is then provided by electrostatic fields. The radial confinement is provided by a uniform axial magnetic field. Because the plasma is non-neutral, there is a radial space charge field, and the plasma undergoes $E \times B$ drift rotation. Here, the cyclotron radii for all of the charges are assumed to be small enough that the cross-magnetic field motion can be described by $E \times B$ dynamics.

A typical equilibrium state is cylindrically symmetrical, with an electric potential, that is, nearly constant along each field line in the interior of the plasma. Axial electric fields are Debye screened out of the plasma interior. Within a few Debye lengths of the plasma end, the magnitude of the potential rises steeply to provide axial confinement for the charged particles.[3,4] Since the Debye length is small compared to the length of the plasma, the plasma length along each field line is relatively well defined. Nevertheless, the cyclotron radii are small compared to the Debye length.

For the equilibrium, the potential in the plasma core and the potential in the end sheath both are independent of the azimuthal angle, where (r, θ, z) is a cylindrical coordinate system with the z-axis coincident with the axis of the trap. Consequently, there is no azimuthal electric field and no radial drift in either region. The drifts in the end sheath are consistent with the drift surfaces in the core; that is, the drifts in the end sheath do not move particles off the drift surfaces in the core.

However, the situation is different when a diocotron mode[5–8] is excited on the plasma. These modes are low-frequency flute-like modes where the cross-magnetic field motion can be described by $E \times B$ drift dynamics. Even when a diocotron mode is excited, the plasma potential remains nearly constant along each field line to within a few Debye lengths of the end.[4] The 2D drift surfaces in the core are the equipotential contours in the rotating frame of the mode, where the potential is time-independent. When the mode is excited, the plasma potential is not cylindrically symmetric, but the potential from the end cylinders is cylindrically symmetric. The proximity of the end cylinders to the end sheaths and the incomplete shielding in the sheath produce a difference in the azimuthal symmetries of the core potential and the end sheath potential. Thus, the drifts during reflection in the end sheaths can lead to steps off the 2D drift surfaces characterizing the cross-field motion in the core. These steps produce particle transport and an associated mode damping; the mechanism is called rotational pumping.

For the special case of an $l = 1$ diocotron mode, previous papers have described the theory[9] and reported the experimental observation[10] of rotational pumping. This paper extends the theory to diocotron modes of arbitrary l number, which may sound like a trivial extension, but, that is, not the case.

The $l = 1$ mode is special in that it is excited simply by displacing the plasma off the trap axis. The displaced plasma retains its cylindrical symmetry, except at the ends, and that symmetry allows the analysis to be carried out using cylindrical coordinates with a displaced axis. In contrast, when a mode with $l > 1$ is excited, the plasma axis remains coincident with the axis of the trap, but the plasma suffers a distortion, and the simplification associated with cylindrical symmetry is no longer available.

As we will see the analysis then is conveniently carried out using the action angle variables (ψ, I) for the drift motion in the core. The drift surfaces in the core are the surfaces of constant I, and the derivative of the end sheath potential with respect to ψ, while holding I constant, yields precisely the drift step off the core surface. This result follows trivially from Hamilton's equation $\dot{I} = -\frac{\partial \overline{H}}{\partial \psi} = -\frac{\partial e\varphi_{end}}{\partial \psi}$, where $\overline{H}$ is the Hamiltonian in the rotating frame of the mode, $\varphi_{end}(z, \psi, I)$ is the end sheath potential, and $e = -|e|$ is the particle charge. Of course, in the interior of the plasma, the potential is only a function of I, so there $\dot{I}$ is zero.

Rotational pumping is particularly strong when the axial bounce motion and the rotational drift motion, in the rotating frame of the mode, satisfy a resonance condition, and we will focus on this case. A particle then takes many steps of the same sign, and the transport and damping are substantially enhanced. This work is similar to previous work on transport in tandem mirrors in the resonant plateau regime.[11,12] The substantial enhancement is an important experimental signature of bounce-resonant rotational pumping. We will see that the predicted flux scales with magnetic field as $1/B^6$, as the magnetic field is decreased and the rotation frequency increases toward the bounce frequency.

Physically, one can think of the transport as arising from a drag torque exerted by the mode on the plasma. We consider the case of an $l = 2$ diocotron mode, that is, excited on a plasma with a top-hat radial density profile. The mode rotates in the same sense as the plasma, but more slowly.[6] The drag torque then opposes the plasma rotation and produces an outward radial drift of the particles; that is, the transport of particles is radially outward. To conserve canonical angular momentum, the mode damps in response.

Next, we suppose that the density profile has a long skirt that extends radially outward beyond the radius where the rotation rate of the mode matches that of the particles. Particles beyond this resonant radius rotate more slowly than the mode, so the drag torque is in the same direction as the rotation and produces a radial drift inward. The response back on the wave from this torque would make the wave grow, but is typically outweighed by the damping produced by the larger number of particles rotating more rapidly than the wave. In both cases, the transport moves the particles toward the resonant radius where the wave rotation rate matches the plasma rotation rate. Very near the resonant radius, the drag force and consequent transport are small.

Again, the case of an $l = 1$ diocotron mode is special in that the resonant radius is located at the wall, so rotational pumping transports all particles radially outward.

The analysis is motivated by current experiments with a non-neutral plasma consisting primarily of electrons but with a small admixture of H^- ions.[13] The characteristic axial bounce frequency for the electrons is very large compared to the drift rotation frequency, so the rotational pumping associated with the electrons is relatively weak. However, the characteristic bounce frequency for the ions is comparable to the drift rotation frequency, so one expects strong rotational pumping to be associated with the ions. In the experiments, the radial distribution of electrons is measured by dumping the electrons out along field lines to a phosphor screen beyond one end of the plasma. The screen is relatively insensitive to the ions, and direct evidence for ion position is not obtained. However, there is an inference that when a diocotron mode is excited, the ions in the plasma are transported outward. Simultaneously, the mode is observed to damp. The purpose of this paper is to find expressions for the transport flux and damping rate.

Other plasmas of recent interest are similar to the electron-H^- plasma and may exhibit similar phenomena. These other plasmas are electron–antiproton plasmas[14] and positron–Beryllium ion plasmas.[15]

The analysis is based on an ordering of length scales and an ordering of frequency scales. The characteristic cyclotron radius for the ions is small compared to the Debye length (i.e., $r_{ci} \ll \lambda_D$), so the cross-magnetic field motion of the ions can be described by $E \times B$ drift dynamics even during reflection from the ends. The characteristic axial bounce frequency for the ions and the $E \times B$ drift rotation frequency ω_r are assumed to be comparable with each other and large compared to an effective collision frequency for the ions (i.e., $\nu \ll |\omega_r|$). As will be discussed later, the effective collision frequency ν is the frequency of small-angle scatterings that are adequate to spoil the resonance between the bounce and rotation motions. Also, the transport time scale τ is assumed to be long compared to effective collision time (i.e., $\nu\tau \gg 1$).

II. ACTION ANGLE VARIABLES FOR THE DRIFT MOTION

As a preliminary to the analysis of a finite length plasma, we establish action angle variables for the 2D $E \times B$ drift motion in a long plasma column on which a diocotron mode of azimuthal wave number l has been excited. We imagine that the end confinement cylinders of the trap are removed to $= \pm\infty$. In the laboratory frame, the 2D $E \times B$ drift motion of a particle guiding center is governed by the drift Hamiltonian[16]

$$H_d = e\varphi_0(p_\theta) + eA\varphi_l(p_\theta)\cos(l\theta - \omega_l t), \tag{1}$$

where (r, θ, z) is a cylindrical coordinate system with the z-axis coincident with the axis of the unperturbed column, $\left(\theta,\, p_\theta = -\frac{eBr^2}{2c}\right)$ are canonically conjugate coordinate and momentum, $\varphi_0(p_\theta)$ is the space charge potential of the equilibrium, A is the dimensionless mode amplitude, and $\varphi_l(p_\theta)\cos(l\theta - \omega_l t)$ is the mode eigenfunction. Here, the magnetic field is taken to be $-B\hat{z}$, so that p_θ is positive for negatively charged species. This choice makes the drift rotation frequency, $\frac{\partial e\varphi_0}{\partial p_\theta}$, negative.

A canonical transformation to a frame that rotates with the mode is given by the generating function

$$F_2(\theta, \overline{p}_\theta) = \left(\theta - \frac{\omega_l}{l}t\right)\overline{p}_\theta, \tag{2}$$

where bars indicate variables in the rotating frame. Here, we use the notation of Goldstein.[17] Following the standard procedure, the new variables are related to the old variables through the relations:

$$\bar{\theta} = \frac{\partial F_2}{\partial \bar{p}_\theta} = \left(\theta - \frac{\omega_l}{l} t\right), \quad p_\theta = \frac{\partial F_2}{\partial \theta} = \bar{p}_\theta, \tag{3}$$

and the new Hamiltonian is related to the old Hamiltonian through the relation

$$\overline{H}_d = H_d + \frac{\partial F_2}{\partial t} = H_d - \frac{\omega_l}{l} \bar{p}_\theta. \tag{4}$$

Thus, the drift Hamiltonian in the rotating frame of the wave is given by the expression

$$\overline{H}_d = e\varphi_0(p_\theta) - \frac{\omega_l}{l} p_\theta + eA\varphi_l(p_\theta)\cos(l\bar{\theta}), \tag{5}$$

where we continue to use the old momentum since the old and new momenta are equal to one another.

Treating A as a constant, the Hamilton–Jacobi equation can be solved, at least formally, to introduce action angle variables.[17] As a first step, one solves Eq. (5) to find $p_\theta = p_\theta(\overline{H}_d, \bar{\theta}, A)$ and then defines the action through the integral

$$I = \frac{1}{2\pi} \int_0^{2\pi} p_\theta(\overline{H}_d, \bar{\theta}, A) d\bar{\theta}, \tag{6}$$

where A and $\overline{H}_d$ are held constant in the integration. In principle, Eq. (6) can be inverted to find $\overline{H}_d(I, A)$. The Hamilton–Jacobi equation for the generating function to action angle variables is then written in the form

$$\frac{\partial F_2'}{\partial \bar{\theta}} = p_\theta\left[\overline{H}_d(I, A), \bar{\theta}, A\right]. \tag{7}$$

One solves this equation for $F_2'(\bar{\theta}, I, A)$ and defines the angle variable

$$\psi = \frac{\partial F_2'(\bar{\theta}, I, A)}{\partial I}. \tag{8}$$

Ultimately, we will need explicit expressions for the transformation, but only to first order in mode amplitude. For a given value of $\overline{H}_d$, we define the momentum p_I through the relation

$$\overline{H}_d = e\varphi_0(p_I) - \frac{\omega_l}{l} p_I. \tag{9}$$

Taylor expanding the right-hand side of Eq. (5) with respect to p_θ about p_I and retaining only first-order terms in $|p_\theta - p_I| \sim A$ yields the relation

$$0 = \left(\frac{\partial e\varphi_0(p_I)}{\partial p_I} - \frac{\omega_l}{l}\right)(p_\theta - p_I) + eA\varphi_l(p_I)\cos(l\bar{\theta}), \tag{10}$$

where use has been made of Eq. (9) to eliminate the zero order terms. Solving for p_θ and substituting into Eq. (6) then yield the result $I = p_I$. Then, solving Eq. (10) for $I = p_I$ yields the relation

$$I = p_\theta + \frac{eA\varphi_l(I)}{\omega_r(I)} \cos(l\bar{\theta}), \tag{11}$$

where $\omega_r(I) = (\frac{\partial e\varphi_0(I)}{\partial I} - \frac{\omega_l}{l})$ is the plasma rotation frequency in the rotating frame of the wave. Likewise, using Eqs. (7) and (8) yields the angle variable

$$\psi = \bar{\theta} - \frac{\partial}{\partial I}\left[\frac{eA\varphi_l(I)}{l\omega_r(I)}\right]\sin(l\bar{\theta}). \tag{12}$$

This linear perturbation theory assumes that the frequency $\omega_r(I)$ is not too small. Near $\omega_r(I) = 0$, the constant I surfaces are nonlinear cat's eye structures involving particle trapping in the wave field, and the present theory is valid only outside of this region. Of course, the I value where $0 = \omega_r(I) = (\frac{\partial e\varphi_0(I)}{\partial I} - \frac{\omega_l}{l})$ is simply the resonant value of I where the rotation rate of the mode matches that of the plasma particles according to linear theory. As mentioned earlier, the transport is small near the resonant value of I.

In Sec. V of this paper, we will show that rotational pumping produces a slow damping of the mode. Under this damping, the mode amplitude does change in time, but the change is slow compared to the period of the angle variable. Although the generating function $F_2'(\bar{\theta}, I, A)$ contains an explicit time dependence, it still generates a valid canonical transformation. However, the new Hamiltonian is not simply $\overline{H}_d(I, A)$, but rather $\overline{H}_d(I, A) + \frac{\partial A}{\partial t}\frac{\partial F_2'}{\partial A}$. Because the second term introduces a dependence on the angle variable ψ, the action I is no longer an exact constant of the motion. However, because the rate of change of $A(t)$ is small compared to the frequency of the angle variable and because I is an action, I is a good adiabatic invariant.[17] The change in I during one cycle of the angle variable is second-order small in the change in A. The ψ dependence in the new term does produce small first-order corrections to the frequency for the angle variable, but away from the resonant layer these corrections are negligible compared to the frequency $\frac{\partial \overline{H}_d}{\partial I} \approx \omega_r$. Consequently, we neglect the second term and continue to use the Hamiltonian $\overline{H}_d(I, A)$, in spite of the slow change in A.

III. HAMILTONIAN FOR ION MOTION IN A FINITE LENGTH TRAP

We consider a plasma consisting primarily of electrons, but also containing a small admixture of H^- ions, confined in a Penning–Malmberg trap. We assume that a diocotron mode of azimuthal mode number l has been excited on the plasma.

Even when the diocotron mode is excited, the electric potential along a given field line is nearly constant within the plasma.[4] Over a few Debye lengths at each end of the plasma, the magnitude of the potential rises dramatically, providing axial confinement. Since the Debye length is small compared to the plasma dimensions, we can define a plasma length along each field line, $L(\bar{\theta}, p_\theta)$. Taking $z = 0$ to be the axial mid-plane of the plasma, we write the rapidly rising potential at the $+z$ end as $e\varphi_{end}\left[z - \frac{L(\bar{\theta}, p_\theta)}{2}\right]$. Changing the minus sign in the end potential expression to a plus sign yields a similar end potential for the other end. However, for future convenience we replace the full-length plasma by a half-length plasma, imposing specular reflection from a plane at $z = 0$, using the potential $e\varphi_{end}(-z)$. From symmetry of the full-length plasma about the $z = 0$ plane, transport for the half-length plasma is the same as that for the full-length plasma. Both end potentials can be thought of as step functions that are high enough that all particles are reflected.

The ion Hamiltonian in the rotating frame of the mode then is given by the expression[14]

$$\overline{H}=\frac{p_z^2}{2m}+\mu B+e\varphi_0(p_\theta)-\frac{\omega_l}{l}p_\theta+eA\varphi_l(p_\theta)\cos(l\overline{\theta})$$
$$+e\varphi_{end}\left[z-\frac{L(\overline{\theta},\,p_\theta)}{2}\right]+e\varphi_{end}(-z). \tag{13}$$

The last term, which effects the specular reflection from the imaginary surface at $z=0$, causes no transport, but must be included to effect the back and forth bounce motion.

Next, we replace the variables $(\overline{\theta},p_\theta)$ with the variables (ψ,I). The set of variables (z,p_z,ψ,I) is an acceptable set of canonical variables, since the canonical transformation obtained above for $(\overline{\theta},p_\theta)$ to (ψ,I) does not depend on (z,p_z) or μ. Rewriting the Hamiltonian in terms of the new variables yields the expression

$$\overline{H}=\frac{p_z^2}{2m}+\mu B+\overline{H}_d(I,A)+e\varphi_{end}\left[z-\frac{L(\psi,I)}{2}\right]+e\varphi_{end}(-z), \tag{14}$$

where use has been made of Eq. (5).

We write the plasma length as $L(\psi,I)=L_0(I)+L_1(\psi,I)$, where $L_1(\psi,I)$ is first order in the mode amplitude. There is a subtle point here. While $L_1(\psi,I)$ is first-order small in the mode amplitude, it is not the only first-order term in the mode amplitude; the definition of I depends on mode amplitude, so $L_0(I)$ contains some dependence on mode amplitude. The important point is that the only length dependence on the variable ψ comes from a term, that is, first-order small. This smallness is used to justify the Taylor expansion

$$e\varphi_{end}\left[z-\frac{L(\psi,I)}{2}\right]\approx e\varphi_{end}\left[z-\frac{L_0(I)}{2}\right]$$
$$-\frac{1}{2}\frac{\partial}{\partial z}\left\{e\varphi_{end}\left[z-\frac{L_0(I)}{2}\right]\right\}L_1(\psi,I). \tag{15}$$

Hamiltonian (13) can then be written in the form

$$\overline{H}=\overline{H}_0(z,p_z,I)+\delta\overline{H}(z,\psi,I), \tag{16}$$

where

$$\overline{H}_0(z,p_z,I)=\frac{p_z^2}{2m}+\mu B+\overline{H}_d(I,A)+e\varphi_{end}\left[z-\frac{L_0(I)}{2}\right]$$
$$+e\varphi_{end}(-z), \tag{17}$$

and

$$\delta\overline{H}(z,\psi,I)=-\frac{1}{2}\frac{\partial}{\partial z}\left\{e\varphi_{end}\left[z-\frac{L_0(I)}{2}\right]\right\}L_1(\psi,I). \tag{18}$$

Let us consider an ion that undergoes a single bounce off the end at $=\frac{L_0(I)}{2}$. First, we consider the bounce according to the Hamiltonian $\overline{H}_0(z,p_z,I)$. Neglecting any variation in mode amplitude during a bounce, this Hamiltonian is independent of t and ψ, so $\overline{H}_0$ and I are constant during the bounce. Also, $e\varphi_{end}\left[z-\frac{L_0(I)}{2}\right]$ is zero both before and after the bounce, so the axial momentum simply changes from p_z^0 to $-p_z^0$ during the bounce.

The rate of change of ψ is given by Hamilton's equation

$$\dot{\psi}=\frac{\partial\overline{H}_0}{\partial I}=\omega_r(I)-\frac{\partial e\varphi_{end}}{\partial z}\left(\frac{1}{2}\right)\frac{\partial L_0(I)}{\partial I}, \tag{19}$$

and integrating the second term over the bounce implies the change due to the bounce

$$\Delta\psi=\left(\frac{1}{2}\right)\frac{\partial L_0(I)}{\partial I}\int dt\left(-\frac{\partial e\varphi_{end}}{\partial z}\right)=\left(\frac{1}{2}\right)\frac{\partial L_0(I)}{\partial I}2p_z^0.$$

Although this step in ψ is zero order in the mode amplitude, it is still small compared to unity, as can be seen from the estimate

$$\Delta\psi=\frac{\partial L_0(I)}{\partial I}p_z^0\simeq\frac{cp_z^0}{eBr}\frac{\partial L_0(r)}{\partial r}\simeq\frac{r_{ci}}{r_p}\frac{\Delta z}{r_p}\ll 1, \tag{20}$$

where we have used a parabolic approximation for the rounded end of the column, $L_0'(r)=L_0-\Delta z\left(1-\frac{r^2}{r_p^2}\right)$. Here, r_p is the characteristic radius of the plasma column and r_{ci} is the characteristic ion cyclotron radius.

In first order, the action undergoes a step given by the expression

$$\Delta I=-\int dt\frac{\partial\delta\overline{H}}{\partial\psi}=\frac{1}{2}\frac{\partial L_1}{\partial\psi}\int dt\frac{\partial e\varphi_{end}}{\partial z}=\frac{1}{2}\frac{\partial L_1}{\partial\psi}2p_z^0, \tag{21}$$

where use has been made of Eq. (18) and of the fact that ψ changes by only a small amount during the reflection. There also is a first-order step in $|p_z|$ during the reflection. Since the total Hamiltonian $\overline{H}=\overline{H}_0(z,p_z,I)+\delta\overline{H}(z,\psi,I)$ is conserved during the reflection, and since $\delta\overline{H}(z,\psi,I)$ is zero both before and after the reflection, we may write the relation

$$0=\Delta\overline{H}_0=\frac{p_z\Delta p_z}{m}+\frac{\partial\overline{H}_d(I)}{\partial I}\Delta I=\frac{p_z\Delta p_z}{m}+\omega_r(I)\Delta I. \tag{22}$$

We will calculate the transport due to the steps ΔI and will not need Δp_z, but it is worth noting that conservation of energy requires Δp_z to be none zero.

To calculate the transport flux to second order in mode amplitude, $\frac{\partial L_1(\psi,I)}{\partial\psi}$ will be needed only to first order in mode amplitude. However, as noted earlier, there is some subtlety in the meaning of $L_1(\psi,I)$. The most natural way to express the length of the plasma along a given field line is the expression $L=L_0'(p_\theta)+L_1'(\overline{\theta},p_\theta)$, where $L_0'(p_\theta)$ is the length of the cylindrically symmetric equilibrium before the mode is excited and $L_1'(\theta,p_\theta)$ is the first-order correction produced by the mode. The primes have been added to emphasize the distinction between this expression for the length and the expression $L=L_0(I)+L_1(\psi,I)$. The chain rule for partial derivatives implies the relation

$$\frac{\partial L_1((\psi,I)}{\partial\psi}=\frac{\partial L((\psi,I)}{\partial\psi}$$
$$=\frac{\partial L_1'(\overline{\theta},p_\theta)}{\partial\overline{\theta}}\left(\frac{\partial\overline{\theta}}{\partial\psi_I}\right)+\frac{\partial}{\partial p_\theta}\left[L_0'(p_\theta)+L_1'(\overline{\theta},p_\theta)\right]\left(\frac{\partial p_\theta}{\partial\psi_I}\right), \tag{23}$$

and by working only to first order in mode amplitude and using Eqs. (11) and (12), the relation reduces to the form

$$\frac{\partial L_1((\psi,I)}{\partial\psi}=\frac{\partial L_1'(\overline{\theta},p_\theta)}{\partial\overline{\theta}}+\frac{\partial L_0'(p_\theta)}{\partial p_\theta}l\frac{eA\varphi_1(I)}{\omega_r(I)}\sin(l\overline{\theta}). \tag{24}$$

To obtain a crude estimate of $L_1(\psi,I)$, we consider the motion of a flux tube in the rotating frame of the wave. As the flux tube moves on a surface of constant I, its value of p_θ oscillates about I as described by Eq. (11). In this motion, the flux tube is subject to radially dependent axial forces which try to make the length of the flux tube conform to the equilibrium length at the appropriate radius, $L_0'(p_\theta)$. Thus, as an estimate we use the expression

$$L_1 \simeq \frac{\partial L_0'}{\partial p_\theta}\bigg]_I (p_\theta - I) = -\frac{\partial L_0'}{\partial p_\theta}\bigg]_I \frac{eA\varphi_l(I)}{\omega_r(I)}\cos(l\overline{\theta}). \tag{25}$$

From Eq. (23), one can see that this approximation is equivalent to neglecting the term $\frac{\partial L_1'(\overline{\theta},p_\theta)}{\partial\overline{\theta}}$ in Eq. (24). In the cosine, the θ may be approximated by ψ since the coefficient of the cosine is already first-order small in the mode amplitude. For the simple case of a rounded end with a length that depends parabolically on r [i.e., $L_0'(r) = L_0 - \Delta z(1-\frac{r^2}{r_p^2})$], the derivative $\frac{\partial L_0'}{\partial p_\theta} = -\frac{2c\Delta z}{eBr_p^2}$ is a constant. The flux and mode damping will be expressed in terms of the ratio $\frac{|L_{1l}|}{L_0} \simeq |\frac{\Delta z}{L_0}\frac{A\varphi_l(r)}{r_p\frac{\partial\varphi_0}{\partial r}}|$. Here, L_{1l} is the Fourier component of $L_1(\theta)$ and $-\frac{\partial\varphi_0}{\partial r}$ is the radial electric field in the rotating frame of the mode. We note that the ratio $\frac{|L_{1l}|}{L_0}$ is independent of the magnetic field.

A more accurate solution can be obtained by using methods developed in Ref. 4. However, the solution then is for specific trap geometry, applied end voltages and number of particles along each field line.

IV. KINETIC THEORY FOR THE TRANSPORT FLUX

The guiding center distribution for the ions can be written in the form $f_i(z, p_z, \overline{\theta}, p_\theta, \mu, t)$ and evolves according to the collisional drift kinetic equation

$$\frac{\partial f_i}{\partial t} + [f_i, \overline{H}] = C[f_i], \tag{26}$$

where $[f_i, \overline{H}]$ is a Poisson bracket and $C[f_i]$ is a collision operator. To discuss transport, we define the density in I-space

$$N_i(I,t) = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dp_z \int_0^{2\pi} d\psi \int_0^{\infty} d\mu f_i(z, p_z, \psi, I, \mu, t), \tag{27}$$

where the total number of ions in the plasma is given by the integral $N_i = \int_0^\infty dI N_i(I,t)$. Integrating over Eq. (26) yields the transport equation

$$\frac{\partial N_i(I,t)}{\partial t} = \frac{\partial}{\partial I}\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dp_z \int_0^{2\pi} d\psi \int_0^{\infty} d\mu \left(f_i \frac{\partial \overline{H}}{\partial \psi}\right), \tag{28}$$

where the Poisson bracket in Eq. (26) was written in the form

$$[f_i, \overline{H}] = \frac{\partial}{\partial z}\left(f_i\frac{\partial \overline{H}}{\partial p_z}\right) - \frac{\partial}{\partial p_z}\left(f_i\frac{\partial \overline{H}}{\partial z}\right) + \frac{\partial}{\partial \psi}\left(f_i\frac{\partial \overline{H}}{\partial I}\right) - \frac{\partial}{\partial I}\left(f_i\frac{\partial \overline{H}}{\partial \psi}\right), \tag{29}$$

and the integrals over z, p_z and ψ in the definition of $N_i(I,t)$ kill the first three terms. Likewise, the integrals over p_z and μ kill the collision operator term. Of course, the collision operator conserves particle number. The integral in Eq. (28) is the negative of the flux in I-space.

First, let us imagine that $\delta\overline{H}$ were zero and look for an equilibrium solution to Eq. (26) with $\overline{H}$ replaced by $\overline{H}_0$. Since $\overline{H}_0$ is independent of ψ, the Poison bracket $[\overline{H}_0, I]$ vanishes, and consequently, the Poisson bracket $[f_{i,0}(\overline{H}_0, I), \overline{H}_0]$ vanishes. For the Maxwellian form

$$f_{i,0} = \frac{N(I)\exp\left[-\frac{1}{T(I)}\left(\overline{H}_0 - \overline{H}_d(I)\right)\right]}{\sqrt{2\pi m T(I)}\left(\frac{T(I)}{B}\right)[2\pi L_0(I)/2]}, \tag{30}$$

the collision operator also vanishes, so $f_{i,0}$ is the desired equilibrium solution. The normalization factors in this distribution follow from Eq. (27).

For the actual situation where $\delta\overline{H}$ is not zero, we write the distribution in the form

$$f_i = f_{i,0}(\overline{H}_0, I) + \delta f_i(z, p_z, \psi, I, \mu, t), \tag{31}$$

where ambiguity in the definition of δf is removed by the requirement that $\int_0^{2\pi} d\psi \delta f_i = 0$.

In transport equation (28), the lack of any ψ dependence in $\overline{H}_0$ and $f_{i,0}$ implies that the equation reduces to the form

$$\frac{\partial N_i(I,t)}{\partial t} = \frac{\partial}{\partial I}\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dp_z \int_0^{2\pi} d\psi \int_0^{\infty} d\mu \left(\delta f_i \frac{\partial \delta\overline{H}}{\partial \psi}\right). \tag{32}$$

Thus, we need to know δf_i only to first order in $\delta\overline{H}$ to obtain the transport flux to second order in $\delta\overline{H}$.

To first order in $\delta\overline{H}$, the perturbation δf_i satisfies the equation

$$\frac{d^{(0)}(\delta f_i)}{dt} - C(\delta f_i) = \frac{\partial \delta f_i}{\partial t} + [\delta f_i, \overline{H}_0] - C(\delta f_i) = [\delta\overline{H}, f_{i,0}], \tag{33}$$

where $\frac{d^{(0)}(\delta f_i)}{dt}$ is the total derivative taken along the orbits specified by the Hamiltonian $\overline{H}_0$. Recalling that $f_{i,0}$ is independent of ψ and that $\delta\overline{H}$ is independent of p_z, the Poisson bracket on the right-hand side can be written in the form

$$[\delta\overline{H}, f_{i,0}] = \frac{\partial\delta\overline{H}}{\partial\psi}\frac{\partial f_{i,0}}{\partial I} + \frac{\partial\delta\overline{H}}{\partial z}\frac{\partial f_{i,0}}{\partial p_z}. \tag{34}$$

By using the relations $\frac{\partial f_{i,0}}{\partial p_z} = -\frac{p_z}{mT}f_{i,0}$ and $\frac{\partial f_{i,0}}{\partial I} = \left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0} - \frac{1}{T}\frac{\partial \overline{H}_0}{\partial I}f_{i,0}$, the right-hand side of Eq. (34) can be rewritten in the form

$$[\delta\overline{H}, f_{i,0}] = \frac{\partial\delta\overline{H}}{\partial\psi}\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0} - \frac{d^{(0)}}{dt}\left(\frac{\delta\overline{H} f_{i,0}}{T}\right), \tag{35}$$

where use has been made of the relations

$$\frac{f_{i,0}}{T}\left(\frac{p_z}{m}\frac{\partial\delta\overline{H}}{\partial z} + \frac{\partial\overline{H}_0}{\partial I}\frac{\partial\delta\overline{H}}{\partial\psi}\right) = \frac{f_{i,0}}{T}\frac{d^{(0)}(\delta\overline{H})}{dt} = \frac{d^{(0)}}{dt}\left(\frac{\delta\overline{H} f_{i,0}}{T}\right). \tag{36}$$

By using the observation that $C\left(\frac{\delta\overline{H} f_{i,0}}{T}\right) = 0$, Eq. (33) can be rewritten in the form

$$\left(\frac{d^{(0)}}{dt} - C\right)\left(\delta f_i + \frac{\delta\overline{H} f_{i,0}}{T}\right) = \frac{\partial\delta\overline{H}}{\partial\psi}\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0}. \tag{37}$$

Finally, we set $\delta g_i = \delta f_i + \frac{\delta\overline{H} f_{i,0}}{T}$ and rewrite Eq. (37) in the form

$$\left(\frac{d^{(0)}}{dt}+\nu\right)\delta g_i=\frac{\partial\delta\overline{H}}{\partial\psi}\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0},\tag{38}$$

where ν is an effective collision frequency that will be discussed later.

For a given phase point (q_j,p_j), the total time derivative $\frac{d^{(0)}}{dt}$ is taken along orbits $q_j(t')$ and $p_j(t')$ that are defined by the equations of motion

$$\frac{dq_j}{dt'}=\frac{\partial\overline{H}_0}{\partial p_j},\quad \frac{dp_j}{dt'}=-\frac{\partial\overline{H}_0}{\partial q_j},\tag{39}$$

subject to the final condition $q_j(t)=q_j$ and $p_j(t)=p_j$. Here, (q_j,p_j) are simply stand-ins for the canonical variables (z,p_z,ψ,I). We recall that μ is constant under the evolution generated by $\overline{H}_0$. The solution to Eq. (38) is given by the integral expression

$$\delta g_i=\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0}\int_0^t dt' e^{\nu(t'-t)}\frac{\partial\delta\overline{H}'}{\partial\psi},\tag{40}$$

where use has been made of the fact that $\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0}$ does not evolve with t'. We recall that $\overline{H}_0$ and I are constant under this evolution.

To further the evaluation of integral (40), we rewrite Eq. (18) in the form

$$\delta\overline{H}(z',\psi',I)=-\frac{1}{2}\frac{\partial}{\partial z'}\left\{e\varphi_{end}\left[z'-\frac{L_0(I)}{2}\right]\right\}\sum_l L_{1l}(I)e^{il\psi}e^{il\omega_r(I)(t'-t)},\tag{41}$$

where $\psi(t')$ has been written as $\psi(t)-\omega_r(I)(t-t')$, neglecting the small increment in ψ during a bounce (see Eq. (20)). To first order in mode amplitude, the sum extends only over $\pm|l|$, where l is the azimuthal mode number of the diocotron mode.

From Eq. (32) and the fact that $\frac{\partial\delta\overline{H}}{\partial\psi}$ is non-zero only during a particle bounce, δf_i, and hence δg_i, are needed only during a bounce. Let t be a time during the bounce, t_0 the time just before the bounce, and the interval from $t=0$ to $t=t_0$ a long time containing many previous bounces. Equation (40) can then be written in the form

$$\delta g_i=-\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0}\sum_l ilL_{1,l}(\mathrm{I})e^{il\psi}\left\{\frac{1}{2}\int_{t_0}^t dt'\frac{\partial}{\partial z'}\left(e\varphi_{end}\left[z'-\frac{L_0(I)}{2}\right]\right)+\frac{1}{2}\int_0^{t_0}dt'\frac{\partial}{\partial z'}\left(e\varphi_{end}\left[z'-\frac{L_0(I)}{2}\right]\right)e^{(\nu+il\omega_r(I))(t'-t)}\right\},\tag{42}$$

where the time-dependent exponential has been replaced by unity in the first integral, since the duration of a bounce is small. The first integral has the value—$[p_z(t)-p_z(t_0)]$. Neglecting the duration of each bounce in the second integral yields the result

$$\delta_i=-\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0}\sum_l ilL_{1,l}(\mathrm{I})e^{il\psi}\left\{-\frac{1}{2}[p_z(t)-p_z(t_0)]+|p_z(t_0)|\int_0^{t_0}\mathrm{d}t'\sum_{j=1}^M\delta\left(t-t'-\frac{L_0(I)j}{|v_z(t_0)|}\right)e^{(\nu+il\omega_r(I))(t'-t)}\right\},\tag{43}$$

where the sum over j is a sum over the previous M bounces.

Carrying out the time integral over the sum of delta functions yields the expression

$$\delta g_i=-\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0}\sum_l ilL_{1,l}(\mathrm{I})e^{il\psi}\left\{-\frac{1}{2}[p_z(t)-p_z(t_0)]+|p_z(t_0)|\sum_{j=1}^M e^{-(\nu+il\omega_r(I))\frac{L_0(I)j}{|v_z(t_0)|}}\right\}.\tag{44}$$

The sum over j is a geometric progression and can be evaluated

$$\sum_{j=1}^M e^{-(\nu+il\omega_r(I))\frac{L_0(I)j}{|v_z(t_0)|}}=\frac{1-e^{-(\nu+il\omega_r(I))\frac{L_0(I)M}{|v_z(t_0)|}}}{e^{+(\nu+il\omega_r(I))\frac{L_0(I)}{|v_z(t_0)|}}-1}.\tag{45}$$

In the numerator, the second term may be neglected since $\nu\frac{L_0(I)M}{|v_z(t_0)|}$ is presumed to be large. The time scale $\tau=\frac{L_0(I)M}{|v_z(t_0)|}$ can be as large as the transport time scale, and this time is assumed to be large compared to the effective collision time scale (i.e., $\nu\tau\gg1$).

By using the mathematical identity

$$\frac{1}{e^{iz}-1}=-\frac{1}{2}-i\sum_{n=-\infty}^{n=\infty}\frac{1}{z-2\pi n},\tag{46}$$

and making the identification $z=\frac{(-i\nu+l\omega_r(I))L_0(I)}{|v_z(t_0)|}$, Eq. (44) can be written in the form

$$\delta g_i=-\left[\frac{\partial f_0}{\partial I}\right]_{\overline{H}_0}\sum_l ilL_{1,l}(\mathrm{I})e^{il\psi}\left\{-\frac{1}{2}[p_z(t)-p_z(t_0)]-|p_z(t_0)|\left(\frac{1}{2}+i\sum_n\frac{|v_z(t_0)|}{L_0(I)}\frac{1}{-i\nu+l\omega_r(I)-2n\frac{\pi|v_z(t_0)|}{L_0(I)}}\right)\right\},\tag{47}$$

where n runs over positive and negative integers.

Substituting $\delta f_i=\delta g_i-\frac{\delta\overline{H}f_{i,0}}{T}$ into Eq. (32) yields the result

$$\frac{\partial N_i(I,t)}{\partial t}=\frac{\partial}{\partial I}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dp_z\int_0^{2\pi}d\psi\int_0^{\infty}d\mu\left(\delta g_i\frac{\partial\delta\overline{H}}{\partial\psi}-\frac{f_{i,0}}{2T}\frac{\partial(\delta\overline{H})^2}{\partial\psi}\right).\tag{48}$$

Here, the second term in the bracket yields zero under the ψ integral, since $f_{i,0}$ is independent of ψ. Dropping the second term and substituting from Eqs. (41) and (47) for $\frac{\partial\delta\overline{H}}{\partial\psi}$ and δg_i into the first term yields the equation

$$\begin{aligned}\frac{\partial N_i(I,t)}{\partial t}=&-\frac{\partial}{\partial I}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dp_z\int_0^{2\pi}d\psi\int_0^{\infty}d\mu\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0}\\&\times\sum_l ilL_{1,l}(\mathrm{I})e^{il\psi}\sum_{l'}il'L_{1,l'}(\mathrm{I})e^{il'\psi}\left\{-\frac{1}{2}[p_z(t)-p_z(t_0)]-|p_z(t_0)|\left(\frac{1}{2}+i\frac{|v_z(t_0)|}{L_0(I)}\sum_n\frac{1}{-i\nu+l'\omega_r(I)-2n\frac{\pi|v_z(t_0)|}{L_0(I)}}\right)\right\}\\&\times\left\{-\frac{1}{2}\frac{\partial}{\partial z}\left\{e\varphi_{end}\left[z-\frac{L_0(I)}{2}\right]\right\}\right\}.\end{aligned}\tag{49}$$

Recalling that the integrand is non-zero only in the end region where the particles are exposed to the plasma sheath potential, we change the z-integral into an integral over the unperturbed bounce motion. Since the duration of the bounce is short, I and ψ can be treated as constant. Let $z = z\left(z^0,\ p_z^0, t - t_0\right)$ and $p_z = p_z\left(z^0,\ p_z^0, t - t_0\right)$ be bounce orbits generated by Hamiltonian $\overline{H}_0$, subject to the initial conditions $z(t_0) = z^0$ and $p_z(t_0) = p_z^0$ and holding ψ and I constant. The Hamiltonian evolution preserves area in phase space, so we may write $dzdp_z = dz^0 d\,p_z^0$. By varying t_0 through a sequence of increments, while holding $t - t_0$ constant, one can sweep z through the whole bounce. Setting $dz^0 = v_z^0 dt_0$ and using $dt = dt_0$ then yields the relation $p_z = d\,p_z^0 v_z^0 dt$. Since the incremental steps all occur before the bounce begins, v_z^0 has the same value for all of the steps and $p_z(t_0) = p_z^0$ has the same value for all of the steps. Also, since $\overline{H}_0$ is constant during this evolution, we may evaluate $\overline{H}_0$ at time t_0, setting $\overline{H}_0 = m(v_z^0)^2/2 + \mu B + \overline{H}_d(I, A)$ in the expression for f_0. We recall that the end potential is negligible at time t_0.

Using the relations $dzdp_z = d\,p_z^0 v_z^0 dt$ and

$$\frac{d}{dt}(p_z) = \frac{d}{dt}[p_z - p_z(t_0)] = -\frac{\partial}{\partial z}\left\{e\varphi_{end}\left[z - \frac{L_0(I)}{2}\right]\right\}, \tag{50}$$

yields the equation

$$\begin{aligned}\frac{\partial N_i(I,t)}{\partial t} &= -\frac{\partial}{\partial I}\int_0^\infty dp_z^0 v_z^0 \int_0^{2\pi} d\psi \int_0^\infty d\mu \left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0} \\ &\quad \times \sum_l ilL_{1,l}(\mathrm{I})e^{il\psi} \sum_{l'} il' L_{1,l'}(\mathrm{I})e^{il'\psi} \\ &\quad \times \frac{1}{2}\left\{-\int_{t_0}^{t_f} dt \frac{d}{dt}\frac{1}{4}\left[p_z(t) - p_z(t_0)\right]^2 - \int_{t_0}^{t_f} dt \frac{dp_z}{dt}|p_z(t_0)| \right. \\ &\quad \left. \times \left(\frac{1}{2} + i\frac{|v_z(t_0)|}{L_0(I)}\sum_n \frac{1}{-i\nu + l'\omega_r(I) - 2n\dfrac{\pi|v_z(t_0)|}{L_0(I)}}\right)\right\},\end{aligned} \tag{51}$$

where use has been made of the fact that p_z^0 is positive for all of the reflections.

Recalling that $p_z(t_f) = -p_z(t_0)$ simplifies the equation to the form

$$\begin{aligned}\frac{\partial N_i(I,t)}{\partial t} &= -\frac{\partial}{\partial I}\int_0^\infty dp_z^0 v_z^0 \int_0^{2\pi} d\psi \int_0^\infty d\mu \left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0} \\ &\quad \times \sum_l ilL_{1,l}(\mathrm{I})e^{il\psi} \sum_{l'} il' L_{1,l'}(\mathrm{I})e^{il'\psi} \\ &\quad \times \left\{|p_z(t_0)|^2\left(+i\frac{|v_z(t_0)|}{L_0(I)}\sum_n \frac{1}{-i\nu + l'\omega_r(I) - 2n\dfrac{\pi|v_z(t_0)|}{L_0(I)}}\right)\right\}.\end{aligned} \tag{52}$$

Carrying out the integrals over ψ and using the fact that n and l run over positive and negative integers yields the equation

$$\begin{aligned}\frac{\partial N_i(I,t)}{\partial t} &= \frac{\partial}{\partial I}\sum_{n,l}\int_0^{+\infty} d\,v_z^0\, v_z^0 4\int_0^\infty d\mu m^3 L_0 2\pi\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0} \\ &\quad \times \frac{|lL_{1,l}(\mathrm{I})|^2}{L_0(I)^2}\frac{\upsilon}{\upsilon^2 + \left[l\omega_r(I) - 2n\dfrac{\pi|v_z(t_0)|}{L_0(I)}\right]^2}.\end{aligned} \tag{53}$$

This is a good point to examine what is meant by the effective collision frequency. First note that the left-hand side of Eq. (38) is a linear operator and that the right-hand side is a sum of terms given by sum (41) for $\delta\overline{H}$. Thus, the solution to Eq. (38) is given by a sum of solutions for the individual terms on the right hand side. Consequently, the effective collision frequency can have a different value for each term in the sum. When the Fokker–Planck collision operator acts on a velocity resonance function, it yields an enhanced effective collision frequency for velocities in the range of the resonance. This enhancement is produced by the velocity diffusion term in the Fokker–Planck operator, which we approximate by $\nu_\parallel \overline{v}^2 \frac{\partial}{\partial v_z^2}$, where $\nu_\parallel$ is a collision frequency characterizing velocity diffusion of the ions and $\overline{v}^2$ is the square of the ion thermal velocity. The velocity width of the resonance function is $\Delta v_z^0 \sim \frac{\nu L_0}{2\pi n}$, so the effective collision frequency at the resonance is $\upsilon \sim \nu_\parallel \overline{v}^2/(\Delta v_z^0)^2$. Here, we have assumed that $\Delta v_z^0 \ll \overline{v}$. Eliminating the resonance width yields the effective collision frequency $\nu \sim \sqrt[3]{\nu_\parallel\left(\frac{\overline{v}2\pi n}{L_0}\right)^2} = \sqrt[3]{\nu_\parallel \overline{\omega}_b^2}(2n)^{2/3}$, where $\omega_b = \frac{\pi|v_z(t_0)|}{L_0(I)}$ is the bounce frequency and $\overline{\omega}_b = \frac{\overline{v}\pi}{L_0(i)}$ is the bounce frequency evaluated at the thermal velocity. The condition that the resonance width be small compared to the thermal velocity can also be stated as the condition $\nu \gg \nu_\parallel$ or equivalently the condition ${\nu_\parallel}^{2/3} \ll (2n\overline{\omega}_b)^{2/3}$. This is also the condition that the resonance function can be replaced by a delta function

$$\frac{\upsilon}{\upsilon^2 + \left[l\omega_r(I) - 2n\dfrac{\pi|v_z(t_0)|}{L_0(I)}\right]^2} \simeq \pi\delta\left[l\omega_r(I) - 2n\frac{\pi|v_z(t_0)|}{L_0(I)}\right], \tag{54}$$

which is insensitive to the exact value of ν.

Here, we are implicitly omitting the term for $n = 0$, which as we will see later yields the non-resonant rotational pumping. For resonant rotational pumping, the sum on n run over positive and negative integer values. The sum can be limited to only positive values of n by inserting a factor of 2. There is then no longer a sum over l since the original sum included only one positive and one negative value, $l = \pm|l|$, and the delta function requires the sign of l to match that of $\omega_r(I)$. Note that the bounce frequency $\omega_b = \pi\frac{|v_z^0|}{L_0(I)}$ is positive. Equation (53) then takes the form

$$\begin{aligned}\frac{\partial N_i(I,t)}{\partial t} &= \frac{\partial}{\partial I}\sum_{n\geq 1} L_0 4\pi^2 m^3\left(\frac{L_0}{\pi}\right)^5\int_0^\infty d\omega_b {\omega_b}^4 \\ &\quad \times \int_0^\infty d\mu\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0}\frac{|lL_{1,l}(\mathrm{I})|^2}{L_0(I)^2}\delta[l\omega_r(I) - 2n\omega_b].\end{aligned} \tag{55}$$

From Eq. (30), one obtains the derivative

$$\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0} = \left\{ \frac{1}{T}\frac{N_i(I)}{L_0(I)}\frac{\partial \overline{H}_d}{\partial I} + \frac{\partial}{\partial I}\left(\frac{N_i(I)}{L_0(I)}\right) + \frac{1}{T(I)^2}\frac{\partial T}{\partial \mathrm{I}}\frac{N_i(I)}{L_0(I)}\left(\frac{m(v_z^0)^2}{2}+\mu B\right) - \frac{3}{2T}\frac{\partial T}{\partial \mathrm{I}}\frac{N_i(I)}{L_0(I)}\right\} \frac{\exp\left[-\frac{1}{T}\left(\frac{m(v_z^0)^2}{2}+\mu B\right)\right]}{\sqrt{2\pi T m}\left({}^T/_B\right)\pi}, \quad (56)$$

and the integral

$$\int_0^\infty d\mu \left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0} = \left\{ \frac{1}{T}\frac{N_i(I)}{L_0(I)}\frac{\partial \overline{H}_d}{\partial I} + \frac{\partial}{\partial I}\left(\frac{N_i(I)}{L_0(I)}\right) + \frac{1}{T(I)^2}\frac{\partial T}{\partial \mathrm{I}}\frac{N_i(I)}{L_0(I)}\left(\frac{m(v_z^0)^2}{2}+T\right) - \frac{3}{2T}\frac{\partial T}{\partial \mathrm{I}}\frac{N_i(I)}{L_0(I)}\right\} \frac{\exp\left[-\frac{1}{T}\left(\frac{m(v_z^0)^2}{2}\right)\right]}{\sqrt{2\pi T m}\pi}. \quad (57)$$

Finally, substituting into Eq. (55) and carrying out the ω_b integral yields the result

$$\frac{\partial N_i(I,t)}{\partial t} = \frac{\partial}{\partial I}4\pi T L_0 \sum_{n\geq 1}\frac{|lL_{1,l}(\mathrm{I})|^2}{L_0^2}\left\{\frac{N_i(I)}{L_0(I)}\frac{\partial \overline{H}_d}{\partial I} + \frac{\partial}{\partial I}\left(\frac{N_i(I)T}{L_0(I)}\right) + \frac{\partial T}{\partial \mathrm{I}}\frac{N_i(I)}{L_0(I)}\left(\frac{1}{2}\left(\frac{l\omega_r}{2n\varpi_b}\right)^2 - \frac{3}{2}\right)\right\}\frac{1}{|2\,n|}\left(\frac{l\omega_r}{2n\varpi_b}\right)^4 \times \frac{\exp\left[-\frac{1}{2}\left(\frac{l\omega_r}{2n\varpi_b}\right)^2\right]}{\sqrt{2\pi\varpi_b^2}}. \quad (58)$$

For completeness, we have retained all of the terms in the squiggly bracket, but the first term is typically larger than the other terms by a factor of order $\frac{\overline{H}_d}{T} \sim \frac{r_p^2}{\lambda_D^2} \gg 1$, where r_p is the plasma radius. Retaining only this term in the squiggly bracket and using the relation $\frac{\partial \overline{H}_d}{\partial I} = \omega_r$ yields the simplified expression

$$\frac{\partial N_i(I,t)}{\partial t} = \frac{\partial}{\partial I}4\pi T N_i(I)\frac{|L_{1,l}(\mathrm{I})|^2}{L_0^2}\sum_{n\geq 1}\frac{l(l\omega_r)^5}{(2n\varpi_b)^5}\frac{\exp\left[-\frac{1}{2}\left(\frac{l\omega_r}{2n\varpi_b}\right)^2\right]}{\sqrt{2\pi}}. \quad (59)$$

We recall that the quantity beyond the derivative $\frac{\partial}{\partial I}$ is the negative of the flux Γ_I.

After all this mathematics, a simple physical model may aid understanding. Let us focus on a case where the $\mathrm{n}=1$ resonance is dominant. From Eq. (21), the change in action during a single bounce is $\frac{\partial L_1}{\partial \psi}p_z$. Because of the resonance, the particle makes $\frac{\omega_b}{\nu}$ bounces of the same sign before the resonance is destroyed by collisions, yielding an overall change $\frac{\partial L_1}{\partial \psi}p_z\frac{\omega_b}{\nu}$. This implies a diffusion coefficient $D = \nu\left\langle\left(\frac{\partial L_1}{\partial \psi}p_z\frac{\omega_b}{\nu}\right)^2\right\rangle_\psi$, and a flux

$$\Gamma_I = \int dz\,dp_z d\psi d\mu\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0} = \frac{L_0}{2}2\pi\Delta p_z D\int_0^\infty d\mu\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0}, \quad (60)$$

where $\Delta p_z = m2\pi\nu/L_0$ is the width of the resonance. Substituting for D and Δp_z, using the approximation $\left[\frac{\partial f_{i,0}}{\partial I}\right]_{\overline{H}_0} \simeq \frac{\omega_r}{T}f_{i,0}$ and doing a small amount of algebra yields the approximate flux

$$\Gamma_I \simeq \frac{2\pi^2}{\sqrt{2\pi}}TN_i(I)l\frac{|L_{1,l}(\mathrm{I})|^2}{L_0^2}\left|\frac{l\omega_r}{2\omega_b}\right|^5\exp\left[-\frac{1}{2}\left(\frac{l\omega_r}{2\omega_b}\right)^2\right], \quad (61)$$

which differs only by a numerical factor from the $n=1$ term in Eq. (59).

Next, we obtain the radial flux, noting first that the total number of particles can be written in the two forms

$$N_i = \int_0^\infty dIN_i(I,t) = L\int_0^{R_W} 2\pi r_I dr_I n_i(r_I,t), \quad (62)$$

where $n_i(r_I,t)$ is the density. By using the relation $I = p_I = -\frac{eBr_I^2}{2c}$, we obtain the relation

$$n_i(r_I,t) = \left(\frac{-eB}{c2\pi L}\right)N_i(I,t). \quad (63)$$

Also using the relation $\frac{\partial}{\partial I} = \frac{\partial}{\partial P_I} = -\frac{c}{eBr_I}\frac{\partial}{\partial r_I}$ allows Eq. (59) to be rewritten in the form

$$\frac{\partial n_i(I,t)}{\partial t} + \frac{1}{r_I}\frac{\partial}{\partial r_I}r_I\left\{\sqrt{\frac{\pi}{2}}\frac{1}{8}\frac{Tn_i(r_I)cl}{eBr_I}\frac{|L_{1,l}(r_I)|^2}{L_0^2} \times \sum_{n\geq 1}\frac{(l\omega_r)^5}{(n\varpi_b)^5}\exp\left[-\frac{1}{8}\left(\frac{l\omega_r}{n\varpi_b}\right)^2\right]\right\} = 0. \quad (64)$$

The quantity in the squiggly brackets is second order in mode amplitude, and r_I reduces to r in zero order in mode amplitude. Thus, to second order in the mode amplitude, the quantity in squiggly brackets is the resonant contribution to the rotational pumping radial flux for the ions

$$\Gamma^i_{\geq 1}(r) = \sqrt{\frac{\pi}{2}}\frac{1}{8}\frac{Tn_i(r)cl}{eBr}\frac{|L_{1,l}(\mathrm{r})|^2}{L_0^2}\sum_{n\geq 1}\frac{(l\omega_r)^5}{(n\varpi_b)^5}\exp\left[-\frac{1}{8}\left(\frac{l\omega_r}{n\varpi_b}\right)^2\right]. \quad (65)$$

Substituting the crude approximation $\frac{|L_{1l}|}{L_0} \simeq \left|\frac{\Delta z}{L_0}\frac{A\varphi_l(r)}{r_p\frac{\partial\varphi_0}{\partial r}}\right|$ from the discussion following Eq. (25), introducing the ion thermal velocity $\overline{v}_i = \sqrt{\frac{T}{m_i}}$ and the ion cyclotron radius $r_{ci} = \frac{\overline{v}_i}{|\Omega_{ci}|}$ yields the approximate flux

$$\Gamma^i_{n\geq 1}(r) \simeq \sqrt{\frac{\pi}{2}}\frac{1}{8}\frac{l\overline{v}_i r_{ci}n_i(r)}{r}\frac{|e|}{e}\left|\frac{\Delta z}{L_0}\frac{A\varphi_l(r)}{r_p\frac{\partial\varphi_0}{\partial r}}\right|^2 \times \sum_{n\geq 1}\frac{(l\omega_r)^5}{(n\varpi_b)^5}\exp\left[-\frac{1}{8}\left(\frac{l\omega_r}{n\varpi_b}\right)^2\right]. \quad (66)$$

The sum over n can be written as the function

$$sum(x) = \sum_{n\geq 1} \left(\frac{x}{n}\right)^5 \exp\left(-\frac{1}{8}\left(\frac{x}{n}\right)^2\right), \tag{67}$$

where $= \frac{l\omega_r}{\omega_b}$, and the *sum(x)* is plotted in Fig. 1 for x values between 0 and 10. This function is odd in x, so values for negative x are given by the negative of the values plotted. The steep rise and then plateau are largely due to the $n = 1$ term, and the subsequent rise is largely due to the $n = 2$ and then $n = 3$ terms. The sum is dominated by first few terms for x values less than about 10. The continued rise in $sum(x)$ for $x > 10$ is given approximately by $sum(x) \approx 32\times$, as can be seen by converting the sum over n to an integral over n. However, a word of caution is needed here since the very high n terms in the sum are artifacts of approximating the reflection from the plasma ends as an impulse, that is, of using the temporal delta functions in Eq. (43). In reality, the reflection is not impulsive, but is smoothed out over a spatial scale of the Debye length, so n and x values should be limited to less than about $L/(2\pi\lambda_D)$.

Since the flux in Eq. (63) is proportional to $\frac{\omega_r^5}{B}$ and since ω_r is proportional to $1/B$, the flux scales as $1/B^6$, as the magnetic field is reduced and the rotation frequency increases toward the bounce frequency.

In contrast, the flux due to the non-resonant, that is, adiabatic, rotational pumping, is independent of magnetic field.[9] This flux is given by the $n = 0$ term in Eq. (53). Evaluating the velocity integral for the $n = 0$ term under the ordering $\nu \ll \omega_r$, using the relation $I = p_I = -\frac{eBr_I^2}{2c}$ and Eq. (53) yields the adiabatic ion flux

$$\Gamma^i_{n=0}(r) = 6\frac{\nu}{l\omega_r}\frac{Tn_i(r)cl}{eBr}\frac{\left|L_{1,l}(\mathrm{r})\right|^2}{L_0^2}$$
$$\simeq 6\frac{\nu}{l\omega_r}\frac{l\bar{v}_i r_{ci} n_i(r)}{r}\frac{|e|}{e}\left|\frac{\Delta z}{L_0}\frac{A\varphi_l(r)}{r_p\dfrac{\partial\varphi_0}{\partial r}}\right|^2. \tag{68}$$

Here, ν is not the effective collision frequency discussed in the paragraph preceding Eq. (54) since the $n = 0$ term does not contain a resonant denominator. The analysis in Ref. 9, carried out for a single species plasma, finds that $\nu = \frac{4}{3}\nu_{\|\perp}$, where $\nu_{\|\perp}$ is the collisional equipartition rate between $T_\|$ and $T_\perp$ [i.e., $\frac{1}{2}\frac{dT_\|}{dt} = \nu_{\|\perp}(T_\perp - T_\|)$].

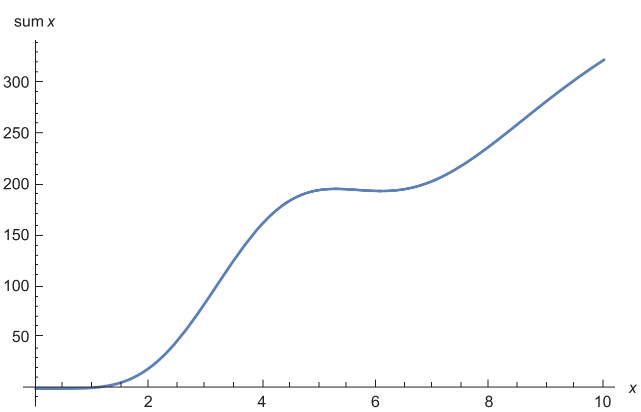

FIG. 1. Plot of function sum(x) vs x.

Physically, the adiabatic transport arises from the periodic axial compression and expansion that each flux tube experiences as the plasma rotates through the asymmetric end surface. During the compression phase, work is done adiabatically on the flux tube and $T_\|$ increases, but then some of the increased parallel kinetic energy scatters into perpendicular kinetic energy, and the energy given back during the expansion phase is slightly less than that gained during the compression phase. The energy difference appears as heat, and Eq. (12) of Ref. 9 provides a simple calculation of the flux by equating the heating rate to the rate of loss of electrostatic energy due to the radial plasma expansion. Here, we are considering a plasma with electrons and ions, so the scattering of parallel ion energy is due to both electrons and ions, and we should set $\nu = \frac{4}{3}\left(\nu^{ii}_{\|\perp} + \nu^{ie}_{\|\perp}\right) \simeq \frac{4}{3}\nu^{ie}_{\|\perp}$. Note that when the ions are in the adiabatic regime, the electrons also are in the adiabatic regime, and the parallel temperatures of the ions and electrons rise and fall together. Consequently, the important scattering is into the perpendicular temperatures of each species. Also, $\nu^{ie}_{\|\perp}$ is larger than $\nu^{ii}_{\|\perp}$ because the electron density is larger than the ion density. Finally, we note that the adiabatic flux in Eq. (68) is independent of the magnetic field because $B\omega_r$ is independent of the magnetic field.

Returning to the question of scaling of the ion flux as the magnetic field is decreased and the rotation frequency increases toward the bounce frequency, the ion flux initially remains constant at the value given by Eq. (68). When the magnetic field has decreased to the point where $64\sqrt{\frac{2}{\pi}}\left|\frac{\nu^{ie}_{\|\perp}}{l\omega_r}\right| \leq |sum(x)| \simeq |x^5|$, the resonant transport begins to dominate with the initial rise in the flux varying as $1/B^6$. As x approaches unity, the rise in the flux is not as steep since the rise in *sum(x)* is less steep. For larger values of x, the resonant transport can be orders of magnitude larger than the initial adiabatic transport.

Note that even when the ions are well into the regime where the resonant transport dominates, the electrons typically are still in the adiabatic regime with a flux given by the expression

$$\Gamma^e_{n=0}(r) = 8\frac{\left(\nu^{ee}_{\|\perp} + \nu^{ei}_{\|\perp}\right)}{l\omega_r}\frac{Tn_e(r)cl}{eBr}\frac{\left|L_{1,l}(\mathrm{r})\right|^2}{L_0^2}$$
$$\simeq 8\frac{\left(\nu^{ee}_{\|\perp} + \nu^{ei}_{\|\perp}\right)}{l\omega_r}\frac{l\bar{v}_i r_{ci}(r) n_e(r)}{r}\frac{|e|}{e}\left|\frac{\Delta z}{L_0}\frac{A\varphi_l(r)}{r_p\dfrac{\partial\varphi_0}{\partial r}}\right|^2. \tag{69}$$

For this flux to be less than the ion flux, it is necessary that $64\sqrt{\frac{2}{\pi}}\left|\frac{\left(\nu^{ee}_{\|\perp}+\nu^{ei}_{\|\perp}\right)}{l\omega_r}\right|\frac{n_e}{n_i} \leq |sum(x)|$, which is more restrictive than the inequality in the previous paragraph because typically $\nu^{ee}_{\|\perp} > \nu^{ie}_{\|\perp}$ and $n_e > n_i$. Of course, at sufficiently low temperature and strong magnetic field the electrons enter the regime of strong magnetization, where $\nu^{ee}_{\|\perp}$ becomes exponentially small.[18]

On physical grounds, we argued earlier that the radial flux is outward where ω_r is negative and inward where ω_r is positive. To see this result in Eqs. (68) and (69), we note that the quantity e is negative and that ω_r is negative for r less than the resonant radius and positive for r greater than the resonant radius.

By making an end distortion that rotates in the same sense as the plasma, but faster, the rotational pumping can transport all ions radially inward. This is an example of the "rotating wall" technique often used to radially compress or expand non-neutral plasmas.[19,20] For example, such a rotating distortion can be produced by using an end electrode with azimuthally separated sectors and applying properly timed voltages to the sectors. In this case, the particle flux is still given by the above equations, but there is a complication. There is no mode in the plasma so the second term in Eq. (24) for L_1 vanishes, and our crude approximation for L_1 vanishes. In this case, one must solve for the perturbed end surface of the plasma. Recent experiments with electron–antiproton plasmas have observed the transport of the antiprotons due to a rotating wall drive in the bounce-rotation resonance regime.[14]

V. MODE DAMPING RATE

For a typical case where the radial particle flux is dominantly outward, that is, increases the particle canonical angular momentum, the mode angular momentum must decrease, that is, the mode must damp. The purpose of this section is to calculate the damping rate.

The total canonical angular momentum in the plasma is given by the expression

$$P_\theta = \int_0^\infty dI \frac{1}{2\pi}\int_0^{2\pi} d\psi p_\theta(\psi, I, t)[N_e(I,t) + N_i(I,t)]. \tag{70}$$

Since the $E \times B$ drift motion is the same for the electrons and the ions, the variables (ψ, I), and equivalently (θ, p_θ), apply to both the ion and the electron density distributions. The function $p_\theta(\psi, I, t)$ is given to first order in mode amplitude by Eqs. (11) and (12). The time dependence in this function is included because we allow the mode amplitude *A(t)* to be a function of time.

Since the angular momentum in the mode turns out to be second order in the mode amplitude, we will need the function $p_\theta(\psi, I, t)$ to second order in mode amplitude. The perturbation treatment leading to Eqs. (11) and (12) can easily be extended to second order, but here we use a trick to avoid that work. We write the second-order solution in the form

$$p_\theta = I - \frac{eA\varphi_l(I)}{\omega_r(I)}\cos(l\overline{\theta}) + \mathrm{O}[A^2]g(\overline{\theta}), \tag{71}$$

where the first two terms on the right-hand side come from Eq. (11) and the third term is an unknown term of second order in A. This relation and the definition of I yield the relation

$$I = \frac{1}{2\pi}\int_0^{2\pi} d\overline{\theta} p_\theta(\overline{\theta}, I, t) = \frac{1}{2\pi}\int_0^{2\pi} I d\overline{\theta} - \frac{eA\varphi_l(I)}{\omega_r(I)}\frac{1}{2\pi}\int_0^{2\pi}\cos(l\overline{\theta})d\overline{\theta} + \mathrm{O}[A^2]\frac{1}{2\pi}\int_0^{2\pi} g(\overline{\theta})d\overline{\theta}. \tag{72}$$

The first term on the right-hand side is simply I and the second term is zero, so the third term must be zero. According to Eq. (12), $\overline{\theta}$ and ψ are equal to zero order in mode amplitude. Since the coefficient in the third term of Eq. (71) is already second order in mode amplitude, $\overline{\theta}$ can be replaced by ψ in the function $g(\theta)$. Thus, we conclude from Eq. (72) that the integral $\frac{1}{2\pi}\int_0^{2\pi} g(\psi)d\psi$ has the value zero. Thus, the unknown third term makes zero contribution to the integral in Eq. (70). In contrast, in the second term on the right-hand side of Eq. (72), $\overline{\theta}$ cannot be replaced by ψ since the coefficient is only first order in mode amplitude.

Equation (70) then reduces to the form

$$P_\theta = \int_0^\infty dII[N_e(I,t) + N_i(I,t)] - \int_0^\infty dI \frac{eA\varphi_l(I,t)}{\omega_r(I)} \times [N_e(I,t) + N_i(I,t)]\frac{1}{2\pi}\int_0^{2\pi} d\overline{\theta}\frac{d\psi}{d\overline{\theta}}\bigg]_{I,t}\cos(l\overline{\theta}), \tag{73}$$

where to first order in mode amplitude Eq. (12) yields the relation

$$\frac{d\psi}{d\overline{\theta}}\bigg]_{I,t} = 1 - \frac{\partial}{\partial I}\left[\frac{eA\varphi_l(I)}{\omega_r(I)}\right]\cos(l\overline{\theta}). \tag{74}$$

With the aid of this relation, Eq. (73) reduces to the form

$$P_\theta = \int_0^\infty dII[N_e(I,t) + N_i(I,t)] - \frac{1}{4}\int_0^\infty dI\left(\frac{eA\varphi_l(I)}{\omega_r(I)}\right)^2\frac{\partial}{\partial I}[N_e(I,t) + N_i(I,t)], \tag{75}$$

where the factor of $\frac{1}{4}$ comes from the product of two factors of $\frac{1}{2}$, the first from the relation $\frac{1}{2\pi}\int_0^{2\pi} d\overline{\theta}\cos^2(l\overline{\theta}) = \frac{1}{2}$, and the second from the relation $\frac{eA\varphi_l(I,t)}{\omega_r(I)}\frac{\partial}{\partial I}\left[\frac{eA\varphi_l(I)}{\omega_r(I)}\right] = \frac{1}{2}\frac{\partial}{\partial I}\left[\frac{eA\varphi_l(I)}{\omega_r(I)}\right]^2$.

To understand the physical meaning of the two terms on the right-hand side of this equation, we first imagine that there is no transport, but that an external agency causes the diocotron mode to slowly grow in amplitude. For example, the diocotron mode is a negative energy mode and is subject to the resistive wall instability,[21] so a small resistivity could cause a slow growth of the mode. If the growth rate is small compared to the rotation frequency of the plasma, as observed in the rotating frame of the mode, a particle on a drift surface I remains on this drift surface as the mode amplitude slowly increases; the action I is a good adiabatic invariant for the particle. Thus, the density distributions $N_e(I)$ and $N_i(I)$ are independent of time during the mode growth. Of course, the function $p_\theta(\overline{\theta}, I, t)$ does change in time as the mode grows.

Thus, the first term on the right-hand side of Eq. (75) is time-independent as the mode grows in amplitude. Initially, when the mode has near zero amplitude, the action I is the same as the canonical momentum p_θ, so the first term on the right is simply the total canonical angular momentum in the absence of the mode. The second term on the right is the angular momentum added to the plasma as a result of the mode, that is, the mode angular momentum. Since $\frac{\partial}{\partial I}[N_e(I,t) + N_i(I,t)]$ is negative, the mode angular momentum is positive; that is, the mean square radius of the plasma increases when the mode is excited.

The transport does introduce time dependence in the density distributions $N_i(I,t)$ and $N_e(I,t)$. During reflection ions and electrons step off the drift surface, breaking the adiabatic invariant I for the particle and introducing time dependence in $N_i(I,t)$ and $N_e(I,t)$.

Including the time dependence in $A(t)\varphi_l(I)$, $N_i(I,t)$, and $N_e(I,t)$, while demanding that the total canonical angular momentum of the plasma is constant yields the relation

$$
\begin{aligned}
0 &= \frac{dP_\theta}{dt} \\
&= \int_0^\infty dII\left[\frac{\partial N_i(I,t)}{\partial t} + \frac{\partial N_e(I,t)}{\partial t}\right] \\
&\quad - \frac{dA(t)^2}{dt}\frac{1}{4}\int_0^\infty dI\left(\frac{e\varphi_l(I)}{\omega_r(I)}\right)^2\frac{\partial}{\partial I}[N_e(I,t)+N_i(I,t)],
\end{aligned}
\tag{76}
$$

where the time derivatives $\frac{\partial N_i(I,t)}{\partial t}$ and $\frac{\partial N_e(I,t)}{\partial t}$ were retained in the first term of Eq. (75) but not the second. Since the time derivatives are second order in mode amplitude and the second term already contains a factor, that is, second order in mode amplitude, the resulting term would be fourth order and has been dropped.

The case of interest is when the ion flux is due to resonant rotational pumping and the electron flux is due to adiabatic rotational pumping. The continuity equations in I-space, $\frac{\partial N_i(I,t)}{\partial t} + \frac{\partial}{\partial I}\Gamma^i_{I,n\geq 1}(I) = 0$ and $\frac{\partial N_e(I,t)}{\partial t} + \frac{\partial}{\partial I}\Gamma^e_{I,n=0}(I) = 0$, define the flux functions in I-space. Substituting for the time derivative terms in the first term of Eq. (76) and integrating by parts with respect to I yields the equation

$$
\begin{aligned}
&\int_0^\infty dI\left[\Gamma^i_{I,n\geq 1}(I) + \Gamma^e_{I,n=0}(I)\right] \\
&\quad = \frac{dA(t)^2}{dt}\frac{1}{4}\int_0^\infty dI\left(\frac{e\varphi_l(I)}{\omega_r(I)}\right)^2\frac{\partial}{\partial I}[N_e(I,t)+N_i(I,t)].
\end{aligned}
\tag{77}
$$

The fluxes in I-space are related to the fluxes in r-space through the equations $\Gamma_I(I) = \Gamma_r(r)(2\pi r L)$ for both electrons and ions. By using this relation, the relation $I = p_I = -\frac{eBr_I^2}{2c}$ and Eq. (63), Eq. (77) can be rewritten in the form

$$
\frac{dA(t)^2}{dt} = \frac{4\int_0^\infty r^2 dr[\Gamma^i_{r,n\geq 1}(r) + \Gamma^e_{r,n=0}(r)]}{\int_0^\infty r^2 dr\left(\frac{c\varphi_l(r)}{Br\omega_r(r)}\right)^2\frac{\partial}{\partial r}[n_e(r)+n_i(r)]}.
\tag{78}
$$

Substituting for $\Gamma^i_{r,n\geq 1}(r)$ from Eq. (65) and for $\Gamma^e_{r,n=0}(r)$ from Eq. (68) then yields the result

$$
\frac{dA(t)^2}{dt} = \frac{\sqrt{\frac{\pi}{2}}\frac{1}{2}\frac{Tcl}{eB}\int_0^\infty rdrn_i(r)\frac{|L_{1,l}(\mathrm{r})|^2}{L_0^2}\left[sum(x_i) + 64\sqrt{\frac{2}{\pi}}\frac{n_e(r)}{n_i(r)}\frac{\left(\nu^{ee}_{\parallel\perp}+\nu^{ei}_{\parallel\perp}\right)}{(l\omega_r)}\right]}{\int_0^\infty r^2 dr\left(\frac{c\varphi_l(r)}{Br\omega_r(r)}\right)^2\frac{\partial}{\partial r}[n_e(r)+n_i(r)]},
\tag{79}
$$

where the first term in the square bracket of the numerator is due to the resonant ion flux and the second to the adiabatic electron flux. These two quantities are negative over most of the density distribution since ω_r is negative there, and e is negative, so the numerator is positive. Since $\frac{\partial}{\partial r}[n_e(r)+n_i(r)]$ is negative, the mode damps. By substituting our crude approximation $\frac{|L_{1l}|}{L_0} \simeq \left|\frac{\Delta z}{L_0}\frac{A\varphi_l(r)}{r_p\frac{\partial\overline{\varphi}_0}{\partial r}}\right|$, we obtain the damping rate

$$
\gamma = \frac{1}{2A^2}\frac{dA(t)^2}{dt} = \frac{\frac{1}{4}\sqrt{\frac{\pi}{2}}\frac{\overline{v}_i r_{ci}}{r_p^2}\frac{|e|}{e}\left(\frac{\Delta z}{L_0}\right)^2 l\int_0^\infty rdrn_i(r)\left(\frac{\varphi_l(r)}{\frac{\partial\overline{\varphi}_0}{\partial r}}\right)^2\left[sum(x_i) + 64\sqrt{\frac{2}{\pi}}\frac{n_e(r)}{n_i(r)}\frac{\left(\nu^{ee}_{\parallel\perp}+\nu^{ei}_{\parallel\perp}\right)}{(l\omega_r)}\right]}{\int_0^\infty drr\left(\frac{\varphi_l(r)}{\frac{\partial\overline{\varphi}_0}{\partial r}}\right)^2 r\frac{\partial}{\partial r}[n_e(r)+n_i(r)]}.
\tag{80}
$$

VI. DISCUSSION

We have generalized the theory of rotational pumping for a diocotron mode to arbitrary azimuthal mode number, calculating the mode induced transport and mode damping. The analysis focused on a non-neutral plasma consisting dominantly of electrons but with a small admixture of H^- ions,[13] but also would be applicable to similar plasmas, such as an electron plasma with an admixture of antiprotons[14] or a positron plasma with an admixture of positive ions.[15] The axial bounce frequency for the light species is typically large compared to the plasma rotation frequency, so the rotational pumping for this species is in the relative weak adiabatic regime. In contrast, the rotational pumping for the heavy species can be enhanced by a resonance between the axial bounce frequency and the plasma rotation frequency, as seen in the rotating frame of the mode.

To obtain numerical estimates for the transport and damping rates, we consider the simple case where the electrons and ions are uniformly mixed and the density profiles are of the top-hat form. The equilibrium electron density has the constant value n_e out to the radius $r = r_p$ and is zero beyond, and likewise, the ion density has the value n_i out to this same radius and is zero beyond.

From Eq. (11), one can see that the radial position of a particle oscillates as it moves along a drift surface according to the equation

$$r(\theta) - r_I = \left(\frac{A\varphi_l(r_I)}{\frac{\partial\bar{\varphi}_0}{\partial r_i}}\right)\cos(l\theta) \equiv D_l(r_I)\cos(l\theta), \qquad (81)$$

where $D_l(r_I)$ is the amplitude of the oscillation at radius r_I. For a top-hat profile, the displacement at the edge of the plasma $D_l(r_p)$ is a convenient measure of the mode amplitude. Experimentally, the displacement can be obtained from an image obtained when the plasma is dumped out along field lines to the phosphor screen, and the image can be used to calibrate the mode signal received on wall electrodes. The experiment of Ref. 13 discusses the damping and transport for the case of an $l = 1$ diocotron mode of scaled amplitude $D_1(r_p)/R_w \approx 10^{-2}$, where $R_w = 3.4$ cm is the radius of the conducting wall for the Penning–Malmberg trap.

Since 2D $E \times B$ drift flow in a uniform magnetic field is incompressible, the mode charge density is limited to the edge of the top-hat density profile. In the interior of the plasma, the mode potential satisfies Laplace's equation, so the mode potential varies as $\varphi_l(r) \sim r^l$. Also, the gradient of the equilibrium potential in the rotating frame of the mode varies as $\frac{\partial\bar{\varphi}_0}{\partial r} \sim r$. Consequently, the displacement at any radius inside the plasma is given by the expression $D_l(r) = D_l(r_p)(\frac{r}{r_p})^{l-1}$.

The radial ion flux can be written as $\Gamma^i_{n\geq 1}(r) = n_i v_r(r)$, where $v_r(r)$ is the radial transport velocity. We define a characteristic transport time through the relation

$$\frac{1}{\tau_T} = \frac{v_r(r_p)}{r_p} = \frac{\Gamma^i_{n\geq 1}(r_p)}{n_i r_p}$$
$$= \sqrt{\frac{\pi}{2}}\frac{1}{8}\frac{l\bar{v}_i r_{ci}}{r_p^2}\left(\frac{\Delta z}{r_p}\right)^2\left(\frac{R_w}{L_0}\right)^2\left(\frac{D_l(r_p)}{R_w}\right)^2 sum(x). \qquad (82)$$

In the experiments of Ref. 13, the magnetic is field 12 kg, and the plasma is at room temperature, yielding the approximate values $\bar{v}_i = 1.6\times 10^5$ cm/s and $r_{ci} = 1.3\times 10^{-3}$ cm. The geometrical factors are approximately $r_p \approx \Delta z \approx 1$ cm, $R_w = 3.4$ cm and $L_0 = 34$ cm. The value $x = \frac{\omega_r}{\omega_b}$ is approximately 5, and as mentioned above, the scaled mode amplitude is about $\frac{D_l(r_p)}{R_w} \approx 10^{-2}$. For these parameters, the characteristic transport time is about $\tau_T \approx 2\times 10^2$ s.

Given enough time, the ions and electrons in a plasma such as that in Ref. 13 undergo centrifugal separation.[16] When an electron and an ion are at the same radius, the ion must rotate slightly faster than the electron to achieve radial force balance. Because of this difference in rotation speed, electron–ion collisions produce a small azimuthal drag force between the two species, causing the ions to drift radially outward and the electrons to drift radially inward. The time scale for this collisional separation of the ions, $\nu_{ie}^{-1}\left(\frac{\Omega_{ci}}{\omega_r}\right)^2$, is two orders of magnitude longer than the transport time for the rotational pumping.

For the top-hat density profile, the ratio of the radial integrals in Eq. (80) for the mode damping rate reduces to the simple result $\frac{n_i}{2l(n_i+n_e)}$, where use has been made of the relation $D_l(r) = D_l(r_p)(\frac{r}{r_p})^{l-1}$. The increment to the damping rate due to the ion rotational pumping is then

$$\gamma_i = \frac{1}{8}\sqrt{\frac{\pi}{2}}\frac{\bar{v}_i r_{ci}}{r_p^2}\left(\frac{\Delta z}{L_0}\right)^2\frac{n_i}{(n_i+n_e)}sum(x)$$
$$= \frac{1}{\tau_T}\frac{n_i}{(n_i+n_e)}\left(\frac{r_p}{D_l(r_p)}\right)^2 \approx 0.2\,\mathrm{s}^{-1}, \qquad (83)$$

where the ratio $\frac{n_i}{(n_i+n_e)}$ has been taken to be 0.1 in accord with Ref. 13. The transport time is longer than the damping time by the ratio of the total canonical angular momentum of the ions to the canonical angular momentum of the diocotron mode

$$\gamma_i\tau_T = \frac{n_i}{(n_i+n_e)}\left(\frac{r_p}{D_l(r_p)}\right)^2 \approx 10^3. \qquad (84)$$

We note that the coefficients of the transport rate and the damping rate, as given by Eqs. (82) and (83), are strongly dependent on temperature, scaling as $T^{3/2}$, if $x_i \propto \sqrt{T}B$ is held constant. The numbers given above for the plasma in Ref. 13 assume the relatively low room temperature. If the temperature were simply raised to 1 eV, while holding $x_i \propto \sqrt{T}B$ constant, both rates would increase by three orders of magnitude.

Because the damping time is short compared to the transport time, the diocotron mode would damp away with only negligible transport, except that in the experiments of Ref. 13 the diocotron mode is continuously driven unstable by a transiting, weak H_2^+ ion beam. This beam is incidental to the rotational pumping process. Before the H^- ions are produced, by dissociative electron attachment to excited H_2 molecules, the diocotron mode grows exponentially at a rate $\gamma^+ \approx .05\,\mathrm{s}^{-1}$. When the H^- density grows to the level assumed in Eq. (83), the H^- increment to the rotational pumping damping rate can stabilize the growth and even damp the mode. Thus, the mode is an intermediary in the angular momentum balance. Ultimately, the angular momentum change associated with pumping the ions out is deposited in the transiting H_2^+ ion beam.

What are some connections of this research to other plasma problems of current interest? Experiments with electron–antiproton plasmas have observed cylindrical separation that proceeds more rapidly than would be expected from the collisional transport time scale,[22] and there is interest in a collective process that could facilitate this fast separation. Although the rotational pumping discussed here does transport out the heavy species faster than the collisionally driven centrifugal separation, it is not an ideal candidate for the desired collective process. The current process relies on an external agency, the H_2^+ ion beam, to provide the angular momentum necessary to move the ions outward. Presumably, what is needed is an instability that is driven by the free energy of the unseparated plasma. The unstable mode would drive the heavy species outward and the light species inward, conserving total plasma canonical angular momentum without invoking the action of an external agency. A possible candidate is the drift wave instability discussed by Dubin.[23]

Another topic of interest is the collective transfer of canonical angular momentum from one species to another, and the bounce-resonant rotational pumping is a good example of such transfer. A diocotron mode supported primarily by a light species transfers angular momentum preferentially to a heavy species, since the bounce-rotation resonance selects for particle mass.

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DATA AVAILABILITY

Data sharing is not applicable for this article as no new data were created or analyzed in this study.

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