Particle fluxes through the separatrix in the trapped particle diocotron mode

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In the trapped particle diocotron mode, the trapped particles undergo $\mathbf{E}\times\mathbf{B}$ drift motion in a uniform $\mathbf{B}$ field. Since such a flow is incompressible one is tempted to assume that the trapped particle density is constant along a fluid element. However, this is not the case since there is interchange of trapped and passing particles through the separatrix. This paper shows that a corrected fluid analysis, taking into account the particle flux through the separatrix, reproduces the same trapped particle density perturbation as obtained from the kinetic theory, thereby resolving confusion in earlier papers. © *2011 American Institute of Physics*. [doi:10.1063/1.3613665]

Electric and magnetic field inhomogeneities in plasma containment devices often cause particles to be trapped in localized regions, and these trapped particles give rise to a class of low frequency modes of oscillation called trapped particle modes[1] and to the important phenomena of neoclassical transport.[2] Recent work using nonneutral plasmas has provided access to the physics of trapped particle modes and neoclassical transport for a simple geometry and quiescent plasma, where well-controlled comparisons of theory and experiment are possible.

Trapped particle diocotron modes (TPDM) are routinely excited on nonneutral plasma columns in which there are classes of trapped and passing particles.[3] To create these classes, the usual Malmberg–Penning trap configuration is modified by applying an azimuthally symmetric potential barrier, the squeeze potential, near the axial mid-point of the column. Particles with relatively low axial velocity are trapped on one side or the other of the barrier, while high velocity particles pass back and forth along the axial magnetic field over the full length of the plasma column.

The mode dynamics is easy to understand qualitatively. The mode potential has odd parity in the axial coordinate and produces bounce-average $\mathbf{E}\times\mathbf{B}$ drift oscillations of the trapped particles that are 180° out of phase on the two sides of the barrier, while the passing particles stream back and forth, partially Debye shielding the perturbation in the trapped particle charge density.

A theoretical description of the TPDM was obtained using the Poisson's equation and the drift kinetic equation with a Fokker–Planck (FP) collision operator.[4,5] Experiments had observed damping of the TPDM, and the theory explained the damping as resulting from collisional velocity diffusion at the separatrix between trapped and untrapped particles. Associated with the damping is a neoclassical particle transport. The damping and associated transport also were investigated using numerical simulations.[6] Much additional work then explored and generalized the neoclassical damping and transport.[7–13] Of course, waves and field asymmetries also produce transport in nonneutral plasmas without separate classes of trapped and passing particles, and in these plasmas the dominant transport mechanism is thought to be resonant particle transport.[14–19]

As described in the abstract, the purpose of this brief communication is to resolve a point of theoretical confusion introduced in the first theoretical papers on the TPDM.[4,5] Using a corrected fluid theory that takes into account the flow of particles back and forth through the separatrix, we show that the fluid description of the trapped particle density perturbation matches that obtained from the kinetic description, and we discuss the consequences of the corrected density perturbation for the mode eigenfunctions.

In Refs. 4 and 5, the plasma potential is written as

$$\phi(\theta,r,z,t)=\phi_0(r,z)+\delta\phi(\theta,r,z,t) \tag{1}$$

where $\phi_0(r,z)$ is the unperturbed potential and

$$\delta\phi(\theta,r,z,t)=\delta\phi_\ell(r,z)\exp[i\ell\theta-i\omega t]+\mathrm{cc} \tag{2}$$

is the mode potential. Here, (θ, r, z) is a cylindrical coordinate system with the z-axis coincident with the axis of the plasma column. Debye shielding forces $\phi_0(r,z)$ and $\delta\phi_\ell(r,z)$ to be nearly z-independent, that is, constant along a magnetic field line, except where the end and squeeze potentials are applied. Here, we consider the simple case where the squeeze potential is applied at the axial mid-point of the column ($z=0$) and is of small axial extent. Then, $\phi_0(r,z)$ can be replaced by $\phi_0(r)$ everywhere except at the ends and in a small axial interval near $z=0$, where $\phi_0(r,z)$ rises to the peak value $\phi_0(r,z=0)=\Phi_s(r)$. The mode potential $\delta\phi_\ell(r,z)$ changes sign in the region of the squeeze potential, so we note that $\delta\phi_\ell(r,z=0)=0$ and set $\delta\phi_\ell(r,z)=\mathrm{sign}(z)\delta\phi_\ell(r)$ for z outside the narrow interval of the squeeze potential.

To define the separatrix velocity, we work in the rotating frame of the mode, where the potential is time-independent and particle energy is conserved. Also, for the drift approximation and uniform magnetic field considered here, the transverse kinetic energy is constant. The separatrix velocity is then the axial velocity separating trapped and passing particles

$$V(r,\theta)=\sqrt{\frac{2q}{m}[\Phi_s(r)-\phi_0(r)-\delta\phi(r,\theta)]}\simeq v_s(r)-\frac{q\delta\phi}{mv_s(r)}, \qquad (3)$$

where q and m are the charge and mass of the particles and $mv_s^2(r)/2=q\Phi_s(r)-q\phi_0(r)$ defines the unperturbed separatrix velocity. Of course, there is a correction to the potentials in the rotating frame, but this correction is z-independent and doesn't enter the expression for the separatrix velocity.

The analysis[4,5] assumes the frequency ordering $\Omega_c \gg \omega_b \gg \omega, \omega_E \gg \nu$, where $\Omega_c = qB/mc$ is the cyclotron frequency, $\omega_b=\pi\bar{v}/L$ the characteristic axial bounce frequency, ω the mode frequency, $\omega_E=(c/Br)(\partial\phi_0/\partial r)$ the $\mathbf{E}\times\mathbf{B}$ drift rotation frequency, and ν the collision frequency. Here, L is the column length and B is the strength of the uniform axial magnetic field. For this frequency ordering, the cross magnetic field motion of the particles is bounce-average $\mathbf{E}\times\mathbf{B}$ drift motion, and the axial motion is an adiabatic (or Debye shielding) response. Because the trapped and passing particles experience different dynamics, the perturbed distribution would be discontinuous in value and slope at the separatrix velocity were it not for collisions. Even though the collision frequency is small, the FP collision operator acts in a narrow boundary layer near the separatrix to smooth the connection, yielding a contribution to the mode dispersion relation that is of order $\sqrt{\nu/\omega}$ rather than ν/ω. The analysis is similar to that used by Rosenbluth, Ross and Kostomarov in analyzing the effect of collisions on the trapped ion mode.[20]

The guiding center distribution function, $f(v, \theta, r, z, t)$, is described by the drift-kinetic equation

$$\frac{\partial f}{\partial t}+v_z\frac{\partial f}{\partial z}-\frac{q}{m}\frac{\partial\phi}{\partial z}\frac{\partial f}{\partial v_z}+\mathbf{V}_D\cdot\nabla_\perp f=C(f), \qquad (4)$$

where $\mathbf{V}_D=(c/B)\hat{z}\times\nabla\phi$ is the drift velocity and $C(f)$ the FP collision operator. The guiding center distribution is written as $f=f_0(v, r, z)+\delta f(v, \theta, r, z, t)$, and the linear analysis of the drift equation yields the results

$$\delta f=\mathrm{sign}(z)\delta f_\ell e^{i\ell\theta-i\omega t}+cc, \quad \delta f_\ell=\begin{cases}\delta f_\ell^{(t)}, & \text{if } \ |v|<v_s,\\ \delta f_\ell^{(p)}, & \text{if } \ |v|>v_s,\end{cases} \qquad (5)$$

where

$$\delta f_\ell^{(t)}=\frac{c\ell}{Br}\frac{\delta\phi_\ell}{\omega'}\frac{\partial f_0}{\partial r}+a\,e^{\chi_t(|v|-v_s)}, \quad \chi_t\equiv(1+i)\sqrt{\frac{\omega'}{2D_v}},$$
$$\delta f_\ell^{(p)}=-\frac{q\delta\phi_\ell}{T}f_0+b\,e^{\chi_p(|v|-v_s)}, \quad \chi_p\equiv-(1+i)\sqrt{\frac{\omega_b}{2D_v}}. \qquad (6)$$

Here, the superscripts (t) and (p) refer to trapped and passing, D_v is the velocity diffusion coefficient in the FP collision operator, T is the plasma temperature in energy units, and $\omega' \equiv \ell\omega_E-\omega$. The first term in the expression for $\delta f_\ell^{(t)}$ is the perturbation due to the bounce-average drift motion of the trapped particles. There is no such term for the passing particles, since the bounce-average of $\mathrm{sign}(z)\delta\phi_\ell(r)$ vanishes for passing particles. The first, term in the expression for $\delta f_\ell^{(p)}$ is the adiabatic response associated with streaming along the field lines. The two terms $a\exp[\chi_t(|v|-v_s)]$ and $b\exp[\chi_p(|v|-v_s)]$ are corrections introduced by the FP collision operator in a small boundary layer near the separatrix. Because velocity derivatives are large in the boundary layer, only the velocity diffusion term need by retained in the FP collision term.

The coefficients a and b are chosen so that $\delta f^{(t)}$ and $\delta f^{(p)}$ are continuous in value and slope at the separatrix, $v=v_s(r)$. For the unperturbed distribution

$$f_0(r,v)=\frac{n_0(r)}{\sqrt{2\pi T/m}}e^{-mv^2/(2T)} \qquad (7)$$

and the frequency ordering $\omega_b \gg \omega' \gg \nu$ one finds the values

$$a=a_s\frac{\chi_p+mv_s/T}{\chi_p-\chi_t}\simeq a_s, \quad b=a_s\frac{\chi_t+mv_s/T}{\chi_p-\chi_t}\simeq 0, \qquad (8)$$

where

$$a_s\equiv-\left(\frac{c\ell}{Br}\frac{1}{\omega'}\frac{\partial f_0(r,v_s)}{\partial r}+\frac{q}{T}f_0(r,v_s)\right)\delta\phi_\ell. \qquad (9)$$

Integrating δf_ℓ over velocity yields the perturbed density

$$\delta n_\ell\simeq\frac{c\ell}{Br}\frac{1}{\omega'}\frac{n_{0t}}{n_0}\frac{\partial n_0}{\partial r}\delta\phi_\ell-\frac{q}{T}n_{0p}\delta\phi_\ell-\frac{2}{\chi_t}\left(\frac{c\ell}{Br}\frac{1}{\omega'}\frac{\partial f_0(r,v_s)}{\partial r}+\frac{q}{T}f_0(r,v_s)\right)\delta\phi_\ell, \qquad (10)$$

where

$$n_{0t}\equiv\int_{-v_s}^{v_s}f_0\,dv=n_0(r)\,\mathrm{erf}\left[v_s(r)\sqrt{m/(2T)}\right],$$
$$n_{0p}\equiv n_0-n_{0t} \qquad (11)$$

are the unperturbed densities of the trapped and passing particles.

References 4 and 5 obtained the correct expression for δf_ℓ, but in writing down an expression for δn_ℓ, the first term on the right-hand side of Eq. (10) was replaced by the term $c\ell/(rB\omega')(\partial n_{0t}/\partial r)\delta\phi_\ell$. One can understand this latter expression as arising from a fluid treatment in which the trapped particle density is assumed to be constant along the $\mathbf{E}\times\mathbf{B}$ drift fluid trajectory, that is, as a solution to the equation

$$0=\frac{dn_t}{dt}=\frac{\partial\delta n_t}{\partial t}+\omega_E\frac{\partial\delta n_t}{\partial\theta}-\frac{c}{Br}\frac{\partial\delta\phi}{\partial\theta}\frac{\partial n_{0t}}{\partial r}. \qquad (12)$$

However, the trapped particle density is not constant along a fluid trajectory because of the particle flux through the velocity separatrix.

To formulate a corrected fluid theory, which takes into account the flux through the separatrix, we first define the trapped particle density (say to the right of the barrier) as the integral

$$n_t(\theta, r, t) = \frac{2}{L}\int_0^{L/2} dz \int_{-V(\theta,r,t)}^{V(\theta,r,t)} dv_z f(v_z, \theta, r, z, t). \qquad (13)$$

Operating with this integral on drift-kinetic Eq. (4) yields the corrected fluid equation

$$\frac{dn_t}{dt} = 2f(V, \theta, r, t)\frac{dV}{dt} + 2D_V \frac{\partial \delta f}{\partial v}\bigg|_{v=V}, \qquad (14)$$

where $d/dt = \partial/\partial t + \mathbf{V}_D \cdot \nabla_\perp$. The terms on the right-hand side arise because derivatives of the limits on the velocity integral [i.e. $v_z = \pm V(\theta, r, t)$] must be subtracted off when the derivative and the integral are commuted. Physically, these terms represent the flux of particles through the separatrix. In the integral over Eq. (4), two terms vanish

$$\int_0^{L/2} dz \int_{-V}^{V} dv_z \left[v_z \frac{\partial f}{\partial z} - \frac{q}{m}\frac{\partial \phi}{\partial z}\frac{\partial f}{\partial v_z}\right] = \int_{\min(q\phi)}^{q\Phi_s(r)} \frac{d\varepsilon}{m}\oint_{\varepsilon=\text{const.}} dz \frac{\partial f}{\partial z}\bigg|_\varepsilon = 0, \qquad (15)$$

where $\varepsilon = mv_z^2/2 + q\phi$ and $\min(q\phi)$ is the minimum value of $q\phi$ in the trapped particle region. The last z-integral is over a closed loop in (z, v_z) space at constant ε.

Dropping nonlinear terms in Eq. (14) yields the equation

$$\frac{dn_t}{dt} = -2i\,\mathrm{sign}(z)\omega'\left\{\left(\frac{q}{mv_s} + \frac{c}{B\omega'}\frac{\ell}{r}\frac{\partial v_s}{\partial r}\right) \times f_0(r, v_s)\delta\phi_\ell - \frac{a}{\chi_t}\right\}e^{i\ell\theta - i\omega t} + cc \qquad (16)$$

for which the solution in terms of complex amplitudes is

$$\delta n_{t,\ell} = \left\{\frac{c\ell n_{0t}}{rB\omega' n_0}\frac{\partial n_0}{\partial r} - \frac{2q}{mv_s}f_0(r, v_s) - \frac{2}{\chi_t}\left(\frac{c\ell}{rB\omega'}\frac{\partial f_0(r, v_s)}{\partial r} + \frac{q}{T}f_0(r, v_s)\right)\right\}\delta\phi_\ell. \qquad (17)$$

A simpler way to calculate the trapped particle density perturbation is to substitute $f = f_0 + \delta f$ into definition (13), where δf is given by Eqs. (5)–(8), and then linearize. Of course, this procedure reproduces the fluid result in Eq. (17).

Likewise the density of passing particles is given by

$$n_p = \frac{2}{L}\int_0^{L/2} dz \left[\int_{V(\theta,r,t)}^{\infty} dv_z f + \int_{-\infty}^{-V(\theta,r,t)} dv_z f\right]. \qquad (18)$$

Substituting the kinetic solution for $f + \delta f$ and then linearizing yields the result

$$\delta n_{p,\ell} = \left(\frac{2q}{mv_s}f_0(r, v_s) - \frac{q}{T}n_{0p}\right)\delta\phi_\ell. \qquad (19)$$

Of course, the sum of Eqs. (17) and (19) reproduces Eq. (10).

Operating on Eq. (4) with the integrals in Eq. (18) yields the modified fluid equation for passing particles

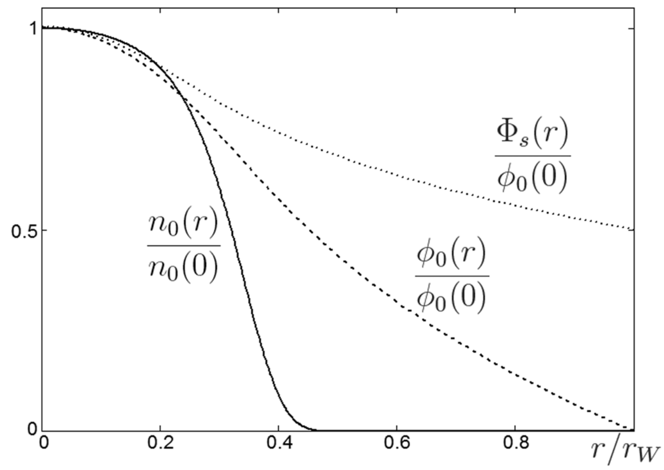

FIG. 1. The solid, dashed, and dotted curves show the radial dependence of the scaled density, potential, and squeeze voltage, respectively [i.e., $n_0(r)/n(0)$, $\phi_0(r)/\phi_0(0)$, and $\Phi_s(r)/\phi_0(0)$].

$$\frac{dn_p}{dt} = -2f(V, \theta, r, t)\frac{dV}{dt} - 2D_V\frac{\partial \delta f}{\partial v}\bigg|_{v=V} + \frac{2}{L}\mathrm{sign}(z)j_z, \qquad (20)$$

where

$$j_z = \int_{V(\theta,rt)}^{\infty} dv_z v_z f|_{z=0} + \int_{-\infty}^{-V(\theta,r,t)} dv_z v_z f|_{z=0} \qquad (21)$$

is the current density of passing particles over the squeeze barrier. Linearizing and using Eq. (19) for $\delta n_{\ell p}$, yields an expression for this "sloshing current"

$$j_z = \frac{i\omega' L}{2}\left\{\frac{c\ell n_{0t}}{rB\omega' n_0}\frac{\partial n_0}{\partial r} - \frac{q}{T}n_{0p} - \frac{2}{\chi_t}\left(\frac{c\ell}{rB\omega'}\frac{\partial f_0(r, v_s)}{\partial r} + \frac{q}{T}f_0(r, v_s)\right)\right\}\delta\phi_\ell e^{i\ell\theta - i\omega t} + cc. \qquad (22)$$

Next we discuss some consequences of the correction to the density perturbation. For simplicity, we consider the limit $D_V \to 0$, focusing on the Real Part of the mode frequency

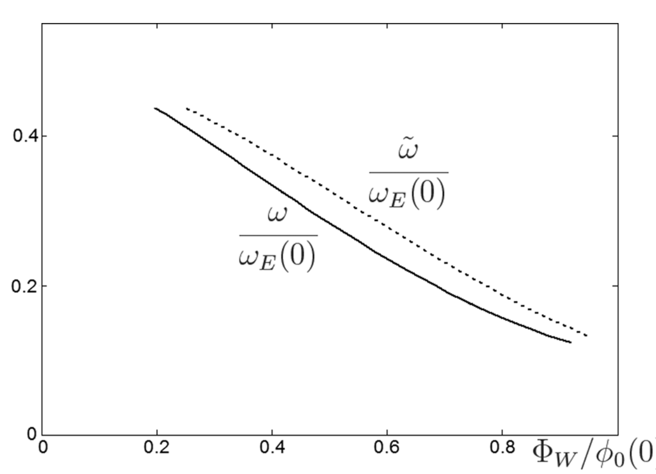

FIG. 2. The scaled frequencies $\omega/\omega_E(0)$ and $\tilde{\omega}/\omega_E(0)$ are plotted versus the scaled squeeze voltage applied at the wall, $\Phi_W/\phi_0(0)$. The ratio of the Debye length to the wall radius is taken to be $\lambda_D(0)/r_W = 0.057$, which corresponds to $T = 1$ eV in the experiments of Ref. 1.

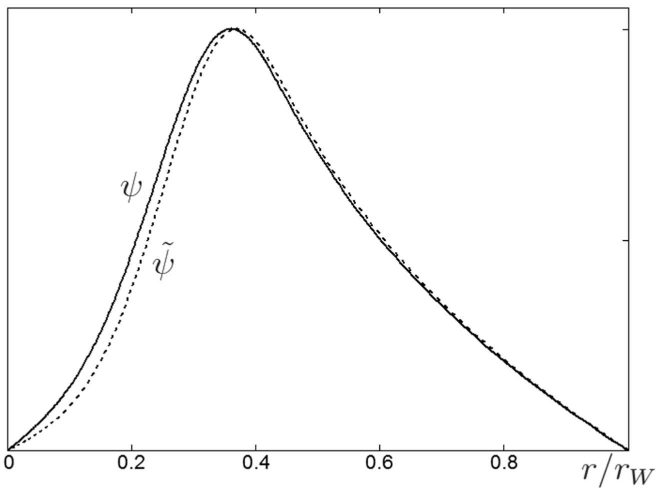

FIG. 3. The radial dependence of the eigenfunctions ψ and $\tilde{\psi}$ is shown for the profiles in Fig. 1 and for $\lambda_D(0)/r_W = 0.057$.

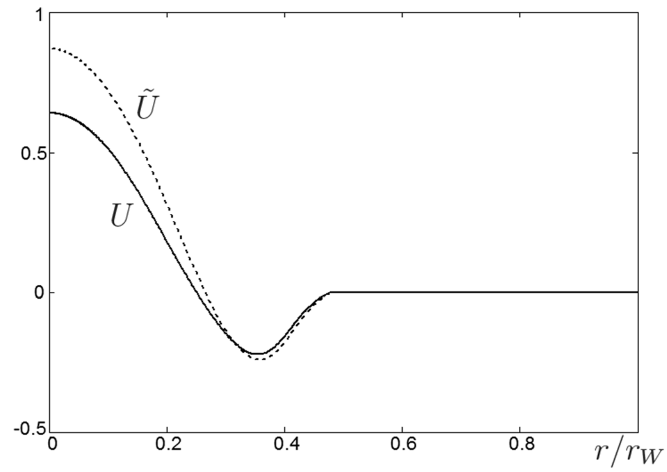

FIG. 4. The radial dependence of the effective potentials U and $\tilde{U}$ for the same conditions as used for Fig. 3.

and eigenfunction. Substituting density perturbation (9) (with $D_V \to 0$) into Poisson's equation yields the mode eigenfunction equation

$$\lambda_D^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - \frac{\ell^2}{r^2}\psi\right] = U(r,\omega)\psi, \qquad (23)$$

where $\psi = \delta\phi_\ell(r)$ is the mode eigenfunction, $\lambda_D = \sqrt{T/4\pi e^2 n(0)}$ is the Debye length on axis, and

$$U(r,\omega) \equiv \frac{n_{0p}(r)}{n_0(0)} + \frac{\omega_E(0)2\lambda_D^2}{\omega_E(r) - \omega/\ell}\frac{n_{0t}(r)}{n_0(r)}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{n_0(r)}{n_0(0)}\right). \qquad (24)$$

For comparison, the uncorrected eigenfunction, say $\tilde{\psi}(r)$, satisfies Eq. (23) with $U(r,\omega)$ replaced by

$$\tilde{U}(r,\tilde{\omega}) = \frac{n_{0p}(r)}{n_0(0)} + \frac{\omega_E(0)2\lambda_D^2}{\omega_E(r) - \tilde{\omega}/\ell}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{n_{0t}(r)}{n(0)}\right). \qquad (25)$$

As we will see, the corrections to the TPDM frequency and eigenfunctions are small for typical experimental conditions. Figure 1 shows plots of $n_0(r)/n(0)$, $\phi_0(r)/\phi_0(0)$, and $\Phi_s(r)/\phi_0(0)$ that are typical for the experiments. For this plot, the squeeze voltage applied at the wall, $\Phi_W = \Phi_s(r_W)$, is chosen to have the value $(0.5)\phi_0(0)$. The 1D (i.e., radial) solution for the squeeze potential assumes that $r_W \ll \Delta L \ll L$, where ΔL is the axial length of the squeeze region. The Debye length and wall radius are taken to have the ratio $\lambda_D/r_W = 0.057$, which corresponds to $T = 1$ eV in the experiments.[3–5]

For azimuthal mode number $\ell = 1$, Fig. 2 shows that plots of the scaled frequencies $\omega/\omega_E(0)$ and $\tilde{\omega}/\omega_E(0)$ versus the scaled squeeze potential applied at the wall, $\Phi_W/\phi_0(0)$. Likewise, Fig. 3 shows $\psi(r)$ and $\tilde{\psi}(r)$ and Fig. 4 shows $U(r)$ and $\tilde{U}(r)$ for the choice of squeeze voltage shown in Fig. 1 [i.e., $\Phi_W/\phi_0(0) = 0.5$].

Although these corrections to the TPDM are small, the uncorrected eigenfunction equation predicts another mode, which rotates slightly faster than the plasma, and the correction eliminates this mode. This "fast mode" could exist only because $\partial n_{0t}/\partial r$ is positive over a range of r values. Since $(n_{0t}/n_0)(\partial n_0/\partial r)$ is uniformly negative, the corrected eigenfunction equation does not support the mode.

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