TEMPERATURE EQUILIBRATION IN A STRONGLY MAGNETIZED PURE ELECTRON PLASMA

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Abstract

For a pure electron plasma in a sufficiently strong magnetic field, an unusual many-body adiabatic invariant constrains the collisional dynamics. To the extent that this adiabatic invariant is preserved by the dynamics, no exchange of energy is possible between the parallel and perpendicular degrees of freedom. The system may then acquire and maintain two different temperatures, $T_{\parallel}$ and $T_{\perp}$. However, since an adiabatic invariant is not an exact constant of the motion, equilibration will eventually take place, but on an exponentially long timescale. The phenomenon is illustrated analytically and numerically, and a derivation of the equipartition rate is outlined.

1. Introduction. Adiabatic Invariants.

The notion of an adiabatic invariant goes back at least to the 1911 Solvay Conference, where Einstein stressed the point that if the length of a simple pendulum is varied sufficiently slowly, the energy E changes only through the frequency ν, in such a way that $E = h\nu$ remains valid throughout. More generally, an action J which is canonically conjugate to an angle θ, where θ varies on a timescale *faster* than any other timescale in the system, is an "almost constant of the motion." Early quantum mechanics evolved around the idea that quantization could be obtained when the values of action variables were restricted to integral amounts of h.

Mechanics textbooks prove the adiabatic invariance of the action by employing some version of the method of averaging.[1] There is another view[2] of the phenomena which may be quite instructive. Consider a 1-dimensional oscillator with slowly varying frequency

$$\ddot{x} + \omega^2(t)x = 0 \quad .$$

If $\omega(t)$ varies between some constant value ω_- in the past and a constant value ω_+ in the future, x has asymptotic solutions

$$x_{\pm} = \frac{1}{2}(C_{\pm}\, e^{i\omega_{\pm}t} + C_{\pm}^{*}\, e^{-i\omega_{\pm}t}) \tag{1}$$

and corresponding well defined values of the action

$$J_{\pm} = \frac{1}{2}\,\omega_{\pm}\ | C_{\pm} |^2 \quad .$$

Letting $t \rightarrow x$ and $\omega^2 \rightarrow k^2 = E - V(x)$ we can consider the quantum mechanical problem

$$\psi'' + (E - V(x))\psi = 0$$

of passage above a smooth potential bump. Here we look for asymptotic states of a transmitted wave to one side; and an incoming plus a reflected on the other:

$$\psi_+ = a_t\ e^{ik_+ x}$$
$$\psi_- = e^{ik_- x} + a_r\ e^{-ik_- x} \quad . \tag{2}$$

The coefficients a_t and a_r are transmission and reflection amplitudes respectively. If we equate the real part of (2) to the solutions (1) we get that $C_+ = a_t$ and $C_- = 1 + a_r^*$. Using this, together with conservation of flux: $k_-(1 - | a_r |^2) = k_+ | a_t |^2$, we can express the difference in action between the two asymptotic states as

$$\Delta J \equiv J_+ - J_- = k_- \left[- | a_r |^2 - \mathrm{Re}\ a_r \right] \quad .$$

Now, from elementary quantum mechanics it is well known[3] that the amplitude for "reflection above the barrier" is vanishingly (in fact exponentially) small.

As we shall see below, for a system such as the strongly magnetized pure electron plasma the entire kinetic energy perpendicular to the magnetic field is an adiabatic invariant for the Hamiltonian describing the system. In particular, this means that one cannot expect equilibration of an anisotropic temperature distribution (say, with $T_\perp < T_\parallel$) to occur on the usual timescale of a few collisions. Rather, a timescale exponentially longer than the collisional can be expected.

2. A Many-Body Adiabatic Invariant.

Consider a gas of N charged classical particles imbedded in a homogeneous magnetic field. The Hamiltonian for such a system is given by

$$H = \sum_{i=1}^{N} \frac{1}{2m} (\mathbf{p}_i - \frac{e}{c} A_i)^2 + \sum_{i<j} \frac{e^2}{| \mathbf{r}_i - \mathbf{r}_j |} \quad .$$

We consider a magnetic field so strong that the typical Larmor radius is small compared to the distance of closest approach (or, equivalently, the cyclotron frequency is the largest frequency which enters the particle dynamics); with this ordering the system is known as a Strongly Magnetized Pure Electron Plasma. For such a plasma, we will show that the total kinetic energy associated with velocity components perpendicular to the magnetic field is an adiabatic invariant. This is a novel invariant, since it involves the velocities of many electrons; it is a many electron adiabatic invariant.

To see this, we make two canonical transformations. First we introduce guiding center $(X,Y) = (p_Y/m\Omega\ ,\ Y)$ and gyro-angle (p_ψ,ψ) coordinates, see fig. 1. In these coordinates,

$$H = \sum_{i=1}^{N} \Omega\, p_{\psi_i} + \frac{p_{z_i}^{\ 2}}{2m} + \sum_{i<j} e^2/\,|\,\mathbf{r}_i - \mathbf{r}_j\,| \quad ,$$

where $\Omega = \dfrac{eB}{mc}$ is the electron cyclotron frequency, and

$$\begin{aligned} |\,\mathbf{r}_i - \mathbf{r}_j\,|^2 = &(X_i + r_{L_i}\cos\psi_i - X_j - r_{L_j}\cos\psi_j)^2 \\ &+ (Y_i + r_{L_i}\sin\psi_i - Y_j - r_{L_j}\sin\psi_j)^2 \\ &+ (z_i - z_j)^2 \quad . \end{aligned}$$

The quantity $r_L = \sqrt{2p_\psi/m\,\Omega}$ is the Larmor radius for an electron. Assuming that the dynamics is that of a many-electron collision, rather than a collective mode of oscillation, the condition for strong magnetization implies that the ψ_i are rapidly varying compared to the other variables. Since there are many fast variables ψ_i, the existence of an adiabatic invariant is not immediately obvious. To uncover the invariant, we make a further canonical transformation to

$$\begin{aligned} &\chi_1 = \psi_1 & &\chi_j = \psi_j - \psi_1 & &j > 1 \\ &p_{\psi_1} = p_{\chi_1} - \sum_{2}^{N} p_{\chi_j} & &p_{\psi_j} = p_{\chi_j} & &j > 1 \quad . \end{aligned}$$

We have made χ_1 the only fast variable and measure all other angles relative to it. The Hamiltonian now appears as

$$H = \Omega\, p_{\chi_1} + \sum_{i=1}^{N} \frac{p_{z_i}^{\ 2}}{2m} + \sum_{i<j} e^2/\,|\,\mathbf{r}_i - \mathbf{r}_j\,|$$

clearly showing that p_{χ_1} and therefore $\Omega p_{\chi_1} = \sum_1^N \Omega p_{\psi_i} = \sum_1^N \frac{m}{2}\, \mathbf{v}_{\perp i}^{\ 2}$ is an adiabatic invariant.

3. Binary Interactions.

A case which can be treated in considerable detail[4] is that of a binary collision in a uniform magnetic field. The equations of motion for the two electrons,

$$\begin{aligned} \frac{d\,\mathbf{v}_1}{dt} + \Omega \mathbf{v}_1 \times \mathbf{e}_z &= \frac{e^2}{m}\,\frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\,\mathbf{r}_1 - \mathbf{r}_2\,|^3} \\ \frac{d\,\mathbf{v}_2}{dt} + \Omega \mathbf{v}_2 \times \mathbf{e}_z &= \frac{e^2}{m}\,\frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\,\mathbf{r}_1 - \mathbf{r}_2\,|^3} \quad , \end{aligned}$$

can be separated into equations for the center of mass and for the relative motion:

$$\frac{d\mathbf{V}}{dt} + \Omega\mathbf{V} \times \mathbf{e}_z = 0$$

$$\frac{d\mathbf{v}}{dt} + \Omega\mathbf{v} \times \mathbf{e}_z = \frac{e^2}{\mu} \frac{\mathbf{r}}{|\mathbf{r}|^3} \quad . \tag{3}$$

Here, $\mu = \frac{m}{2}$ is the reduced mass. Notice that (3) describes the motion of an electron in a uniform magnetic field and the field of a fixed charge (see fig. 2). The solution for the center of mass motion is trivial; the interesting part of the dynamics is in the relative motion. In fact, since $V_\perp^2$ is a constant of the motion, and $mv_{1_\perp}^2/2 + mv_{2_\perp}^2/2 = \mu v_\perp^2/2 + mV_\perp^2$, we can calculate the change in the total perpendicular kinetic energy in one collision by calculating the change in $\mu v_\perp^2/2$.

From (3), one obtains

$$\Delta\left(\frac{\mu}{2} v_\perp^2\right) = \int_{-\infty}^{\infty} dt \; \frac{e^2 \mathbf{v}_\perp(t) \cdot \mathbf{r}_\perp(t)}{|\mathbf{r}(t)|^3} \quad . \tag{4}$$

Denote by $v_\parallel$ and $v_\perp$ the parallel and perpendicular velocities of the moving charge at infinity, and let $b = \frac{e^2}{1/2\; mv_\parallel^2}$ be a measure of the distance of closest approach. The quantity $\epsilon = v_\parallel/\Omega b$ is a small parameter, the condition $\epsilon << 1$ expressing the condition that the Larmor radius is much smaller than the distance of closest approach or, equivalently, that the cyclotron frequency is much larger than the characteristic frequency associated with the parallel dynamics. Following the usual practice in the theory of adiabatic invariants, we use the lowest-order orbits in evaluating the time integral, that is, we rewrite (4) as

$$\Delta\left(\frac{\mu}{2} v_\perp^2\right) \simeq e^2 v_\perp \rho \cos(\delta) \int_{-\infty}^{\infty} dt \; \frac{\cos(\Omega t)}{[\rho^2 + z^2(t)]^{3/2}} \quad , \tag{5}$$

where (ρ, z) is the guiding center approximation for $(\mathbf{r}_\perp, z)$, δ is a phase, and $z(t)$ is determined by

$$\left(\frac{dz}{dt}\right)^2 + \frac{2e^2/\mu}{[\rho^2 + z^2(t)]^{1/2}} = v_\parallel^2 \quad . \tag{6}$$

In terms of the scaled variables

$$\tilde{z} = z/b \quad , \quad \tilde{t} = v_\parallel t/b \quad , \quad \tilde{\rho} = \rho/b$$

we can rewrite (5) and (6) as

$$\Delta\left(\frac{\mu}{2} v_\perp^2\right)^2 \simeq \frac{e^2}{b} \frac{v_\perp}{v_\parallel} \tilde{\rho} \; \cos(\delta) \int_{-\infty}^{\infty} \frac{d\tilde{t}\; \cos(\tilde{t}/\epsilon)}{[\tilde{\rho}^2 + \tilde{z}^2]^{3/2}} \tag{7}$$

$$\left(\frac{d\tilde{z}}{d\tilde{t}}\right)^2 + \left[\tilde{\rho}^2 + \tilde{z}^2\right]^{-1/2} = 1 \quad . \tag{8}$$

The integrand in (7) is a rapidly varying function, and the integral is exponentially small. In fact, by analytic continuation one can argue [4] that the value of the integral has the form

$$\int_{-\infty}^{\infty} \frac{d\tilde{t}\, \cos(\tilde{t}/\epsilon)}{[\tilde{\rho}^{\,2} + \tilde{z}^2]^{3/2}} = h(\tilde{\rho}, \epsilon) \exp[-g(\tilde{\rho})/\epsilon] \quad , \tag{9}$$

where $g(\tilde{\rho})$ is a monotone function with $g(0) = \pi/2$ and $g(\tilde{\rho}) \to \tilde{\rho}$ for $\tilde{\rho} \uparrow \infty$, and h is a non-exponential function.

We thus have an expression for the exchange of kinetic energy between the parallel and perpendicular degrees of freedom in a single binary collision. It depends on $\tilde{\rho}$, which can be thought of as the impact parameter, and on ϵ, the adiabadicity-parameter. The largest amount of exchange evidently occurs for low impact parameter, high-velocity collisions.

4. Temperature Equilibration.

We now discuss the influence of the adiabatic invariant on the long-time collisional evolution of the electron velocity distribution. On the time scale of a few collisions there is negligible exchange of energy between the parallel and the perpendicular degrees of freedom, and the distribution of parallel velocities and the distribution of perpendicular velocities become Maxwellian separately, with the $T_{\parallel}$ not necessarily equal to $T_{\perp}$.

The evolution does not stop at this stage, however. Each collision produces an exponentially small exchange of energy between the parallel and the perpendicular degrees of freedom, and these collisions act cumulatively in such a way that $T_{\perp}$ and $T_{\parallel}$ relax to a common value on an exponentially long timescale. From the observation that small impact parameter collisions are most effective in producing an exchange of parallel and perpendicular energy, we deduce that the most important collisions are well separated binary events. Such collisions can be treated with a Boltzmann-like collision operator[5]

$$\begin{aligned} \partial_t f(\mathbf{v}_1,t) = n \int 2\pi \rho\, d\rho \int d\mathbf{v}_2 \; |\mathbf{e}_z \cdot (\mathbf{v}_2 - \mathbf{v}_1)| \\ \times \Big[f(\mathbf{v}_1',t)\, f(\mathbf{v}_2',t) - f(\mathbf{v}_1,t)\, f(\mathbf{v}_2,t) \Big] \quad . \end{aligned} \tag{10}$$

The first integral in (10) replaces the usual integral over scattering cross section, and the term $|\mathbf{e}_z \cdot (\mathbf{v}_2 - \mathbf{v}_1)|$ replaces the usual factor $|(\mathbf{v}_2 - \mathbf{v}_1)|$ since particles stream toward one another along field lines.

We use the Boltzmann-like operator to evaluate the integral

$$\frac{dT_{\perp}}{dt} = \int d\mathbf{v}_1 \, \frac{m\, \mathbf{v}_{1\perp}^2}{2} \, \partial_t f(\mathbf{v}_1,t) \quad . \tag{11}$$

Substituting (10) into (11) and using symmetry arguments yields the result

$$\frac{dT_\perp}{dt} = \frac{n}{4} \int 2\pi\, \rho\, d\rho \int d\,\mathbf{v}_1\, d\,\mathbf{v}_2\ \mid e_z \cdot (\mathbf{v}_2 - \mathbf{v}_1) \mid$$
$$\times \left[f\,(\mathbf{v}_1', t)\, f\,(\mathbf{v}_2', t) - f\,(\mathbf{v}_1, t)\, f\,(\mathbf{v}_2, t) \right]$$
$$\times \left(\frac{m}{2} v_{1\perp}^2 + \frac{m}{2} v_{2\perp}^2 - \frac{m}{2} v_{1\perp}'^{\,2} - \frac{m}{2} v_{2\perp}'^{\,2} \right) \quad .$$

If we assume that the distribution functions are of the form

$$f\,(\mathbf{v}, t) = \left(\frac{m}{2\pi T_\parallel}\right)^{1/2} \left(\frac{m}{2\pi T_\perp}\right) \exp\left(- \frac{m v_\parallel^2}{2T_\parallel} - \frac{m v_\perp^2}{2T_\perp}\right)$$

and furthermore change variables in the integral to relative and center of mass velocities, the center of mass part can be integrated out, and we obtain

$$\frac{dT_\perp}{dt} = \frac{n}{4} \int 2\pi\, \rho\, d\rho \int d\,\mathbf{v}\ \mid v_\parallel \mid$$
$$\times \left(\frac{m}{4\pi T_\parallel}\right)^{1/2} \left(\frac{m}{4\pi T_\perp}\right) \exp\left(- \frac{m v_\parallel^2}{2T_\parallel} - \frac{m v_\perp^2}{2T_\perp}\right)$$
$$\times \left(\exp\left[\left(\frac{1}{T_\perp} - \frac{1}{T_\parallel}\right) \Delta\left(\frac{\mu}{2} v_\perp^2\right)\right] - 1\right) \Delta\left(\frac{\mu}{2} v_\perp^2\right) \quad .$$

Taylor expanding the exponential, integrating out $v_\perp$, and inserting the expressions (7) and (9) finally yields

$$\frac{dT_\perp}{dt} = (T_\parallel - T_\perp)\, n \bar{b}^2 \bar{v}_\parallel \cdot I(\bar{\epsilon}) \quad ,$$

where barred quantities involve thermal velocities, and

$$I(\bar{\epsilon}) = \frac{\sqrt{2\pi}}{24} \int_0^\infty \frac{d\,\epsilon}{\epsilon} \int_0^\infty \tilde{\rho}\ d\,\tilde{\rho}\ \exp\left(- \frac{1}{2} (\epsilon / \bar{\epsilon})^{2/3}\right) h^2(\tilde{\rho}, \epsilon) \exp\left(-2g(\tilde{\rho})/\epsilon\right) \quad .$$

Using the saddle-point method yields the result $I(\bar{\epsilon}) \sim \exp\left(- \frac{2}{\bar{\epsilon}^{2/5}}\right)$.

The main point to note is that the equilibration rate is larger than one might have guessed. Since the exchange of parallel and perpendicular energy for an isolated collision between two electrons is exponentially small in $(\epsilon)^{-1}$, one might have guessed that the equilibration rate would be exponentially small in $(\bar{\epsilon})^{-1}$. However, the equilibration rate turns out to be exponentially small in $(\bar{\epsilon}^{2/5})^{-1}$, and this destinction is important since $(\bar{\epsilon}^{2/5})^{-1} \ll (\bar{\epsilon})^{-1}$ for $\bar{\epsilon} \ll 1$.

The $(\bar{\epsilon}^{2/5})^{-1}$ dependence is determined by a competition between the velocity dependence of $\exp(-\pi/\epsilon)$ and the velocity dependence of the distribution of relative velocities, $\exp\left(- \frac{1}{2} (\epsilon / \bar{\epsilon})^{2/3}\right)$. Collisions characterized by large relative velocities are particularly effective at producing an exchange of parallel and perpendicular energy, but there are relatively few such collisions.

5. A Numerical Analysis.

The approximation (5) to the exact expression (4) for the change in perpendicular kinetic energy involves substituting the lowest-order orbits into the time integral. Although such an approximation is the traditional method of calculating the exponentially small change in an adiabatic invariant, it is not obvious that an approximation valid only to algebraic accuracy [i.e. $O(\epsilon)$] can accurately determine an exponentially small quantity. To investigate (5) and its analytical evaluation (9) one can integrate numerically the equations of motion keeping track of the kinetic energy in the parallel and in the perpendicular degrees of freedom.

Recall that the total perpendicular kinetic energy is proportional to an action associated with the gyrotron motion. The statement that the action is an adiabatic invariant is only strictly true in the $\epsilon \downarrow 0$ limit. For finite ϵ, the action is the first term in an asymptotic series for a "true" invariant, $\bar{J}$. This series can be constructed using Lie perturbation techniques.[6] For the system (3) one can obtain

$$\bar{J} = \frac{\mu}{2} v_\perp^2 - \epsilon \frac{\tilde{\rho} \frac{\mu}{2} v_\perp v_\parallel}{[\tilde{\rho}^2 + \tilde{z}^2]^{3/2}} \cos(\chi)$$

$$+ \epsilon^2 \left[\frac{3/8\, \mu\, v_\perp^2 \tilde{\rho}^2 \cos(2\chi)}{[\tilde{\rho}^2 + \tilde{z}^2]^{5/2}} - \frac{3/2\, \mu\, v_\perp v_\parallel(\tilde{z})\, \tilde{\rho} \sin(\chi)}{[\tilde{\rho}^2 + \tilde{z}^2]^{5/2}} + \frac{\mu/\rho\, v_\parallel^2\, \tilde{\rho}^2}{[\tilde{\rho}^2 + \tilde{z}^2]^3} \right] + O(\epsilon^3) \quad .$$

Here, χ is essentially the gyroangle in the relative motion. For a collision where both the initial and final state has $|z| = \infty$, we see that the change in $\bar{J}$ between these two limits is actually given by the change in $\frac{\mu}{2} v_\perp^2$, since higher order terms contain powers of the Coulomb denominator $[\tilde{\rho}^2 + \tilde{z}^2]^{-1/2}$. For finite $\tilde{z}$, i.e. during the collision, $\frac{\mu}{2} v_\perp^2$ is not even approximately conserved. In an accurate computer simulation for a given ϵ (fig. 3a), we can monitor the evolution of the series with more and more terms added. For slightly higher ϵ (fig. 3b), we can even detect the asymptotic "breaking" of the adiabatic invariant, i.e. the finite change in $\frac{\mu}{2} v_\perp^2$.

For a number of such runs with different values of ϵ, one can plot $\Delta(\frac{\mu}{2} v_\perp^2)$ as a function of $1/\epsilon$ to check the exponential character; one such plot is shown in fig. 4. Computed values of $\Delta(\frac{\mu}{2} v_\perp^2)$ are compared to the expression (9); one can see that the agreement is quite good.

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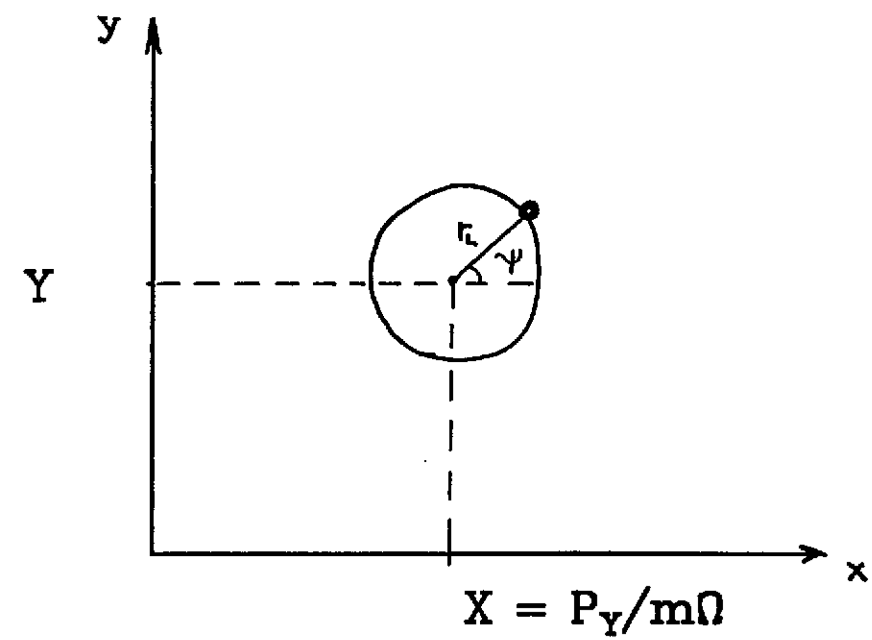


fig. 1 Guiding center (X,Y) and gyro-angle coordinates for a single electron.

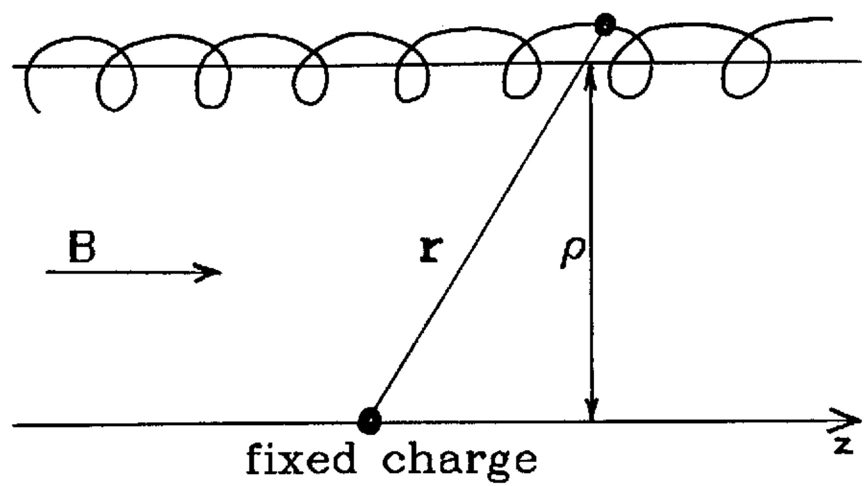


fig. 2 The reduced motion (eq. (3)) is equivalent to that of an electron in a uniform magnetic field and the field of a fixed charge.

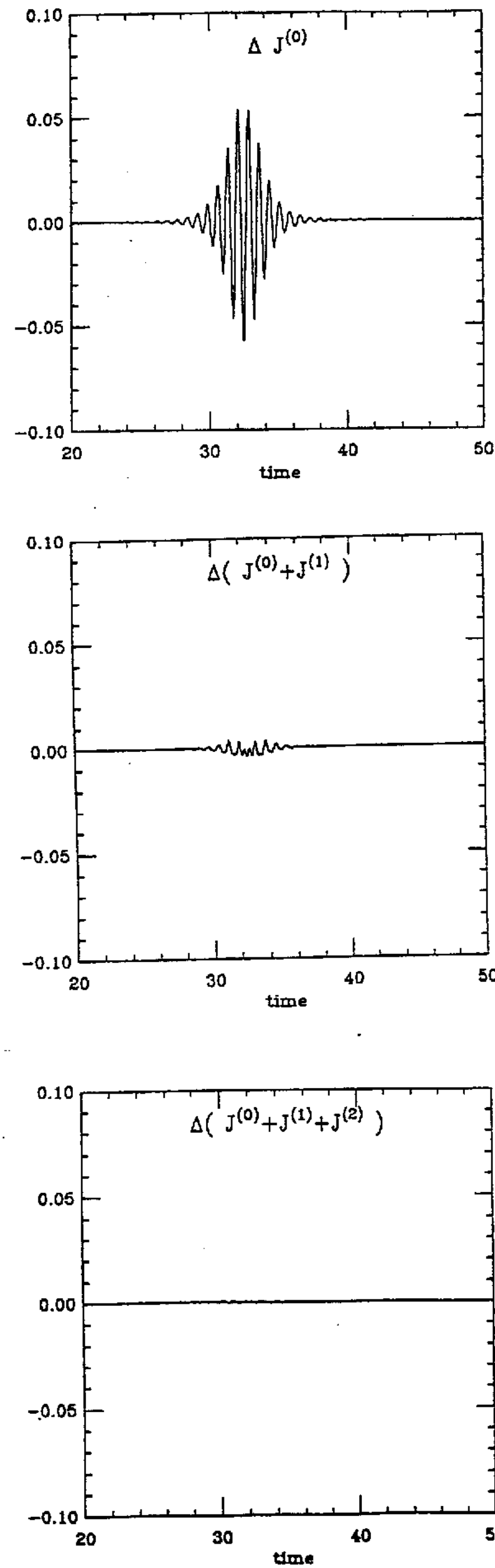
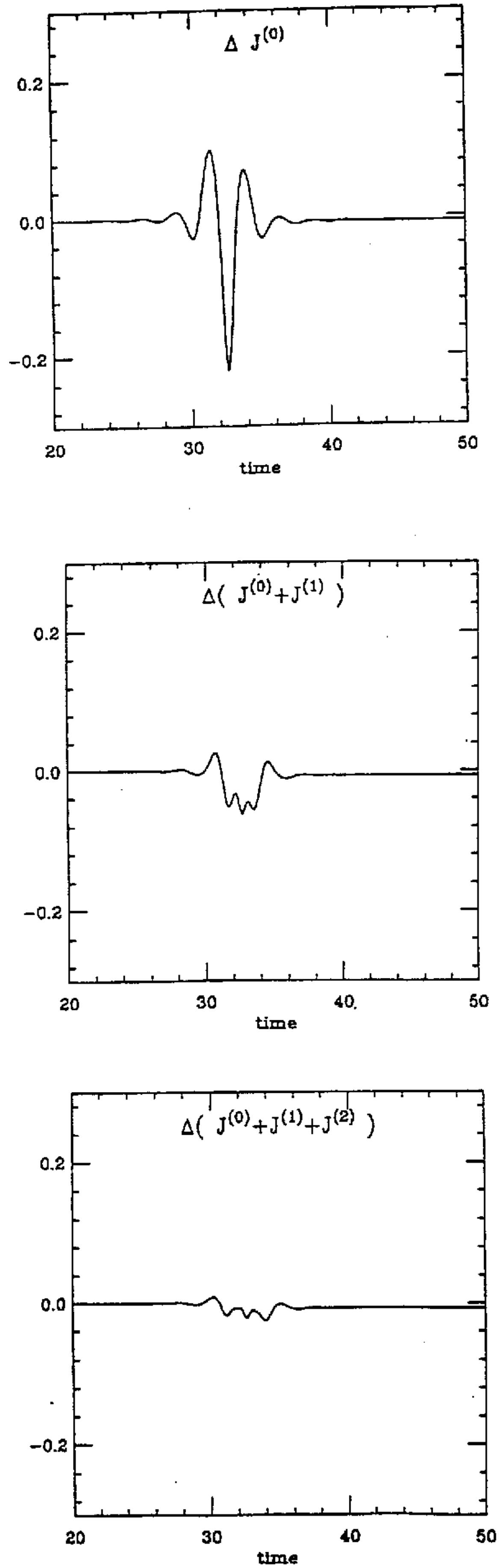
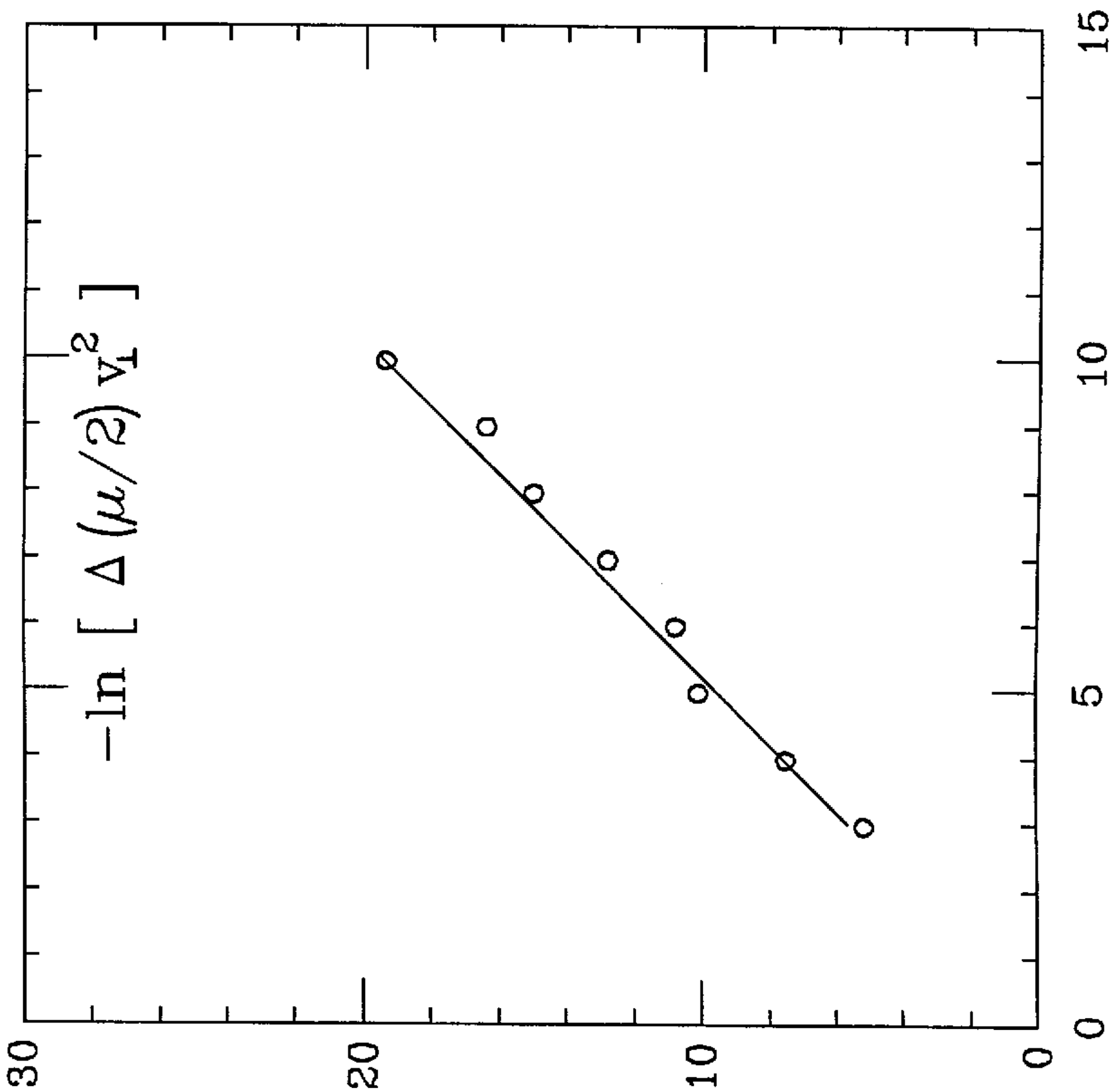


fig. 3a-b The variation of the asymptotic series expression for J as more terms are added.



Δ J(0)
0.2
0.0
−0.2
20
30
40
50
time
Δ(J(0)+J(1))
0.2
0.0
−0.2
20
30
40
50
time
Δ(J(0)+J(1)+J(2))
0.2
0.0
−0.2
20
30
40
50
time



<u>fig. 4</u> The analytical expression (9) - the solid line - compared to $(\frac{\mu}{2} v_\perp^2)$ - the o's - computed by direct integration of the equations of motion, for several values of ε.