

Classical Derivation of Manley Rowe relation for 3-wave instability of a pump wave

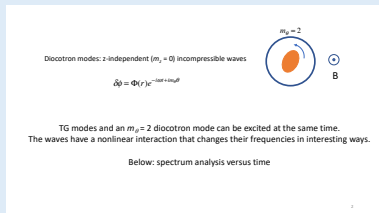
pump frequency ω_0, θ' wavenumber m_0

resonant daughter waves have frequencies $\omega_1 = \omega_2 = \omega_0$ in the frame of the pump wave and θ' wavenumbers m_1, m_2 , respectively, with $m_1 = \pm m_0$ for a 3 wave instability in the lab frame the daughter wave frequencies are $\omega_1^* = \omega_0 + m_1 \frac{\partial \omega_0}{\partial m_0}, \omega_2^* = \omega_0 + m_2 \frac{\partial \omega_0}{\partial m_0}$

so the resonance condition in the lab frame is $\omega_1^* - \omega_2^* = (\omega_0 - m_0 v_{\theta'}) \frac{\partial \omega_0}{\partial m_0} = \pm 2m_0 v_{\theta'} \frac{\partial \omega_0}{\partial m_0}$

instability condition: in frame of pump wave the daughter wave energies E_1, E_2 are opposite in sign: $E_1 E_2 < 0$ if positive energy wave gains energy resonantly from negative energy wave, both waves grow in amplitude

translate this condition to the lab frame using Galilean transformation of wave energy $\frac{dE}{dt} = \frac{dE^*}{dt} + v_{\theta'} \frac{dE^*}{d\theta'}$

$$E_1 E_2 = E_1^* E_2^* \frac{\partial E_1^* / \partial \omega_1^*}{\partial E_1^* / \partial \omega_1^*} \frac{\partial E_2^* / \partial \omega_2^*}{\partial E_2^* / \partial \omega_2^*} < 0$$


The 3 nearly degenerate TG waves are strongly mixed by $m_0=2$ diocotron "pump" with low frequency ω_0

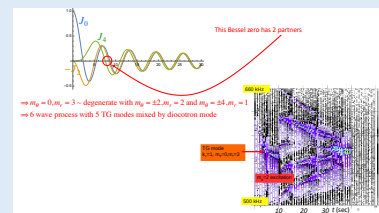
For infinite B the mixing is simplified: for diocotron mode which has potential $\propto \cos 2\theta'$, where $\theta' = \theta - \omega_0 t / 2$,

$$\delta\Phi_1 = \cos \pi z / L [A_0 \Phi_0(r) + B_0 \Phi_2(r) \cos 2\theta'] e^{-i m_0 \omega_0 t}$$

$$\delta\Phi_2 = \cos \pi z / L [A_1 \Phi_1(r) + B_1 \Phi_3(r) \cos 2\theta'] e^{-i m_1 \omega_1 t}$$

$$\delta\Phi_3 = \cos \pi z / L [B_2 \Phi_1(r) \sin 2\theta'] e^{-i m_2 \omega_2 t}$$

- r, θ dependence similar to modes of a slightly elliptical drumhead
- For infinite B, θ antisymmetry of $\delta\Phi_3$ implies it does not couple to $m_0=0$ detector or excitor electrodes, which explains why observed central mode is weaker and its frequency is ω_3 independent of diocotron amplitude
- Upper and lower frequencies ω_1, ω_2 exhibit "avoided crossing" behavior, with $\omega_1 - \omega_2$ proportional to pump amplitude



if $E_1^* E_2^* > 0$, (usual case of positive energy daughter waves), then instability requires $\omega_1^* \omega_2^* < 0$.

say $\omega_1^* < 0, \omega_2^* > 0$.

then the resonance condition $\omega_1^* - \omega_2^* = \pm 2m_0 v_{\theta'}$ can be written as $\omega_1^* + 1 + \omega_2^* = \pm 2m_0 v_{\theta'}$: Manley Rowe condition

pump wave is highest frequency and "decays" into lower frequency daughter waves

(same condition obtained if instead $\omega_1^* > 0, \omega_2^* < 0$)

(same condition also obtained if both daughter waves are negative energy)

(or if the pump wave is negative energy)

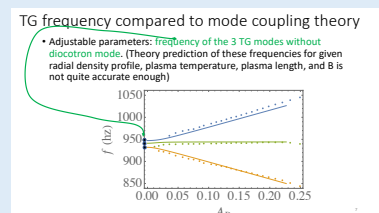
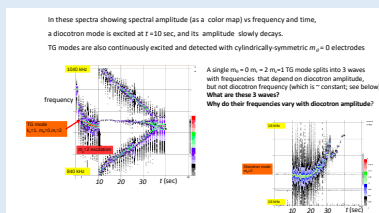
BUT if $E_1^* E_2^* < 0$, (one negative energy wave), then instability requires $\omega_1^* \omega_2^* > 0$.

say $\omega_1^* \geq \omega_2^* > 0$.

then the resonance condition $\omega_1^* - \omega_2^* = \pm 2m_0 v_{\theta'}$ can be written as $\omega_1^* + 1 - \omega_2^* = \pm 2m_0 v_{\theta'}$: modified Manley-Rose condition

pump wave is NOT the highest frequency (negative energy "photon" absorbed rather than emitted)

(same condition also obtained if $\omega_1^* \geq \omega_2^* > 0$)



These TG mode degeneracies also predict decay instabilities involving the diocotron mode.

Example:

TG pump wave: $m_0 = -1, m_1 = 1, m_2 = 1$, frequency ω_0^*

TG daughter wave 1: $m_0 = 1, m_1 = 1, m_2 = 1$, frequency $\omega_1^* = \omega_0^*$ (large magnetic field)

Daughter wave 2: $m_0 = 2, m_1 = 0, m_2 = 1$, frequency $\omega_2^* = \omega_0^*$ (diocotron mode)

Prediction: On resonance, growth rate of daughters scales as $1/B^2$

NOTE: The usual Manley-Rose Frequency relation is NOT satisfied for this decay process (when viewed in lab frame)

$$|\omega_0^*| + |\omega_1^*| + |\omega_2^*| \text{ on resonance}$$

instead $|\omega_0^*| = |\omega_1^*| = |\omega_2^*|$

So for this decay process the pump wave is NOT the highest frequency wave

The correct frequency resonance relation follows from a more general relation, that for instability the daughter wave energies must be opposite in sign in the frame of the pump wave, and their frequencies must be equal in that frame.

Manley-Rose is NOT satisfied because the diocotron mode is a negative energy wave (in the lab frame); Manley-Rose works only for positive energy daughter waves

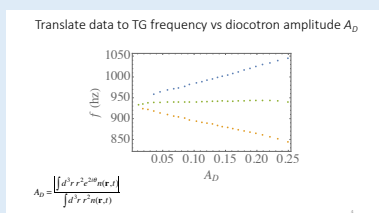
"Quantum" approach to Manley-Rose relation gives same answer

- Let Pump wave action I_0 be reduced by one unit in decay process: $\Delta I_0 = -\hbar = \frac{\Delta E_0}{1\omega_0}$
- and daughter wave actions I_1, I_2 each increase by one unit (why? Shut up and calculate!):

$$M_1 = \hbar = \frac{\Delta E_1}{1\omega_1}, M_2 = \hbar = \frac{\Delta E_2}{1\omega_2}$$

Assume this wave is negative energy

- Energy conservation in the decay: $-\Delta E_0 = \Delta E_1 + \Delta E_2$

$$\Rightarrow \hbar 1\omega_0 = -\hbar 1\omega_1 - \hbar 1\omega_2$$


Mode coupling of Trivelpiece-Gould and diocotron modes in a nonneutral plasma column

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Conclusions

- Plasma waves in nonneutral plasmas allow a great many mode degeneracies that lead to strong nonlinear mode coupling effects
- These effects produce frequency splittings as well as the possibility of decay instabilities, some of which involve the low-frequency diocotron modes
- Because diocotron modes are negative energy, the usual Manley-Rose relation for 3-wave decay instabilities involving these modes must be modified