UNIVERSITY OF CALIFORNIA, SAN DIEGO

Rotational Pumping Transport in Magnetized, Non-Neutral Plasmas

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by

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This thesis is dedicated to Mom and Dad.

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Finally, I am most grateful for the love and support of my wife, Lisa. She has given my life balance and has made the last five years happy ones. She has patiently listened to many of my ideas and has helped me to clarify them. Her simple questions about physics were often the most difficult to answer.

Vita, Publications and Fields of Study

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Publications

1. H. Donis-Keller, P. Green, C. Helms, S. Cartinhour, B. Weiiffenbach, K. Stephens, T. P. Keith, D. W. Bowden, D.R. Smith, E. S. Lander, D. Botstein, G. Akots, K. S. Rediker, T. Gravius, V. A. Brown, M. B. Rising, C. Parker, J.A. Powers, D.E. Watt, E. R. Kauffman, A. Bricker, P. Phipps, H. Muller-Kahle, T. R. Fulton, S. Ng, J. W. Schumm, J. C. Braman, R. G. Knowlton, D. F. Barker, S. M. Crooks, S.E. Lincoln, M. J. Daly, and J. Abrahamson, "A Genetic Linkage Map of the Human Genome", Cell **51**, 319 (1987)

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3. S.M. Crooks and T.M. O'Neil, "Rotational Pumping and Damping of the $m = 1$ Diocotron Mode" "A New Theory for the Damping of the $l = 1$ Diocotron Mode in Non-Neutral Plasmas" *Bull. Am. Phys. Soc.* **38**, 1972 (1993)

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Fields of Study

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Abstract of the Dissertation

Rotational Pumping Transport in Magnetized, Non-Neutral Plasmas

by

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Rotational pumping causes cross-field transport in non-neutral plasmas when the end confinement potentials are non-axisymmetric. Because the Debye length is small the asymmetries are screened out within the plasma, but cause the end shape of the plasma to distort. The basic idea can be understood by considering a single flux tube of plasma. As the flux undergoes $\mathbf{E} \times \mathbf{B}$ drift rotation about the center of the column, the length of the tube oscillates about some mean value, and this produces a corresponding oscillation in $T_{\|}$. In turn, the collisional relaxation of $T_{\|}$ toward $T_{\perp}$ produces a slow dissipation of electrostatic energy into heat and a consequent radial expansion of the plasma. Formally, the transport is calculated by solving the drift-kinetic Boltzmann equation in the limit where the axial bounce frequency of a thermal particle is large compared to its $\mathbf{E} \times \mathbf{B}$ drift rotation frequency and in the limit where bounce-rotation resonances are the dominate effect.

The transport is applied to three problems. When the asymmetry is produced by displacing the plasma column off-axis, that is by creating an $m = 1$ diocotron

mode, conservation of total canonical angular momentum is used to relate rotational pumping transport to a damping rate. The predicted damping rate is in good agreement with the experimental observations of Cluggish and Driscoll. Next, the theory is generalized to include time-dependent confinement potentials. When the potentials take the form of an asymmetry that rotates more rapidly than the plasma, the particle flux is directed radially inward and the plasma is compressed. Finally, transport for a non-neutral plasma confined in a toroidal magnetic field is calculated. In this geometry, the transport can be understood by considering a toroidal flux tube of plasma that undergoes a poloidal $\mathbf{E} \times \mathbf{B}$ drift. As the flux tube drifts, its length oscillates. Since the toroidal action $I = (2\pi)^{-1} \oint dl \, mv_{\parallel}$ is a good adiabatic invariant, $T_{\parallel}$ oscillates. Also, the magnetic field is non-uniform in this geometry, and conservation of the adiabatic invariant $\mu = mv_{\perp}{}^2/2B$ implies that $T_{\perp}$ oscillates. These oscillations have different magnitudes and the collisional relaxation of $T_{\parallel}$ toward $T_{\perp}$ produces a slow dissipation of electrostatic energy into heat and a corresponding cross-field transport of the plasma.

Chapter 1

General Introduction

This thesis discusses a new transport theory for non-neutral plasmas. By analogy with magnetic pumping the transport mechanism is called rotational pumping. The theory is applied to three problems of current interest: in Chapter 2 the theory is used to explain the observed damping of the $m = 1$ diocotron mode[1], in Chapter 3 inward transport due to a rotating asymmetry is described and in Chapter 4 rotational pumping transport is calculated for a plasma confined in toroidal geometry. These chapters are presented as free standing papers. This chapter contains an overview of rotational pumping transport and its applications.

1.1 Overview of Rotational Pumping Transport

In typical experiments the plasma is confined in a Penning trap as shown in Figure 1.1. A conducting cylinder is divided axially into three sections, the two end sections being held at a negative (positive) potential relative to the central section. There is a uniform magnetic field directed along the axis of the cylinder. The electron (ion) plasma resides in the central section with axial confinement provided by the end sections and radial confinement by the magnetic field. The Larmor radius is typically small so the cross-field motion may be described by $\mathbf{E} \times \mathbf{B}$ drift dynamics[2, 3].

The most important concepts necessary for understanding transport in these

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plasmas are angular momentum and torque. The canonical angular momentum of a single particle is

$$p_\theta = m v_\theta r + \frac{e}{c} A_\theta(r) r \qquad (1.1)$$

where v_θ is the θ component of the particle's velocity, r is the radial position of the particle, and e carries a sign. For a uniform magnetic field the θ component of the vector potential is $A_\theta(r) = Br/2$. In guiding center theory, the mechanical part of the angular momentum is negligible compared to the angular momentum in the field[4] and Eq. (1.1) may be rewritten as

$$p_\theta = \frac{eB}{2c} r^2. \qquad (1.2)$$

Torques acting on an individual particle cause its angular momentum and radial position to change.

The total canonical angular momentum of the plasma is

$$P_\theta = \sum_{j=1}^{N} \frac{eB}{2c} r_j^2 = \frac{eB}{2c} N \langle r^2 \rangle. \qquad (1.3)$$

where $\langle r^2 \rangle$ is the mean square radius of the plasma. Internal torques cause a rearrangement of the plasma which conserves P_θ and $\langle r^2 \rangle$. External torques cause radial particle transport which does not conserve angular momentum and results in a net expansion of the plasma column. At high neutral pressures the dominant external torque arises from neutrals which act as stationary scattering centers in the lab frame[5 − 7]. At lower neutral pressures the dominant torque is believed to be due to azimuthal asymmetries in the confining fields[5, 7]. In this thesis the transport due to asymmetries in the end confinement potentials is calculated.

Within the plasma, the end confinement potentials fall of exponentially on the scale of a Debye length[8]. When the Debye length is small the non-axisymmetric end confinement potentials have the effect of changing the shape of the plasma. This is an important starting point in the theoretical development. The shape of the

plasma is assumed to be known and is expressed in terms of the length of the plasma parallel to the magnetic field

$$L = L_0(r) + \delta L(r, \theta) \tag{1.4}$$

as shown in Fig. 1.2. Here (r, θ) is a cylindrical coordinate system centered on an axis through the center of charge. The length may be estimated by considering the shape of the vacuum equipotential contours [9] or may be calculated numerically with a 3-D equilibrium Poisson solver [1].

Since the Debye length is small the plasma particles are assumed to reflect specularly off each end of the plasma. When a particle reflects off a non-axisymmetric surface it experiences a force in the $\hat{\theta}$ direction which causes its canonical angular momentum $p_\theta = \frac{eB}{2c} r^2$ and radial position to change. This is the fundamental process underlying rotational pumping transport. The transport is calculated in two limits: when the bounce frequency for a thermal particle is much larger than the $\mathbf{E} \times \mathbf{B}$ drift rotation frequency, adiabatic transport dominates; when these two frequencies become comparable, bounce rotation resonances increase the particle flux.

1.1.1   Adiabatic Transport

In the adiabatic limit a transport flux due to the end asymmetries can be derived by considering a single flux tube of plasma as shown in Figs. 1.3 and 1.4. The flux tube has length $L(r, \theta)$ as given by Eq. (1.4), cross section area δA, and contains δN particles. The dominant cross field motion of the flux tube is the $\mathbf{E} \times \mathbf{B}$ drift $\mathbf{v_D} = (c/B)\hat{z} \times \nabla\Phi$. We assume that the azimuthal asymmetries are small and so to lowest order the electric potential is of the form $\Phi = \Phi(r)$. The flux tube drifts in a circular orbit about the center of the column with the frequency

$$\omega_R = \frac{c}{Br} \frac{\partial \Phi}{\partial r}. \tag{1.5}$$

Setting $\theta = \omega_R t$ in Eq. (1.4) then implies that the length of the flux tube varies temporally as $L(r,t) = L_0(r) + \delta L(r, \omega_R t)$. The cyclic axial compression and expansion produces a cyclic variation in the parallel temperature, and this is coupled collisionally to the perpendicular temperature. The full temperature evolution is governed by the equations

$$\frac{dT_\parallel}{dt} = -T_\parallel \frac{2}{L}\frac{dL}{dt} + 2\nu_{\perp,\parallel}\left(T_\perp - T_\parallel\right) \ . \tag{1.6}$$

and

$$\frac{dT_\perp}{dt} = -\nu_{\perp,\parallel}\left(T_\perp - T_\parallel\right), \tag{1.7}$$

where $\nu_{\perp,\parallel}$ is the collisional equipartition rate. We have used the fact that δN is constant in deriving these equations. The first term on the right hand side of Eq. (1.6) describes the compressional heating (or expansion cooling) of the parallel degrees of freedom, and the second term describes the collisional coupling to the perpendicular degrees of freedom. The perpendicular degrees of freedom are not directly affected by the change in length, so the R.H.S. of Eq. (1.7) contains only the collisional coupling term. The factor of two difference in the collisional coupling term for Eq. (1.7) relative to Eq. (1.6) simply reflects the fact that there are 2 perpendicular degrees of freedom and 1 parallel.

A two time scale analysis of Eqs. (1.6) and (1.7) based on the smallness of $\delta L/L$ and on the frequency ordering $\omega_R \gg \nu_{\perp,\parallel}$ yields the result

$$\frac{d\langle T_\parallel \rangle}{dt} = 8\nu_{\perp,\parallel}\frac{\langle T_\parallel \rangle}{L_0^2}\langle \delta L^2 \rangle + 2\nu_{\perp,\parallel}\left(\langle T_\perp \rangle - \langle T_\parallel \rangle\right), \tag{1.8}$$

$$\frac{d\langle T_\perp \rangle}{dt} = -\nu_{\perp,\parallel}\left(\langle T_\perp \rangle - \langle T_\parallel \rangle\right), \tag{1.9}$$

where $\langle \cdot \rangle$ indicates an average over the fast time scale, that is over one rotation period. In addition to the energy conserving terms, the first term on the right hand side of Eq. (1.8) represents a secular increase in $T_\parallel$. Physically, this term arises because collisions cause a small phase shift in the parallel temperature fluctuations so

that the parallel temperature and pressure are slightly larger in the compression stage than in the expansion stage. More work is done on the plasma during compression than is done by the plasma during expansion. The result is that the plasma in the flux tube is heated. This effect is similar to magnetic pumping,[10] and by analogy, we refer to it as rotational pumping.

Since the confinement potentials are time independent, the total energy in the plasma is conserved, and the increase in thermal energy must be balanced by a corresponding decrease in the electrostatic energy. The particle flux is found by equating the increase in the thermal energy of the flux tube to local Joule heating. That is,

$$n\frac{d}{dt}\left(\frac{1}{2}\langle T_{\parallel}\rangle + \langle T_{\perp}\rangle\right) = -e\frac{\partial\Phi}{\partial r}\Gamma_r, \tag{1.10}$$

where Γ_r is the radial particle flux and n is the density. The R.H.S. of this equation is the Joule heating per unit volume, and again we have used the fact that $\delta N = CONST$. Equations (1.8)-(1.10) are solved for the flux and yield

$$\Gamma_r = 4\nu_{\perp,\parallel}n(r)\frac{T}{-e\partial\Phi/\partial r}\frac{\langle \delta L^2\rangle}{L_0^2}, \tag{1.11}$$

where

$$\langle \delta L^2\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \delta L^2(r,\theta). \tag{1.12}$$

It is striking that Γ_r depends on magnetic field strength only through $\nu_{\perp,\parallel}$. In the regime of weak magnetization (i.e., $r_c \gg b$, where $r_c = \bar{v}/\Omega_c$ and $b = e^2/m\bar{v}^2$), this dependence is very weak, $\nu_{\perp,\parallel} \propto \ln(r_c/b)$. In the regime of strong magnetization (i.e., $r_c \ll b$) $\nu_{\perp,\parallel}$ becomes exponentially small[11, 12] and the theory predicts that the flux becomes exponentially small. This scaling is a unique signature of rotational pumping transport in the adiabatic limit. A formal solution of the Drift-Kinetic Boltzmann equation in the adiabatic limit yields the same result found with this simple flux tube model.

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1.1.2 Resonant Particle Transport

When the bounce frequency of a thermal particle is comparable to the $\mathbf{E} \times \mathbf{B}$ drift rotation frequency, bounce-rotation resonances greatly enhance the transport. The basic idea behind resonant particle transport is easy to understand. When a particle is reflected off the non-axisymmetric end potential it experiences a force in the $\hat{\theta}$ direction causing its angular momentum, $p_\theta = \frac{eB}{2c}r^2$, and radial position to change. The magnitude and sign of the radial step depend on the particle's azimuthal position at the point of reflection. A resonant particle reflects at the same θ position for many bounces and consequently takes many radial steps in the same direction. For non-resonant particles the radial steps tend to cancel. A solution to the drift-kinetic Boltzmann equation in the "resonant-plateau" regime [13, 14] yields the result

$$\Gamma_r = 4 \left(\sum_{n,l} \frac{1}{2}\sqrt{\pi/2} \frac{(l\omega_R)^6}{(n\bar{\omega}_B)^5} \exp\left[-\frac{(l\omega_R)^2}{2(n\bar{\omega}_B)^2} \right] \right) \frac{|\delta L_l|^2}{L_0^2} \frac{T}{-e\partial\Phi/\partial r} n, \qquad (1.13)$$

where $\bar{\omega}_B = \frac{\pi}{L_0}\sqrt{T/m}$ is the bounce frequency for a thermal particle. Note that in this equation we have assumed that only one end of the plasma is non-axisymmetric. In chapter 2 both ends of the plasma are distorted and therefore in this equation $\bar{\omega}_B \to 2\bar{\omega}_B$.

The quantity in parenthesis replace $\nu_{\perp,\parallel}$ in the adiabatic flux. One can see that when $\bar{\omega}_B \gg \omega_R$ the resonant particle flux is negligible. The unusual scaling of this quantity with magnetic field may be understood physically as follows. A particle is resonant when $n\omega_B = l\omega_R$, or expressed in terms of its velocity

$$v_{res} = \frac{L}{\pi}\frac{l}{n}\omega_R \sim \frac{1}{B}. \qquad (1.14)$$

The exponential factor in Eq. (1.13) is just the Maxwellian distribution function evaluated at the resonant velocity. Simply stated, the transport is proportional to the number of particles that participate. In addition there is a coefficient that scales

as $\omega_R{}^6 \sim B^{-6}$. One should expect a diffusion coefficient to scale as the square of a step size. Here a resonant particle takes an irreversible step when it undergoes a collision and becomes non-resonant. The number of bounces a particle makes before undergoing a collision scales as the bounce frequency divided by the collision frequency

$$\#\text{Bounces} \sim \frac{\omega_B}{\nu} \sim \frac{\omega_R}{\nu} \sim \frac{1}{B}. \tag{1.15}$$

In each reflection, a particle takes a small radial step which is proportional to the impulse required to reflect the particle and, since this is an $\mathbf{E} \times \mathbf{B}$ drift, inversely proportional to the magnetic field. The impulse is proportional to the particles momentum and therefore the radial step for each bounce scales as

$$\frac{\Delta r}{\#\text{Bounces}} \sim \frac{1}{B} m v_{res} \sim \frac{1}{B^2}. \tag{1.16}$$

The irreversible step that a particle takes after a collision is

$$\Delta r \sim \#\text{Bounces} \left(\frac{\Delta r}{\#\text{Bounces}} \right) \sim \frac{1}{B^3}. \tag{1.17}$$

The square of the step size and the diffusion coefficient scale as B^{-6} as is evident in Eq. (1.13).

1.2 The $m = 1$ Diocotron Mode

The first application of rotational pumping transport is a calculation of the damping rate of the $m = 1$ diocotron mode[1, 2, 15 − 17]. One can think of this mode as a rigid displacement of the plasma column away from the central axis of the trap. The image charges induced in the conducting wall cause the column as a whole to $\mathbf{E} \times \mathbf{B}$ drift about the central axis of the trap while the space charge field causes the column to rotate about an axis through its center of charge.

In a frame that rotates with the mode the off-axis column is a stationary state (an equilibrium) except for the slow evolution on the transport timescale. The

connection between damping and transport follows from conservation of angular momentum. As shown in Figs. 1.5 and 1.6 $\mathbf{D}$ is defined as the displacement of the center of charge and $\mathbf{r_j}$ is measured from the center of charge ($\mathbf{R_j} = \mathbf{D} + \mathbf{r_j}$). The canonical angular momentum from Eq. (1.3) can then be written as

$$P_\theta \simeq \frac{eB}{2c} N \left[D^2 + \frac{1}{N} \sum_{j=1}^{N} r_j^2 \right],$$
(1.18)

Note that the cross term, $\sum_{j=1}^{N} 2\mathbf{r_j} \cdot \mathbf{D}$, vanishes because $\mathbf{D}$ defines the center of charge. Since the apparatus is cylindrically symmetric, P_θ is conserved. This implies a relation between plasma expansion and mode damping. Differentiating Eq. (1.18) with respect to time yields the relation

$$\frac{\partial}{\partial t}(D^2) = -\frac{\partial}{\partial t} \left[\frac{1}{N} \sum_{j=1}^{N} r_j^2 \right] = -\frac{\partial}{\partial t} \langle r^2 \rangle.$$
(1.19)

which can be written in terms of a particle flux as

$$\frac{\partial}{\partial t} D = \left[-\frac{1}{2D^2} \frac{1}{N} \int d^3r \, r\Gamma_r \right] D.$$
(1.20)

The quantity in brackets is the damping rate, γ. Internal torques which conserve angular momentum in the frame centered on the plasma cancel out in this integral. Note that the plasma is also in a sheared field in the off-axis frame which causes its cross section to be slightly non-circular. These distortions are negligible when D is small.

The displaced plasma column sees non-axisymmetric end potentials and these cause the end shape of the plasma to be non-axisymmetric. This asymmetry is illustrated in Fig. 1.6 and can be characterized by the length of the plasma parallel to the magnetic field as given by Eq. (1.4). A simple analytic theory[9] suggests that the asymmetric component of the plasma length may be approximated by $\delta L(r, \theta) = \kappa \frac{D}{R_w} r \sin \theta$, where κ is a numerical constant of order 2.4. For a constant density, isothermal plasma in the the adiabatic regime, the integral expression in Eq. (1.20)

yields a damping rate

$$\gamma = -2\kappa^2 \nu_{\perp,\parallel} \frac{\lambda_D{}^2 r_p{}^2}{L_0^2 R_w{}^2} \frac{1}{(1 - r_p{}^2/R_w{}^2)} \tag{1.21}$$

where $\lambda_D = \sqrt{T/4\pi e^2 n}$ is the Debye length, r_p is the radius of the plasma and R_w is the radius of the conducting cylinder.

Fig. 1.7 contains a plot of Cluggish and Driscoll's experimentally measured damping rate vs. temperature[1] along with the predicted damping rate from Eq. (1.21) (the dashed curve). Cluggish and Driscoll have also numerically calculated the damping rate by using an equilibrium Poisson solver to determine the length of the plasma, and then numerically integrating the expression for the damping rate in Eq. (1.20) with the adiabatic particle flux from Eq. (1.11). This numerically calculated damping rate is indicated in the figures by γ_{num} (the solid curve). The damping rate decreases dramatically when the plasma becomes strongly magnetized ($r_c < b$) because $\nu_{\perp,\parallel}$ becomes exponentially small in this regime[11,12]. In Fig. 1.8 the damping rate is plotted as a function of magnetic field strength in the regime of weak magnetization where $\nu_{\perp,\parallel}$ and γ depend weakly on the strength of the magnetic field, $\nu_{\perp,\parallel}, \gamma \propto \ln(r_c/b)$. The close agreement between theory and experiment is convincing evidence of rotational pumping transport.

1.3   The "Rotating Wall"

In Chapter 3 rotational pumping theory is applied to the "rotating wall" problem. This refers to a technique of using time-dependent confinement voltages to produce inward radial transport and compression of a non-neutral plasma[18 − 22]. When the voltages are applied in such a way as to simulate an asymmetry that rotates faster than the plasma the resultant torques cause inward transport. In Chapter 3 general thermodynamic arguments are used to show that the plasma is transported inward so long as the asymmetry rotates faster than the plasma. This result can also

be understood in terms of a simple heuristic argument. An asymmetry that rotates more slowly than the plasma exerts a drag on the plasma and causes $\mathbf{F} \times \mathbf{B}$ drifts that are directed radially outward. An asymmetry that rotates more rapidly than the plasma exerts a drag in the same direction that the plasma rotates and causes $\mathbf{F} \times \mathbf{B}$ drifts that are directed radially inward.

Motivated by the good agreement between theory and experiment found for the damping of the $m = 1$ diocotron mode, we consider the case where time-dependent voltages are applied at one end of the plasma. An expression for the radial particle flux can be derived easily for the case that the potential on one of the confinement cylinders has the form

$$\Phi_c = V_c + \delta V \cos[m\theta - \omega t]. \tag{1.22}$$

Here V_c is a constant voltage that provides confinement and the time dependent voltage is in the form of an asymmetry rotating at frequency ω/m. The calculation is easiest if we work in a frame that rotates with the asymmetry. In this frame the confinement potentials are static and we can use the results of the rotational pumping calculation valid for static end asymmetries.

In the adiabatic limit the flux is

$$\Gamma_r = 4\nu_{\perp,\parallel} n\langle r \rangle \frac{T}{-e\partial\Phi'/\partial r} \frac{\langle \delta L^2 \rangle}{L_0^2}, \tag{1.23}$$

where Φ' is the potential in the frame that rotates with the asymmetry. The transformation to the rotating frame gives a $1/c$ $\mathbf{v} \times \mathbf{B}$ contribution to the electric field so that

$$-\frac{\partial\Phi'}{\partial r} = -\frac{\partial\Phi}{\partial r} + \frac{Br}{c}\frac{\omega}{m} \tag{1.24}$$

where Φ is the potential in the lab frame and is related to the density through Poisson's equation. Using this expression the flux may be written as

$$\Gamma_r = 4\nu_{\perp,\parallel} n(r) \frac{T}{\frac{-eBr}{c}[\omega_R - \omega/m]} \frac{\langle \delta L^2 \rangle}{L_0^2}, \tag{1.25}$$

where we have used $\omega_R = c/Br \; \partial\Phi/\partial r$. For a pure electron plasma with $\omega/m > \omega_R$ the flux is negative, that is, the plasma is compressed.

The inward particle flux implied by Eq. (1.25) has a simple physical interpretation in terms of conservation of energy. Rotational pumping causes the plasma to heat. Since energy is conserved in the frame that rotates with the asymmetry, the electrostatic energy must decrease as the plasma temperature increases. If ω is sufficiently large we see from Eq. (1.24) that the direction of the electric field is reversed in the rotating frame. To liberate electrostatic energy, the plasma shrinks rather than expands. A similar analysis can be used to calculate the flux due to resonant particles.

In recent experiments pure ion plasmas have been observed to be stationary for a period of days[21]. In these experiments the plasma is under the influence of an applied torque due to a rotating field asymmetry and an ambient torque due to collisions with neutrals and stationary field asymmetries. For the plasma to be stationary, the torques must cancel. In addition the heating caused by the applied torque must be balanced by a cooling mechanism. In chapter 3 we show that a stable stationary plasma is possible when the background torque and cooling is due to neutrals.

1.4   Transport in a Toroidally Confined Plasma

In chapter 4 the transport theory is modified to describe a non-neutral plasma in a toroidal magnetic field geometry[23]. O'Neil and Smith[24] recently showed that such a plasma can be stable if the dynamical frequencies are ordered as $\Omega_c \gg \bar{\omega}_T \gg \omega_E$. Here, Ω_c is the cyclotron frequency, $\bar{\omega}_T$ is the toroidal rotation frequency for a thermal particle ($\bar{\omega}_T$ replaces $\bar{\omega}_B$), and ω_E is the $\mathbf{E} \times \mathbf{B}$ drift rotation frequency in the poloidal direction. We assume this ordering here (the adiabatic

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limit), and take $\mu = mv_\perp{}^2/2B$ and $I = (2\pi)^{-1} \oint dl\, mv_\parallel = (2\pi)^{-1} mv_\parallel L(\rho)$ to be good adiabatic invariants. Here, $L(\rho) = 2\pi\rho$ is the length of a flux tube and ρ is the major radius. The main difference between this calculation and the calculation in chapters 2 and 3 is that the magnetic field is not uniform. The field strength is given by $B = B_0 \rho_0 / \rho$ where B_0 and ρ_0 are constants. Thus, as a flux tube undergoes poloidal drift rotation (and ρ varies cyclically about some mean value), both the parallel and the perpendicular temperature fluctuate. The temperature evolution is described by

$$\frac{\partial T_\parallel}{\partial t} = -\frac{2}{L}\frac{\partial L}{\partial t}T_\parallel + 2\nu_{\perp,\parallel}(T_\perp - T_\parallel) \qquad (1.26)$$

and

$$\frac{\partial T_\perp}{\partial t} = \frac{1}{B}\frac{\partial B}{\partial t}T_\perp - \nu_{\perp,\parallel}(T_\perp - T_\parallel) \qquad (1.27)$$

where $\nu_{\perp,\parallel}$ is the equipartition rate.

In the small inverse aspect ratio limit, a two-time scale analysis is again used to calculate the heating rate. Conservation of energy then leads to the particle flux

$$\Gamma_r = \frac{1}{2}\nu_{\perp,\parallel}n(r)\frac{T}{-e\partial\Phi/\partial r}\left(\frac{r}{\rho_0}\right)^2, \qquad (1.28)$$

where ρ_0 is the major radius at the plasma center and r is the minor radius measured from the plasma center. The reader may wish to refer to Fig. 4.1 in chapter 4 where these coordinates are illustrated. This simple calculation assumes that the intersections of the $\mathbf{E} \times \mathbf{B}$ drift surfaces with the $r\theta$-plane are circular. A kinetic treatment in poloidal action-angle variables yields a result which is valid for more general plasma profiles. This transport sets an upper limit on the confinement time of non-neutral plasmas confined in toroidal geometry.
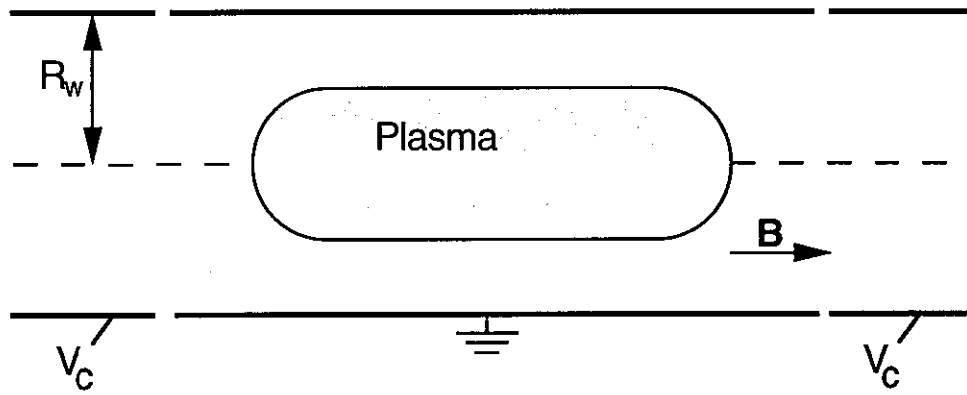
Figure 1.1: The Confinement Geometry
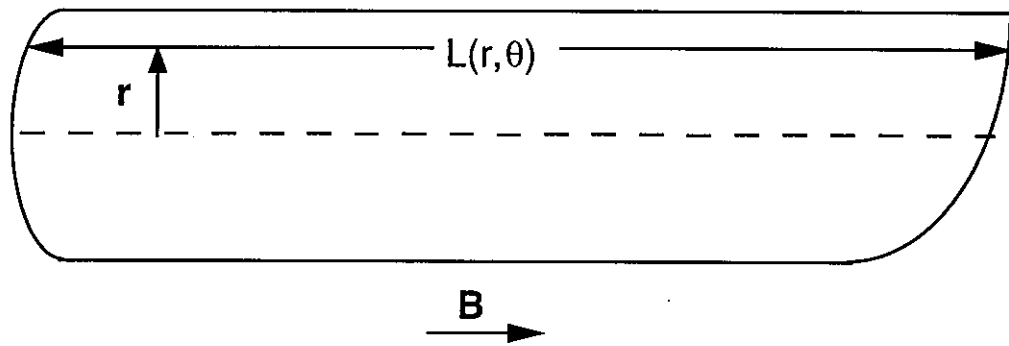
Figure 1.2: The Length of the Plasma
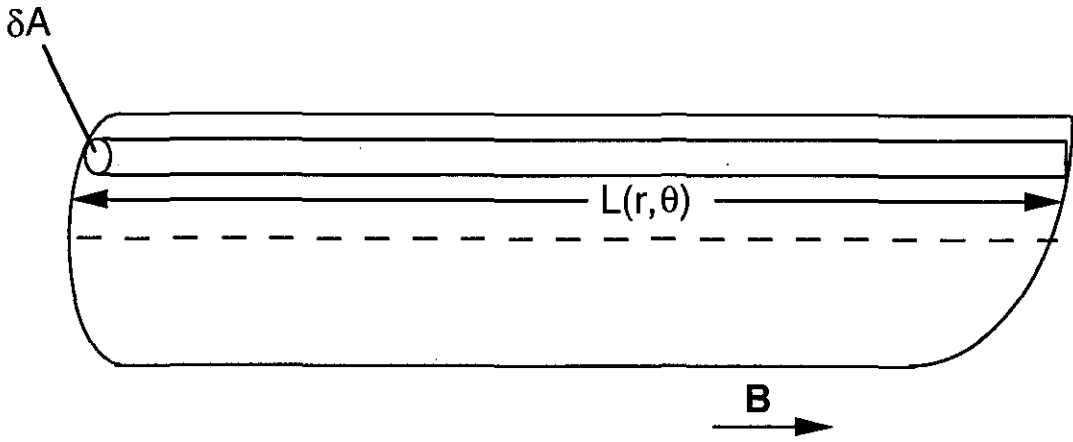
Figure 1.3: Side view of flux tube.
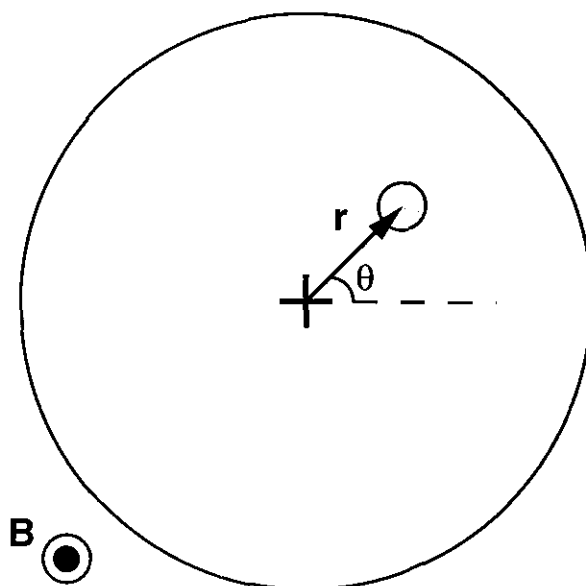
Figure 1.4: End view of flux tube.
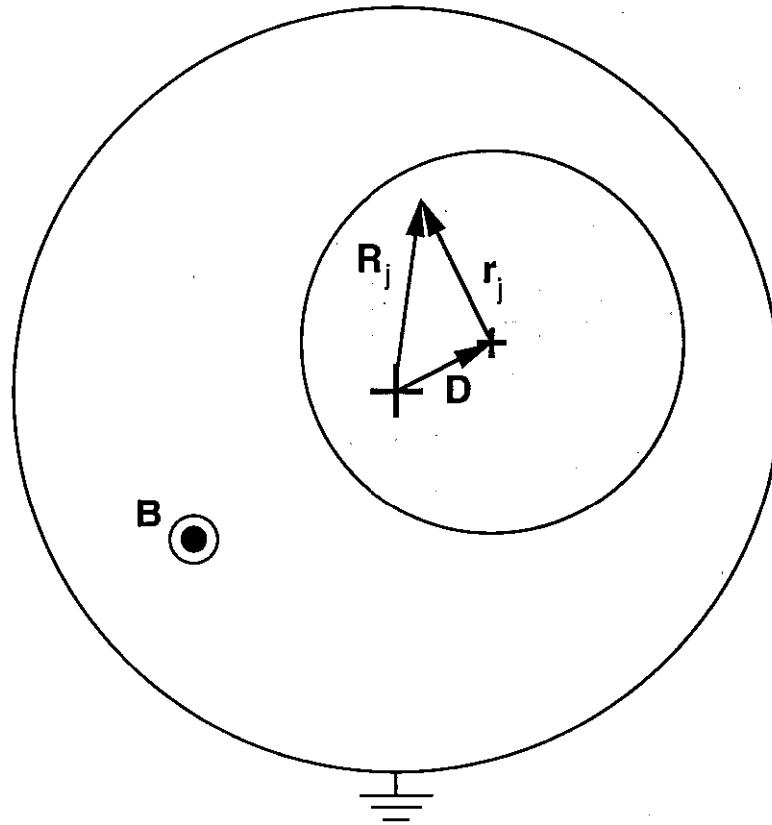
Figure 1.5: Coordinate system for the off-axis plasma (end view).
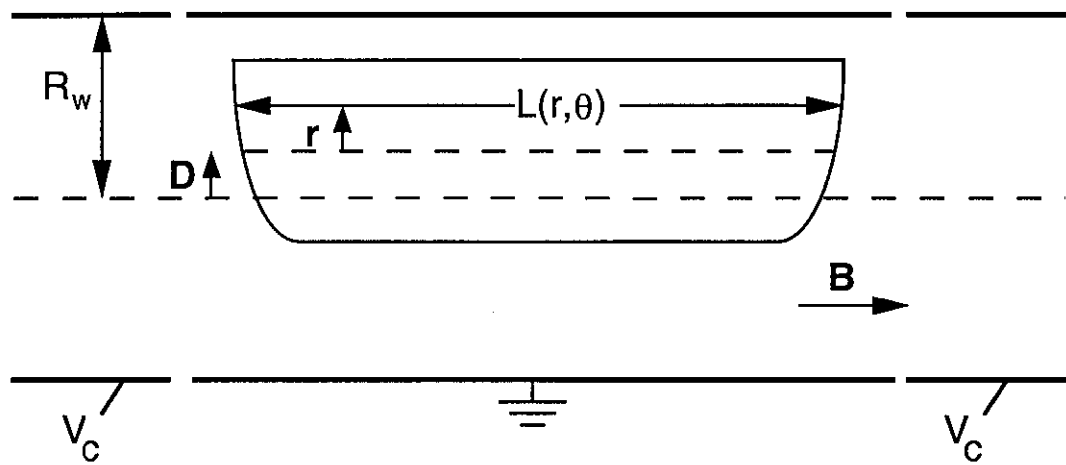
Figure 1.6: Length of the off-axis plasma.
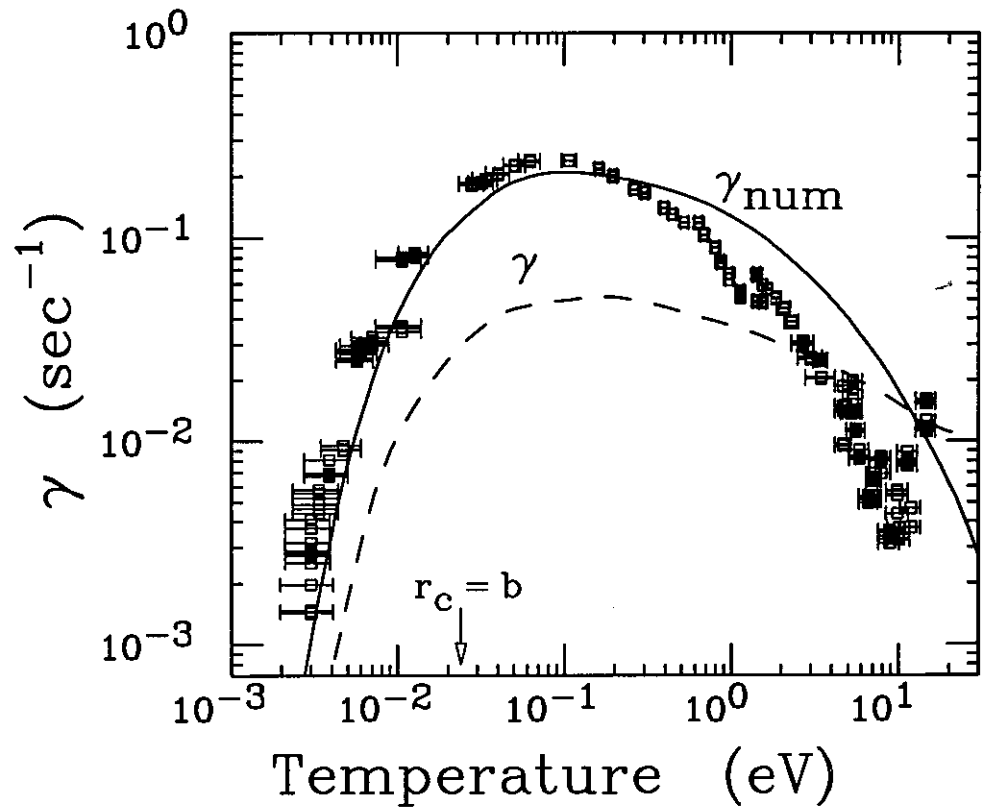
Figure 1.7: Damping rate of the $m = 1$ diocotron mode vs. temperature. Experimental measurements and numerical calculations (γ_{num}) by Cluggish and Driscoll[1]
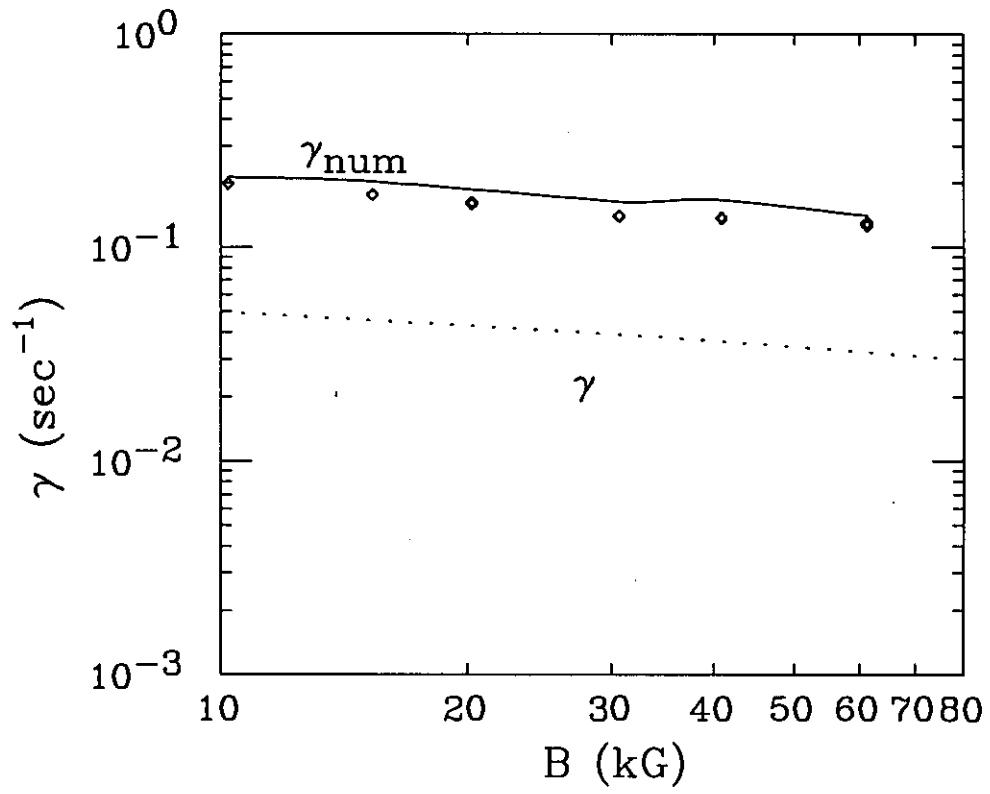
Figure 1.8: Damping rate of the $m = 1$ diocotron mode vs. magnetic field strength. Experimental measurements and numerical calculations (γ_{num}) by Cluggish and Driscoll[1]

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Chapter 2

Rotational Pumping and Damping of the m=1 Diocotron Mode

2.1 Abstract

An effect which we call rotational pumping (by analogy with magnetic pumping) causes a slow damping of the $m = 1$ diocotron mode in non-neutral plasmas. In a frame centered on the plasma and rotating at the diocotron mode frequency, the end confinement potentials are non-axisymmetric. As a flux tube of plasma undergoes $\mathbf{E} \times \mathbf{B}$ drift rotation about the center of the column, the length of the tube oscillates about some mean value, and this produces a corresponding oscillation in $T_{\parallel}$. In turn, the collisional relaxation of $T_{\parallel}$ toward $T_{\perp}$ produces a slow dissipation of electrostatic energy into heat and a consequent radial expansion (cross-field transport) of the plasma. Since the canonical angular momentum is conserved, the displacement of the column off-axis must decrease as the plasma expands. In the limit where the axial bounce frequency of an electron is large compared to its $\mathbf{E} \times \mathbf{B}$ drift rotation frequency theory predicts the damping rate $\gamma = -2\kappa^2 \nu_{\perp,\parallel}(r_p{}^2/R_w{}^2)(\lambda_D{}^2/L_0^2)/(1 - r_p{}^2/R_w{}^2)$ where κ is a numerical constant, λ_D is the Debye length, R_w is the radius of the cylindrical conducting wall, r_p is the

effective plasma radius, L_0 is the mean length of the plasma, and $\nu_{\perp,\parallel}$ is the equipartition rate. A novel aspect of this theory is that the magnetic field strength enters only through $\nu_{\perp,\parallel}$. As the field strength is increased, the damping rate is nearly independent of the field strength until the regime of strong magnetization is reached (i.e., $\Omega_c > \bar{v}/b = (kT)^{3/2}/\sqrt{m}\,e^2$), and then the damping rate drops off dramatically. This signature has been observed in recent experiments. For completeness the theory is extended to the regime where the bounce frequency is comparable to the rotation frequency and bounce-rotation resonances are included.

2.2 Introduction

Recent experiments have involved the confinement of pure electron plasmas in Penning traps.[1 − 4] A schematic diagram for such a trap is shown in Figure 2.1. A conducting cylinder is divided axially into three sections, the two end sections being held at a negative potential relative to the central section. There is a uniform magnetic field, B, directed along the axis of the cylinder. The electron plasma resides in the central section, with axial confinement provided by the negatively biased end sections and radial confinement by the magnetic field. The Larmor radius is typically small, so the cross field motion may be described by $\mathbf{E} \times \mathbf{B}$ drift dynamics.[3, 5]

The most commonly observed excitation of such a plasma is the diocotron mode of azimuthal wave number $m = 1$.[3, 6 − 8] One can think of this mode as a rigid displacement of the plasma column away from the central axis of the trap. The image charges induced in the conducting wall cause the column as a whole to $\mathbf{E} \times \mathbf{B}$ drift about the central axis of the trap while the space charge field causes the column to rotate about an axis through its center of charge. The main reason that this mode plays such a prominent role in the dynamics of pure electron plasmas is that it is damped only weakly; in typical experiments it is observed to survive 10^5 periods. In

spite of this, Cluggish and Driscoll have been able to measure the damping rate and characterize its parameter dependence over a wide range.[9] In this paper we present a theory of the damping which agrees with the parameter dependence observed in the experiments. This theory is closely related to the work of Ryutov and Stupukov on transport in magnetic mirror traps.[10]

Diocotron modes of azimuthal wave number $m > 1$ typically damp due to a wave-particle resonance.[11] The resonance is spatially localized at the resonant radius, r_s, defined by $m\omega_R(r_s) = w$, where ω_R is the single particle $\mathbf{E} \times \mathbf{B}$ rotation frequency and ω is the mode frequency. For a monotonically decreasing density profile, the $m = 1$ diocotron mode is special because $r_s = R_w$, the radius of the conducting wall.[11] There are no resonant particles because the density is zero at the wall and therefore a different mechanism is required to explain the observed damping of the $m = 1$ diocotron mode.

Since the damping timescale is characteristic of collisional transport timescales, we look for an explanation which involves collisional transport. It is convenient to work in a frame that rotates with the mode so that the off-axis column is a stationary state (an equilibrium) except for the slow evolution on the transport timescale. The connection between damping and transport follows from conservation of angular momentum. In guiding center theory, the canonical angular momentum of the plasma is approximately[12]

$$P_\theta \simeq \frac{eB}{2c} \sum_{j=1}^{N} R_j^2, \tag{2.1}$$

where $\mathbf{R_j}$ is the position of the jth particle measured from the trap axis (see Fig. 2.2) and e carries a sign. If $\mathbf{D}$ is the displacement of the center of charge and $\mathbf{r_j}$ is measured from the center of charge ($\mathbf{R_j} = \mathbf{D} + \mathbf{r_j}$) the canonical angular momentum can be written as

$$P_\theta \simeq \frac{eB}{2c} N \left[D^2 + \sum_{j=1}^{N} r_j^2 \right], \tag{2.2}$$

Note that the cross term, $\sum_{j=1}^{N} 2\mathbf{r_j} \cdot \mathbf{D}$, vanishes because $\mathbf{D}$ defines the center of charge. Since the apparatus is cylindrically symmetric, P_θ is conserved. This implies a relation between plasma expansion and mode damping. Differentiating Equation (2.2) with respect to time yields the relation

$$\frac{\partial}{\partial t}(D^2) = -\frac{\partial}{\partial t}\left[\frac{1}{N}\sum_{j=1}^{N} r_j^2\right] = -\frac{\partial}{\partial t}\langle r^2 \rangle. \tag{2.3}$$

Given a transport theory which describes the radial expansion of the plasma, Eq. (2.3) can be used to calculate the damping rate.

The angular momentum calculated about an axis through the center of charge is not conserved. If it were, $\langle r^2 \rangle$ would be constant in time and the mode would not damp. We therefore restrict our attention to transport processes which depend on the non-axisymmetric nature of the confining fields in a frame centered on the plasma. In particular, we consider the effect of the end confinement potentials.

When the Debye length is small the plasma has a well defined edge. A displaced column sees non-axisymmetric end potentials and these cause the end shape of the plasma to be non-axisymmetric. This asymmetry can be characterized by the length of the plasma parallel to the magnetic field,

$$L(r, \theta) = L_0(r) + \delta L(r, \theta), \tag{2.4}$$

where (r, θ) is a cylindrical coordinate system centered on an axis through the center of charge (see Fig. 2.3). A simple analytic theory as well as numerical studies[13] indicate that the asymmetric component is well represented by a uniform tilt at an angle proportional to D/R_w, where R_w is the radius of the conducting cylinder and D is the displacement of the center of charge off axis. Therefore the asymmetric part may be written as

$$\delta L(r, \theta) = \kappa \frac{D}{R_w} r \sin \theta, \tag{2.5}$$

where κ is a numerical constant.

A simple transport equation can be derived by considering a single flux tube of plasma as shown in Figs. 2.4 and 2.5. The flux tube has length $L(r, \theta)$ as given by Eq. (2.4), cross section area δA, and contains δN particles. The dominant cross field motion of the flux tube is the $\mathbf{E} \times \mathbf{B}$ drift $\mathbf{v_D} = (c/B)\hat{z} \times \nabla\Phi$. Assuming that the plasma column has a circular cross section the electric potential is of the form $\Phi = \Phi(r)$, and the flux tube drifts in a circular orbit about the center of the column with the frequency

$$\omega_R = \frac{c}{Br}\frac{\partial\Phi}{\partial r}. \tag{2.6}$$

Setting $\theta = \omega_R t$ in Eq. (2.4) then implies that the length of the flux tube varies temporally as $L(r, t) = L_0(r) + \delta L(r, \omega_R t)$. From Eq. (2.5) it then follows that the length of the flux tube undergoes a sinusoidal variation about the length $L_0(r)$. The cyclic axial compression and expansion produces a cyclic variation in the parallel temperature, and this is coupled collisionally to the perpendicular temperature. The full temperature evolution is governed by the equations

$$\frac{dT_\parallel}{dt} = -T_\parallel \frac{2}{L}\frac{dL}{dt} + 2\nu_{\perp,\parallel}\left(T_\perp - T_\parallel\right) \tag{2.7}$$

and

$$\frac{dT_\perp}{dt} = -\nu_{\perp,\parallel}\left(T_\perp - T_\parallel\right) \tag{2.8}$$

where $\nu_{\perp,\parallel}$ is the collisional equipartition rate. We have used the fact that δN is constant in deriving these equations. The first term on the right hand side of Eq. (2.7) describes the compressional heating (or expansion cooling) of the parallel degrees of freedom, and the second term describes the collisional coupling to the perpendicular degrees of freedom. The perpendicular degrees of freedom are not directly affected by the change in length, so the R.H.S. of Eq. (2.8) contains only the collisional coupling term. The factor of two difference in the collisional coupling term for Eq. (2.8) relative to Eq. (2.7) simply reflects the fact that there are 2 perpendicular degrees of freedom and 1 parallel.

A two time scale analysis of Eqs. (2.7) and (2.8) based on the frequency ordering $\omega_R \gg \nu_{\perp,\parallel}$ yields the result

$$\frac{d\langle T_\parallel \rangle}{dt} = 8\nu_{\perp,\parallel}\frac{\langle T_\parallel \rangle}{L_0^2}\langle \delta L^2 \rangle + 2\nu_{\perp,\parallel}\left(\langle T_\perp \rangle - \langle T_\parallel \rangle\right) \tag{2.9}$$

$$\frac{d\langle T_\perp \rangle}{dt} = -\nu_{\perp,\parallel}\left(\langle T_\perp \rangle - \langle T_\parallel \rangle\right) \tag{2.10}$$

where $\langle \cdot \rangle$ indicates an average over the fast time scale, that is over one rotation period. In addition to the energy conserving terms, the first term on the right hand side of Eq. (2.9) represents a secular increase in $T_\parallel$. Physically, this term arises because collisions cause a small phase shift in the parallel temperature fluctuations so that the parallel temperature and pressure are slightly larger in the compression stage than in the expansion stage. More work is done on the plasma during compression than is done by the plasma during expansion. The result is that the plasma in the flux tube is heated. This effect is similar to magnetic pumping,[14] and by analogy, we refer to it as rotational pumping.

Since the confinement potentials are time independent, the total energy in the plasma is conserved, and the increase in thermal energy must be balanced by a corresponding decrease in the electrostatic energy. The particle flux is found by equating the increase in the thermal energy of the flux tube to local Joule heating. That is,

$$n\frac{d}{dt}\left(\frac{1}{2}\langle T_\parallel \rangle + \langle T_\perp \rangle\right) = -e\frac{\partial \Phi}{\partial r}\Gamma_r, \tag{2.11}$$

where Γ_r is the radial particle flux and n is the density. The R.H.S. of this equation is the Joule heating per unit volume, and again we have used the fact that $\delta N = CONST$. Equations (2.9)-(2.11) are solved for the flux and yield

$$\Gamma_r = 4\nu_{\perp,\parallel}n(r)\frac{T}{-e\partial\Phi/\partial r}\frac{\langle \delta L^2 \rangle}{L_0^2}, \tag{2.12}$$

where

$$\langle \delta L^2 \rangle = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \delta L^2(r,\theta). \tag{2.13}$$

Finally, the damping rate is calculated by using conservation of angular momentum. After introducing the particle flux, Eq. (2.3) becomes

$$2D\frac{dD}{dt} = -\frac{\partial}{\partial t}\langle r^2 \rangle = -\frac{1}{N}\int d^3r\, 2r\Gamma_r. \tag{2.14}$$

Using Eq. (2.12) and the assumed form of the asymmetry from Eq. (2.5), we find the result

$$\frac{\partial}{\partial t}(D) = \left[-\frac{1}{N}\int d^3r\, 2\nu_{\perp,\parallel} n(r)\frac{T}{-e\frac{\partial \Phi}{\partial r}}\frac{\kappa^2}{L_0^2 R_w{}^2}r^3 \right] D. \tag{2.15}$$

The quantity in brackets is γ, the damping rate of the mode. In this equation it should be remembered that Φ is the potential in a frame rotating at the mode frequency ω_D. [6] It is related to the potential in the lab frame through

$$\Phi = \Phi_L - \frac{BR^2}{2c}\omega_D, \tag{2.16}$$

where the second term, $BR^2/2c\,\omega_D$, arises from the motion of the plasma column through the magnetic field. The potential in the lab frame, Φ_L, is composed of two parts: the space charge potential, Φ_0, and the potential due to the image charges induced on the conducting wall, Φ_I. Changing variables to a coordinate system centered on the plasma using $\mathbf{R} = \mathbf{D} + \mathbf{r}$ yields the result

$$\Phi = \Phi_0 - \frac{Br^2}{2c}\omega_D + \left[\Phi_I - \frac{B}{c}\mathbf{r}\cdot\mathbf{D}\,\omega_D\right] - \frac{BD^2}{2c}\omega_D. \tag{2.17}$$

The two terms in brackets cancel (this condition may be used to determine ω_D) and the last term may be dropped as it is just an additive constant. This leaves

$$\Phi = \Phi_0 - \frac{Br^2}{2c}\omega_D. \tag{2.18}$$

Of course, Φ_0 is related to the density through Poisson's equation.

For simplicity, we consider an isothermal, constant density plasma of radius r_p. The integral in Eq. (2.15) is then trivial and yields the result

$$\gamma = -2\kappa^2\nu_{\perp,\parallel}\frac{\lambda_D{}^2 r_p{}^2}{L_0^2 R_w{}^2}\frac{1}{(1 - r_p{}^2/R_w{}^2)} \tag{2.19}$$

where $\lambda_D = \sqrt{T/4\pi e^2 n}$ is the Debye length. It is striking that γ depends on magnetic field strength only through $\nu_{\perp,\parallel}$. In the regime of weak magnetization (i.e., $r_c \gg b$, where $r_c = \bar{v}/\Omega_c$ and $b = e^2/m\bar{v}^2$), this dependence is very weak, $\nu_{\perp,\parallel} \propto \ln(r_c/b)$. In the regime of strong magnetization (i.e., $r_c \ll b$) $\nu_{\perp,\parallel}$ becomes exponentially small[15, 16] and our theory predicts that γ becomes exponentially small.

Fig. 2.6 contains a plot of Cluggish and Driscoll's experimentally measured damping rate vs. temperature[9] along with the predicted damping rate from Eq. (2.19) (the dashed curve). Here, κ was taken to be 2.4 as given in reference [13]. Cluggish and Driscoll also numerically calculated the damping rate by using an equilibrium Poisson solver to determine the shape of the plasma more accurately, and then numerically integrating the expression for the damping rate in Eq. (2.15). This numerically calculated damping rate is indicated in the figures by γ_{num} (the solid curve). The dramatic decrease in the observed damping rate when r_c becomes small compared to b is rather convincing evidence that our theory focuses on the relevant damping mechanism. In Fig. 2.7 the damping rate is plotted as a function of magnetic field strength in the regime of weak magnetization where $\nu_{\perp,\parallel}$ and γ depend weakly on the strength of the magnetic field, $\nu_{\perp,\parallel}, \gamma \propto \ln(r_c/b)$. Other parameter dependences in Eq. (2.19) have also been checked by Cluggish and Driscoll and very good agreement between theory and experiment was found[9]. Moreover, they verified two of the fundamental assumptions underlying the theory: the canonical angular momentum and the total energy are nearly constant during the plasma evolution.

In Section 2.3 we present a more rigorous calculation of the transport by solving the drift-kinetic Boltzmann equation in the limit that $\omega_B \gg \omega_R$, where ω_B is the single particle bounce frequency parallel to the magnetic field, and ω_R is the rotation frequency. In Section 2.4 we consider the effect of bounce-rotation

resonances and show that in some regimes resonant particles enhance the damping rate.

2.3 Kinetic Treatment in the Adiabatic Limit

In this section and in Section III we assume the following frequency ordering:

$$\Omega_c \gg \omega_R, \omega_B \gg \nu \gg \gamma \qquad (2.20)$$

where Ω_c is the cyclotron frequency, ω_B is the axial bounce frequency, ω_R is the rotation frequency, ν is the collision frequency, and γ is the damping rate. Since Ω_c is the largest frequency, we may describe the collisionless single particle dynamics with a guiding center Hamiltonian of the form[16]

$$H = \frac{P_z^2}{2m} + \mu B + e\Phi(p_\theta) + e\Phi_e(\theta, p_\theta, z) \qquad (2.21)$$

where $p_\theta = eB/2cr^2$ is the canonical angular momentum conjugate to θ and (r, θ) is a cylindrical coordinate system centered on axis through the center of charge. We break up the potential into two parts: $\Phi(p_\theta)$ is the space charge potential in a frame rotating at the diocotron frequency and $\Phi_e(\theta, p_\theta, z)$ is the Debye-screened end potential. Since the Debye length is small we let

$$e\Phi_e(\theta, p_\theta, z) = \begin{cases} 0; & |z| < 1/2\, L(\theta, p_\theta) \\ \infty; & \text{otherwise} \end{cases} \qquad (2.22)$$

where $L(\theta, p_\theta)$ is the length of the plasma parallel to the magnetic field as discussed in the introduction. The term $\mu B = 1/2mv_\perp^2$ is the perpendicular kinetic energy of the particle. In the guiding center limit, $\mu = CONST.$, and since the magnetic field is assumed to be uniform, μB enters the Hamiltonian as an additive constant. We retain this term in the Hamiltonian because it is useful to write Maxwellian distribution functions as a function of H.

In the experiments, it is typically the case that the bounce frequency, $\omega_B = 2\pi |v_z|/2L$, is much larger than the rotation frequency, $\omega_R = \partial/\partial p_\theta(e\Phi)$, for the vast

majority of the particles. In this section we assume that this is true for all of the particles. (In Section III we allow for bounce-rotation resonances.) In the limit $\omega_B >> \omega_R$, the bounce action

$$I = \frac{1}{2\pi} \oint p_z dz = \frac{1}{2\pi} \oint \sqrt{2m \left(H - B\mu - e\Phi - e\Phi_e \right)}, \qquad (2.23)$$

is a good adiabatic invariant. As noted by J.B. Taylor,[17] an equation of this form implicitly defines H in terms of I, θ, and p_θ. Given the simple form of the end potential, this equation is easily inverted to give

$$H(I, \theta, p_\theta) = \frac{\pi^2 I^2}{2mL^2(\theta, p_\theta)} + B\mu + e\Phi(P_\theta) \qquad (2.24)$$

We represent the plasma with a distribution of guiding centers,

$$f = f(I, \psi, p_\theta, , \theta, \mu, t), \qquad (2.25)$$

where ψ is the angle conjugate to I and indicates the phase of a particle in its bounce motion (i.e. its position along the magnetic field). This distribution function evolves according to the drift-kinetic Boltzmann equation,

$$\frac{\partial f}{\partial t} + [f, H] = C(f), \qquad (2.26)$$

where $C(\cdot)$ is the collision operator, and the Poisson-bracket is given by

$$[f, H] = \frac{\partial f}{\partial \psi} \frac{\partial H}{\partial I} + \frac{\partial f}{\partial \theta} \frac{\partial H}{\partial p_\theta} - \frac{\partial f}{\partial p_\theta} \frac{\partial H}{\partial \theta}. \qquad (2.27)$$

In the adiabatic limit, $\omega_B = \partial H/\partial I$ is large and so $\partial f/\partial \psi$ must be small. Otherwise, $\partial f/\partial t$ would be large and the distribution would evolve rapidly along the magnetic field. Physically, this corresponds to the fact that any initially large ψ variations are rapidly phase mixed by the bounce motion. The small ψ variations are uninteresting from the standpoint of cross-field transport and may be eliminated by integrating Eq. (2.26) over ψ. The result is

$$\frac{\partial \bar{f}}{\partial t} + \frac{\partial \bar{f}}{\partial \theta} \frac{\partial H}{\partial p_\theta} - \frac{\partial \bar{f}}{\partial p_\theta} \frac{\partial H}{\partial \theta} = C(\bar{f}) \qquad (2.28)$$

where

$$\bar{f}(I,p_\theta,\theta,\mu,t) = \int_0^{2\pi} d\psi\, f(I,\psi,p_\theta,\theta,\mu,t). \tag{2.29}$$

Rewriting Eq. (2.28) as

$$\frac{\partial \bar{f}}{\partial t} + \frac{\partial}{\partial \theta}\left[\frac{\partial H}{\partial p_\theta}\bar{f}\right] - \frac{\partial}{\partial p_\theta}\left[\frac{\partial H}{\partial \theta}\bar{f}\right] = C(\bar{f}) \tag{2.30}$$

and integrating over I, μ, and θ yields the transport equation

$$\frac{\partial N(p_\theta)}{\partial t} = \frac{\partial}{\partial p_\theta}\left[\int \frac{d\theta}{2\pi}\int dI d\mu \frac{\partial H}{\partial \theta}\bar{f}\right] \tag{2.31}$$

where

$$N(p_\theta) = \int \frac{d\theta}{2\pi}\int dI d\mu \bar{f} \tag{2.32}$$

The integral over the collision operator vanishes because collisions conserve the number of particles.

To obtain a transport equation accurate to second order in $\partial H/\partial \theta$ we must obtain $\bar{f}$ accurate to first order in $\partial H/\partial \theta$. Thus we look for solution to Eq. (2.28) in the form

$$\bar{f} = \bar{f}_0(H,p_\theta) + \delta\bar{f}(I,p_\theta,\theta,\mu) \tag{2.33}$$

where $\delta\bar{f}/\bar{f}_0 \sim \delta L/L_0$ and

$$\bar{f}_0 = N(p_\theta)(2\pi T/m)^{-3/2} e^{e\Phi/T} e^{-H/T} \tag{2.34}$$

Written in velocity variables, $\bar{f}_0$ is just a Maxwellian times the θ-averaged 2-D density. Taking $\bar{f}_0$ in this form makes use of the frequency ordering, $\nu \gg \gamma$. Collisions are assumed to occur more rapidly than transport, so the zeroth order distribution is a Maxwellian along the magnetic field.

$\delta\bar{f}$ is obtained from Eq. (2.28) written to first order in $\delta L/L_0$,

$$\frac{\partial \delta\bar{f}}{\partial t} + \bar{\omega}_R \frac{\partial \delta\bar{f}}{\partial \theta} + \left(\frac{\pi^2 I^2}{2mL^2}\right)\frac{2}{L}\frac{\partial \delta L}{\partial \theta}\left[\frac{\bar{\omega}_R}{T} + \frac{1}{\bar{f}_0}\frac{\partial \bar{f}_0}{\partial p_\theta}\right]\bar{f}_0 = C\left[\bar{f}_0 + \delta\bar{f}\right], \tag{2.35}$$

34

where

$$\bar{\omega}_R = \frac{\partial H}{\partial p_\theta} = \frac{\partial e\Phi}{\partial p_\theta} - \frac{\pi^2 I^2}{2mL^2}\frac{2}{L}\frac{\partial L_0}{\partial p_\theta} \tag{2.36}$$

is the total bounce averaged rotation frequency and

$$
\begin{aligned}
\frac{1}{\bar{f}_0}\frac{\partial \bar{f}_0}{\partial p_\theta} &= \frac{1}{N}\frac{\partial N}{\partial p_\theta} - \frac{3}{2}\frac{1}{T}\frac{\partial T}{\partial p_\theta} + \left(\frac{\pi^2 I^2}{2mL^2} + B\mu\right)\frac{1}{T^2}\frac{\partial T}{\partial p_\theta} \\
&\quad + \frac{1}{T}\left(\frac{\pi^2 I^2}{2mL^2}\right)\frac{2}{L}\frac{\partial L_0}{\partial p_\theta}.
\end{aligned} \tag{2.37}
$$

In general, the solutions to Eq. (2.35) consist of a sum of a driven response and free oscillations of the form

$$\delta \bar{f} = \sum_l \delta \bar{f}_l e^{il\left(\theta - \bar{\omega}_R t\right)} \tag{2.38}$$

Since the system has finite shear $(\partial \bar{\omega}_R/\partial p_\theta \neq 0)$ these free oscillation terms become rapidly oscillating functions of p_θ at large t and are rapidly damped by any diffusive transport processes. Since the driving terms vary on the slow transport timescale, the term $\partial \delta \bar{f}/\partial t$ may be dropped from Eq. (2.35), leaving

$$\bar{\omega}_R\frac{\partial \delta \bar{f}}{\partial \theta} + \left(\frac{\pi^2 I^2}{2mL^2}\right)\frac{2}{L}\frac{\partial \delta L}{\partial \theta}\left[\frac{\bar{\omega}_R}{T} + \frac{1}{\bar{f}_0}\frac{\partial \bar{f}_0}{\partial p_\theta}\right]\bar{f}_0 = C\left[\bar{f}_0 + \delta \bar{f}\right], \tag{2.39}$$

Given the the frequency ordering $\nu \ll \omega_R$, this equation may be solved perturbatively in the effective collision frequency. Dropping the collision operator term and integrating yields

$$\delta \bar{f}^{(0)} = -\frac{1}{\bar{\omega}_R}\left(\frac{\pi^2 I^2}{2mL^2}\right)\frac{2\delta L}{L}\left[\frac{\bar{\omega}_R}{T} + \frac{1}{\bar{f}_0}\frac{\partial \bar{f}_0}{\partial p_\theta}\right]\bar{f}_0, \tag{2.40}$$

where the superscript indicates the ordering in collisions. The collisional response is obtained by inserting $\delta \bar{f}$ into the collision operator on the R.H.S. of Eq. (2.39). This yields

$$\frac{\partial}{\partial \theta}\left(\delta \bar{f}^{(1)}\right) = \frac{1}{\bar{\omega}_R}C\left[\bar{f}_0\left(1 - \frac{1}{\bar{\omega}_R}\left(\frac{\pi^2 I^2}{2mL^2}\right)\frac{2\delta L}{L}\left[\frac{\bar{\omega}_R}{T} + \frac{1}{\bar{f}_0}\frac{\partial \bar{f}_0}{\partial p_\theta}\right]\right)\right]. \tag{2.41}$$

Substituting $\bar{f} = \bar{f}_0 + \delta \bar{f}^{(0)} + \delta \bar{f}^{(1)}$ into the transport equation [Eq.(2.30)] yields

$$\frac{\partial N(p_\theta)}{\partial t} = \frac{\partial}{\partial p_\theta}\left[\int \frac{d\theta}{2\pi}\int dI d\mu \frac{\pi^2 I^2}{2mL^2}\left(-\frac{2}{L_0}\frac{\partial \delta L}{\partial \theta}\right)\delta \bar{f}^{(1)}\right] \tag{2.42}$$

where the collisionless terms have vanished in the integral over θ. Integrating by parts and substituting from Eq. (2.41) results in

$$\frac{\partial N(p_\theta)}{\partial t} = \frac{\partial}{\partial p_\theta}\left[\int \frac{d\theta}{2\pi}\int dI d\mu \frac{\pi^2 I^2}{2mL^2}\frac{2\delta L}{L_0}\frac{1}{\bar{\omega}_R}\right.$$
$$\left.\times C\left[\bar{f}_0\left(1 - \frac{1}{\bar{\omega}_R}\left(\frac{\pi^2 I^2}{2mL^2}\right)\frac{2\delta L}{L}\left[\frac{\bar{\omega}_R}{T} + \frac{1}{\bar{f}_0}\frac{\partial \bar{f}_0}{\partial p_\theta}\right]\right)\right]\right] \tag{2.43}$$

After changing variables of integration from (μ, I) to $(v_z, v_\perp)$, this equation may be written as

$$\frac{\partial N(p_\theta)}{\partial t} = \frac{\partial}{\partial p_\theta}\left[\int \frac{d\theta}{2\pi}\int dI d\mu \left(\frac{1}{2}mv_z^2\right)\frac{2\delta L}{L_0}\frac{1}{\bar{\omega}_R}\right.$$
$$\left.\times C\left[\bar{f}_0\left(1 - \frac{1}{\bar{\omega}_R}\left(\frac{1}{2}mv_z^2\right)\frac{2\delta L}{L}\left[\frac{\bar{\omega}_R}{T} + \frac{1}{\bar{f}_0}\frac{\partial \bar{f}_0}{\partial p_\theta}\right]\right)\right]\right] \tag{2.44}$$

where

$$\bar{\omega}_R = \frac{\partial e\Phi}{\partial p_\theta} - \left(\frac{1}{2}mv_z^2\right)\frac{2}{L_0}\frac{\partial L_0}{\partial p_\theta}, \tag{2.45}$$

$$\frac{1}{\bar{f}_0}\frac{\partial \bar{f}_0}{\partial p_\theta} = \frac{1}{N}\frac{\partial N}{\partial p_\theta} - \frac{3}{2}\frac{1}{T}\frac{\partial T}{\partial p_\theta} + \left(\frac{1}{2}mv^2\right)\frac{1}{T^2}\frac{\partial T}{\partial p_\theta} + \left(\frac{1}{2}mv_z^2\right)\frac{1}{T}\frac{2}{L_0}\frac{\partial L_0}{\partial p_\theta}, \tag{2.46}$$

and

$$\bar{f}_0 = N(p_\theta)f_M \tag{2.47}$$

with f_M a Maxwellian distribution. An expression very similar to the second term on the R.H.S. of Eq. (2.45) was previously derived by Peurrung and Fajans.[18]

The velocity integral in this equation is simplified by assuming that the Debye length is small. This is typically the case in the experiments and is consistent with our assumed form of the end potential. The ratio of the two terms in Eq. (2.45) scales as

$$\frac{\frac{1}{2}mv_z^2}{\partial e\Phi/\partial p_\theta}\frac{2}{L_0}\frac{\partial L_0}{\partial p_\theta} \sim \frac{T}{e\Phi} \sim \frac{\lambda_D^2}{r_p^2} \tag{2.48}$$

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where r_p is the radius of the plasma. We therefore neglect the velocity dependent term in the rotation frequency and take

$$\bar{\omega}_R \simeq \frac{\partial e\Phi}{\partial p_\theta} = \omega_R \qquad (2.49)$$

the usual $\mathbf{E} \times \mathbf{B}$ rotation frequency. Similarly, the ratio of the two terms inside the collision operator in Eq. (2.43) scales as

$$\frac{1}{(\bar{\omega}_R/T)} \frac{1}{\bar{f}_0} \frac{\partial \bar{f}_0}{\partial p_\theta} \sim \frac{T}{e\Phi} \sim \frac{\lambda_D{}^2}{r_p{}^2} \qquad (2.50)$$

and therefore we will neglect the term, $1/\bar{f}_0 \partial \bar{f}_0/\partial p_\theta$

We take the collision operator in the general form

$$C[f] = \int d^3 v_1 \, d\sigma |v_{rel}| \left(f(v_1')f(v') - f(v)f(v_1) \right), \qquad (2.51)$$

where $d\sigma$ is the differential cross section and $v_{rel} = v - v_1$. Using this form and the small Debye length approximation, we obtain

$$\frac{\partial N(p_\theta)}{\partial t} = -\frac{\partial}{\partial p_\theta} \left[\int \frac{d\theta}{2\pi} \left(\frac{2\delta L}{L_0} \right)^2 \frac{N}{\omega_R T} N \int d^3 v \left(\frac{1}{2}mv_z^2 \right) \int d^3 v_1 d\sigma |v_{rel}| \quad (2.52) \right.$$

$$\left. \times \left[\left(\frac{1}{2}mv_{z1}'^2 + \frac{1}{2}mv_z'^2 \right) f_{M1}'f_M' - \left(\frac{1}{2}mv_{z1}^2 + \frac{1}{2}mv_z^2 \right) f_{M1}f_M \right] \right]$$

To evaluate the velocity integral in this equation, it is instructive to consider the collisional relaxation of an anisotropic Maxwellian distribution

$$f_A(v) = \left(\frac{2\pi T_\parallel}{m} \right)^{-1/2} \left(\frac{2\pi T_\perp}{m} \right)^{-1} \exp \left[\frac{\frac{1}{2}mv_z^2}{T_\parallel} - \frac{\frac{1}{2}mv_\perp^2}{T_\perp} \right]. \qquad (2.53)$$

The change in the parallel temperature due to collisions is given by

$$\frac{d}{dt} \left(\frac{T_\parallel}{2} \right) = N \int d^3 v \left(\frac{1}{2}mv_z^2 \right) \int d^3 v_1 \, d\sigma |v_{rel}| \left(f_A(v_1')f_A(v') - f_A(v)f_A(v_1) \right)$$

$$= \nu_{\perp,\parallel} \left(T_\perp - T_\parallel \right), \qquad (2.54)$$

and may be used as a definition of the equipartition rate, $\nu_{\perp,\parallel}$. Consider the case $T_\perp = T$ and $T_\parallel = (1 - \alpha)T$. Substituting this into Eq. (2.53) and taking the limit

$\alpha \to 0$, one can easily show

$$N \int d^3v \left(\frac{1}{2}mv_z^2\right) \int d^3v_1 d\sigma |v_{rel}| \left[\left(\frac{1}{2}mv'^2_{z1} + \frac{1}{2}mv'^2_z\right) f'_{M1} f'_M \right.$$
$$\left. - \left(\frac{1}{2}mv_{z1}^2 + \frac{1}{2}mv_z^2\right) \dot{f}_{M1} f_M\right] = -T^2 \nu_{\perp,\parallel}. \tag{2.55}$$

This is precisely the integral that appears in the transport equation. Eq. (2.52) can now be written in the simple form

$$\frac{\partial N(p_\theta)}{\partial t} = -\frac{\partial}{\partial p_\theta} \left[4\nu_{\perp,\parallel} \frac{\langle \delta L^2 \rangle}{L_0^2} \left(\frac{T}{-\omega_R}\right) N\right]. \tag{2.56}$$

Changing variables from (p_θ, θ) to $\langle r, \theta \rangle$ yields the result

$$\frac{\partial N(r)}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r} r \left[4\nu_{\perp,\parallel} \frac{\langle \delta L^2 \rangle}{L_0^2} \left(\frac{T}{-e\partial\Phi/\partial r}\right) N\right]. \tag{2.57}$$

The quantity in brackets is the radial particle flux and is identical to the flux given by Eq. (2.14) in the introduction.

2.4 Resonant Particle Transport

In the previous section, we derived a transport equation in the adiabatic limit by assuming that $\omega_B \gg \omega_R$ for every particle in the system. This approach neglects the effect of particles that satisfy the resonance condition, $l\omega_R = 2n\omega_B$, where n and l are small integers. When the bounce frequency for a thermal particle, $\bar{\omega}_B = \frac{\pi}{L_0}\sqrt{T/M}$, is comparable to the rotation frequency and collisions are sufficiently weak, resonant particle transport dominates. In this regime, the damping rate of the $m = 1$ diocotron mode is larger and scales differently than the damping rate given in Eq. (2.19). A similar effect occurs for transport in tandem mirrors in the "resonant-plateau" regime.[19, 20]

The basic idea behind resonant particle transport is easy to understand. When a particle is reflected off the non-axisymmetric end potential it experiences a force in the $\hat{\theta}$ direction causing its angular momentum, $p_\theta = \frac{eB}{2c}r^2$, and radial position

to change. The magnitude and direction of the radial step depends on the particle's azimuthal position at the point of reflection. In addition, fast particles take larger steps because a larger force is required to reflect them. Consider a particle satisfying the lowest order resonance condition $\omega_R = 2\omega_B$. Such a particle reflects off each end of the plasma at the same θ position for many bounces and consequently takes many radial steps in the same direction. For non-resonant particles the radial steps tend to cancel.

When collisions are sufficiently weak resonant particle transport is always present. The size of the transport is determined by the relative number of resonant particles and by the contribution from each resonant particle. In the adiabatic limit low order bounce-rotation resonances are at low velocities. Although there are a large number of resonant particles, the contribution from each particle is small because the radial steps are small and infrequent. In this regime resonant particle transport is negligible. In the opposite limit where $\omega_R \gg \bar{\omega}_B$ and the resonance is located at a large velocity, the contribution from each particle is large but there are few particles on the tail of the Maxwellian which interact resonantly. We will see from the formal treatment that resonant particle transport is important when $\bar{\omega}_B \simeq \omega_R$.

Our starting point is again the drift-kinetic Boltzmann equation,

$$\frac{\partial f}{\partial t} + [f, H] = C(f), \tag{2.58}$$

where

$$H = \frac{P_z^2}{2m} + \mu B + e\Phi(p_\theta) + e\Phi_e(\theta, p_\theta, z). \tag{2.59}$$

The calculation is simplified by considering a system that consists of only one half of the plasma, that is, we take

$$e\Phi_e(\theta, p_\theta, z) = \begin{cases} 0; & 0 < z < 1/2\, L(\theta, p_\theta) \\ \infty; & \text{otherwise} \end{cases} \tag{2.60}$$

Since the particles specularly reflect off a plane at $z = 0$ without changing θ, p_θ , z or $|v_z|$, the transport equation will be the same for this system as for the full length system.

Writing the Poisson bracket in Eq. (2.58) as

$$[f, H] = \frac{\partial}{\partial z}\left(f\frac{\partial H}{\partial p_z}\right) - \frac{\partial}{\partial p_z}\left(f\frac{\partial H}{\partial z}\right) + \frac{\partial}{\partial \theta}\left(f\frac{\partial H}{\partial p_\theta}\right) - \frac{\partial}{\partial p_\theta}\left(f\frac{\partial H}{\partial \theta}\right) \qquad (2.61)$$

and integrating over all variables except p_θ gives the result

$$\frac{\partial N(p_\theta)}{\partial t} = \frac{\partial}{\partial p_\theta}\left[\int \frac{2\theta}{2\pi} \int d^2 v_\perp \int dv_z \int dz f \frac{\partial H}{\partial \theta}\right], \qquad (2.62)$$

where

$$N(p_\theta) = \int \frac{2\theta}{2\pi} \int d^2 v_\perp \int dv_z \int dz f. \qquad (2.63)$$

Solutions to Eq. (2.58) are assumed to take the form

$$f = f_0(H, p_\theta, t) + \delta f(p_z, z, p_\theta, \theta, t), \qquad (2.64)$$

where $\delta f / f_0 \sim \delta L / L_0$ and

$$f_0 = \frac{N(p_\theta)}{(1/2L_0)(2\pi T/m)^{3/2}} e^{e\phi/T} e^{-H/T}. \qquad (2.65)$$

In velocity variables, this is a Maxwellian distribution. As in Section II we neglect terms higher order in $\lambda_D{}^2/r_p{}^2$ by assuming L_0, T, and N are constant in p_θ .

To first order in $\delta L / L_0$, the drift-kinetic Boltzmann equation is

$$\frac{\partial(\delta f)}{\partial t} + [\delta f, H_0] - C(f_0 + \delta f) = -[f_0, H], \qquad (2.66)$$

where H_0 is the Hamiltonian with $\delta L = 0$. The first two terms on the left hand side can be thought of as a derivative along the unperturbed orbit,

$$\frac{\partial(\delta f)}{\partial t} + [\delta f, H_0] = \frac{d^{(0)}}{dt}(\delta f) \qquad (2.67)$$

We approximate the collision operator by

$$C(f_0 + \delta f) = -\nu \delta f \qquad (2.68)$$

where ν is an effective collision frequency. Clearly, this oversimplifies the transport problem in the adiabatic limit because there is no reason to expect $\nu = \nu_{\perp,\parallel}$. For resonant particle transport, however, we will let $\nu \to 0$ at the end of the calculation and the effective collision frequency drops out. We do not expect this transport to depend sensitively on the detailed nature of the collision operator.

Evaluating the R.H.S. of Eq. (2.66) and using Eqs. (2.67) and (2.68) gives

$$\frac{d^{(0)}}{dt}(\delta f) + \nu \delta f = \frac{\omega_R}{T} f_0 \frac{\partial(e\Phi_e)}{\partial \theta}. \tag{2.69}$$

The solution to this equation is

$$\delta f(t) = \int_0^t dt' e^{-\nu(t-t')} \left[\frac{\omega_R}{T} f_0 \frac{\partial(e\Phi_e)}{\partial \theta} \right]', \tag{2.70}$$

where the prime indicates evaluation along the unperturbed orbit. To this order in δL f_0 is constant along the unperturbed orbit and may be factored out of the integral so that

$$\delta f(t) = \frac{\omega_R}{T} f_0 \int_0^t dt' e^{-\nu(t-t')} \frac{\partial(e\Phi_e)}{\partial \theta'}. \tag{2.71}$$

Consider Eq. (2.62), the transport equation. Since $\partial H/\partial \theta = 0$ everywhere except at the end of the plasma, we need to find δf only at the end of the plasma where the particles are reflected. Therefore, it is convenient to write Eq. (2.71) as

$$\delta f(t) = \frac{\omega_R}{T} f_0 \int_0^{t_0} dt' e^{-\nu(t-t')} \frac{\partial(e\Phi_e)}{\partial \theta'} + \frac{\omega_R}{T} f_0 \int_{t_0}^t dt' e^{-\nu(t-t')} \frac{\partial(e\Phi_e)}{\partial \theta'}. \tag{2.72}$$

where t_0 indicates the time just before the reflection. The first term consists of a sum of many reflections and the second term is due to a single partial reflection.

We delay evaluation of the second term until δf is inserted into the transport equation. We evaluate the first term by calculating the effect of a single reflection and then summing over many reflections. The axial impulse exerted by the end potential on a particle during reflection is

$$-2|p_z^0| = -\int_{turn} dt' \frac{\partial e\Phi_e}{\partial z'}. \tag{2.73}$$

Since the end potential is only a function of the quantity $(z - \frac{1}{2}L(\theta, p_\theta))$, this may be rewritten as

$$-2|p_z^0| = \frac{2}{\partial L/\partial \theta} \int_{turn} dt' \frac{\partial e\Phi_e}{\partial \theta'}. \tag{2.74}$$

Since $-\partial(e\Phi)/\partial \theta = \dot{p}_\theta$, the change in p_θ due to the reflection is

$$\Delta p_\theta = -\int_{turn} dt' \frac{\partial e\Phi_e}{\partial \theta'} = +|p_z^0| \frac{\partial \delta L}{\partial \theta}. \tag{2.75}$$

This equation has a simple physical interpretation. The force which reflects a particle is normal to the surface at the end of the plasma. Due to the asymmetry, there is a small component of this force in the $\hat{\theta}$ direction which exerts a torque on the particle causing p_θ to change. A larger force is needed to reflect fast particles and therefore these particles take larger steps in p_θ .

The first term in Eq. (2.72) can now be written as

$$\delta f_1 = \frac{-\omega_R}{T} f_0 \int_0^{t_0} dt' e^{-\nu(t-t')} \sum_{j=1}^{N} |p_z^0| \frac{\partial(\delta L)}{\partial \theta'} \delta(t' - t_j), \tag{2.76}$$

where the index j is summed over past reflections. The time at the jth reflection is given simply by

$$t_j = t - \frac{L_0}{|v_z^0|} j, \tag{2.77}$$

and along unperturbed orbits

$$\theta' = \theta - \omega_R(t - t') \tag{2.78}$$

After substituting in a Fourier series for δL, Eq. (2.76) becomes

$$\delta f_1 = \frac{-\omega_R}{T} f_0 |p_z^0| \sum_l il\delta L_l(p_\theta) e^{il\theta} \sum_{j=1}^{N} e^{-j(il\omega_R+\nu)L_0/|v_z^0|}. \tag{2.79}$$

This contribution to δf arises from a series of discrete "kicks" acting on f_0. Note that those kicks which occurred in the distant past (large j) are collisionally damped.

The sum over j can be evaluated exactly so that

$$\delta f_1 = \frac{-\omega_R}{T} f_0 |p_z^0| \sum_l il\delta L_l(p_\theta) e^{il\theta} \left(\frac{e^{-(il\omega_R+\nu)L_0/|v_z^0|N} - 1}{e^{-(il\omega_R+\nu)L_0/|v_z^0|} - 1} \right). \tag{2.80}$$

The exponential in the numerator can be dropped because

$$\nu \frac{L_0}{|v_z^0|} N \sim \nu t \sim \frac{\nu}{\gamma} \gg 1. \tag{2.81}$$

Using the mathematical identity

$$\frac{1}{e^{iz} - 1} = -\frac{1}{2} - i \sum_{n=-\infty}^{\infty} \frac{1}{1 - 2\pi n}, \tag{2.82}$$

the full fluctuation distribution is written as

$$
\begin{aligned}
\delta f &= \frac{\omega_R}{T} f_0 |p_z^0| \sum_l il\delta L_l(p_\theta) e^{il\theta} \left(\frac{1}{2} + i \sum_n \frac{1}{(l\omega_R - i\nu)L_0/|v_z^0| - 2\pi n} \right) \\
&+ \frac{\omega_R}{T} \int_{t_0}^{t} dt' \frac{\partial e\Phi_e}{\partial \theta'}.
\end{aligned}
\tag{2.83}
$$

Inserting $f = f_0 + \delta f$ into Eq. (2.62) gives

$$\frac{\partial N(p_\theta)}{\partial t} = \frac{\partial}{\partial p_\theta} \left[\int \frac{d\theta}{2\pi} \int d^2 v_\perp \int dv_z \int dz \delta f \frac{\partial e\Phi_e}{\partial \theta} \right], \tag{2.84}$$

where f_0 has vanished in the integral over θ. Since we have calculated δf as a function of time, it is useful to transform this integral into an integral over unperturbed orbits. First note that the integrand is nonzero only at the end of the plasma and so the transformation need only be valid for the short turning time. If a particle has phase space coordinates $(\theta^0, p_\theta{}^0, z^0, p_z^0)$ at time t_0 at some position just before the end of the plasma, the unperturbed orbits valid during the turn are

$$\theta = \theta^0, \tag{2.85}$$

$$p_\theta = p_\theta{}^0, \tag{2.86}$$

$$p_z = p_z(p_z^0, z^0, t), \tag{2.87}$$

$$z = z(p_z^0, z^0, t). \tag{2.88}$$

We change variables in the integral using

$$dz \, dp_z = \left| \frac{\partial(z, p_z)}{\partial(t, p_z^0)} \right| dt \, dp_z^0 = \left| \frac{\partial z}{\partial t} \frac{\partial p_z}{\partial p_z^0} - \frac{\partial p_z}{\partial t} \frac{\partial z}{\partial p_z^0} \right| dt \, dp_z^0. \tag{2.89}$$

The equations of motion are derived from the unperturbed Hamiltonian, H_0, and therefore this may be written as

$$dz \; dp_z = \left| \frac{\partial H_0}{\partial p_z} \frac{\partial p_z}{\partial p_z^0} + \frac{\partial H_0}{\partial z} \frac{\partial z}{\partial p_z^0} \right| dt \; dp_z^0. \tag{2.90}$$

The quantity in absolute value bars is just $\partial H_0 / \partial p_z^0$ and therefore

$$dz \; dp_z = \left| \frac{\partial H_0}{\partial p_z^0} \right| dt \; dp_z^0 = \left| v_z^0 \right| dt \; dp_z^0. \tag{2.91}$$

The transport equation can now be written in the form

$$\frac{\partial N(p_\theta)}{\partial t} = \frac{\partial}{\partial p_\theta} \left\{ \int \frac{d\theta}{2\pi} \int d^2 v_\perp \int_0^\infty dv_z^0 v_z^0 \int_{t_0}^{t_f} dt' \delta f' \frac{\partial H}{\partial \theta'} \right\}, \tag{2.92}$$

where the primes indicate evaluation along the unperturbed orbit and t_f is the time just after the turn.

Using δf from Eq. (2.83) this becomes

$$\begin{aligned}
\frac{\partial N(p_\theta)}{\partial t} &= \frac{\partial}{\partial p_\theta} \left\{ \int \frac{d\theta}{2\pi} \int d^2 v_\perp \int_0^\infty dv_z^0 v_z^0 \int_{t_0}^{t_f} dt' \frac{\partial e\Phi_e}{\partial \theta'} \left[\frac{\omega_R}{T} f_0' \int_{t_0}^{t_f} dt'' \frac{\partial e\Phi_e}{\partial \theta''} \right. \right. \\
&\quad + \frac{\omega_R}{T} f_0' p_z^0 \sum_l i l \delta L_l(p_\theta) e^{il\theta} \left(\frac{1}{2} \right. \\
&\quad \left. \left. \left. + i \sum_n \frac{1}{(l\omega_R - i\nu)L_0/v_z^0 - 2\pi n} \right) \right] \right\}
\end{aligned} \tag{2.93}$$

To this order, f_0 is constant along the orbit and may be pulled out of the integrals. In velocity variables, $f_0(t = t_0)$ is given by

$$f_0 = \frac{N(p_\theta)}{\frac{1}{2} L_0} f_M \tag{2.94}$$

where f_M is a Maxwellian.

The time integrals can be expressed in terms of Δp_θ as

$$\int_{t_0}^{t_f} dt' \frac{\partial e\Phi_e}{\partial \theta'} = -\Delta p_\theta \tag{2.95}$$

and

$$\int_{t_0}^{t_f} dt' \frac{\partial e\Phi_e}{\partial \theta'} \int_{t_0}^{t_f} dt'' \frac{\partial e\Phi_e}{\partial \theta''} = \frac{1}{2} \Delta p_\theta{}^2 \tag{2.96}$$

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Using Δp_θ from Eq. (2.75), Eq. (2.90) becomes

$$
\begin{aligned}
\frac{\partial N(p_\theta)}{\partial t} &= \frac{\partial}{\partial p_\theta}\left\{\int\frac{d\theta}{2\pi}\int d^2v_\perp\int_0^\infty dv_z^0 m^2(v_z^0)^3 f_M N(p_\theta)\frac{\omega_R}{T}\right.\\
&\quad\left.\times\sum_l il\delta L_l e^{il\theta}\sum_{l'} il'\delta L_{l'}e^{il'\theta}\sum_n\frac{i}{(l\omega_R-i\nu)L_0/v_z^0-2\pi n}\right\}.\quad(2.97)
\end{aligned}
$$

After integrating over $v_\perp$ and θ and dropping the imaginary part of the integral, this becomes

$$
\frac{\partial N(p_\theta)}{\partial t} = -\frac{\partial}{\partial p_\theta}\left\{-\frac{2\omega_R}{T}N(p_\theta)\int_0^\infty dv_z^0 m^2(v_z^0)^4 f_M\sum_{l,n}l^2\frac{|\delta L_l|^2}{L_0^2}\frac{\nu}{(l\omega_R-2n\omega_B)^2+\nu^2}\right\},
$$
$$(2.98)$$

where we have introduced $\omega_B = \pi|v_z|/L_0$.

First consider the $n = 0$ term in the sum. This may be thought of as the transport due to a resonance located at $v_z = \infty$ and corresponds physically to transport in the adiabatic limit. Evaluating the integral over v_z for this term yields

$$
\frac{\partial N^{\text{adiabatic}}}{\partial t} = -\frac{\partial}{\partial p_\theta}\left\{3\nu\frac{\langle\delta L^2\rangle}{L_0^2}\left(\frac{T}{-\omega_R}\right)N(p_\theta)\right\}.\quad(2.99)
$$

Identifying ν with $4/3\nu_{\perp,\parallel}$, we have the same result obtained in Section II. If we had kept ν as an operator, we would have recovered $\nu_{\perp,\parallel}$.

Now consider the terms in the sum for $n > 0$. Since we are working in the small ν limit we approximate

$$
\frac{\nu}{(l\omega_R-2n\omega_B)^2+\nu^2} \simeq \pi\delta(l\omega_R-2n\omega_B).\quad(2.100)
$$

The factor of 2 appears because particles are reflected at both ends of the plasma. Particles with $\omega_B = \omega_R$, for example may step radially outward at one end of the plasma, but will step inward at the other end.

Using the approximation of Eq. (2.97) the transport equation becomes

$$
\frac{\partial N(p_\theta)}{\partial t} = -\frac{\partial}{\partial p_\theta}\left\{4\left(\frac{1}{64}\sqrt{\pi/2}\sum_{l,n}\frac{(l\omega_R)^6}{(n\bar\omega_B)^5}\frac{|\delta L_l|^2}{L_0^2}\exp\left[-\frac{(l\omega_R)^2}{8(n\bar\omega_B)^2}\right]\right)\frac{T}{-\omega_R}N(p_\theta)\right\},
$$
$$(2.101)$$

where $\bar{\omega}_B = \pi\sqrt{T/m}/L_0$ is the mean bounce frequency in the plasma. The size of the resonant particle transport is determined by two competing effects. Consider the case where the resonance is located at large v_z. The resonant particles take large radial steps as the end potential must exert a large force in order to reflect them. Furthermore, these fast moving particles are reflected very frequently. While these effects tend to increase the transport, the location of the resonance on the tail of the Maxwelllian insures that there are relatively few resonant particles. Similarly, when the resonance is located at small v_z, the contribution from each particle is small, but there are many particles which interact resonantly.

For moderate temperature plasmas the $n = 1$ term in Eq. (2.98) is largest. To determine which terms in the sum over l are largest we estimate the size of the Fourier components δL_l. The end shape of the plasma is axisymmetric about the central axis of the trap and by considering the shape of the vacuum equipotential contours we expect the radius of curvature of the end shape to be proportional to R_w the radius of the conducting wall. If we model the end of the plasma as the intersection of a cylinder and a hemisphere of radius R_w we find that the length of the plasma parallel to the magnetic field is

$$L(r, \theta) = L_0 - \sqrt{R_w{}^2 - (D^2 + 2Dr\cos\theta + r^2)} \tag{2.102}$$

where D is the displacement of the center of charge off axis and (r, θ) is a cylindrical coordinate system centered on an axis through the center of charge. Taylor expanding this expression in the limit $Dr \ll R_w{}^2$ we find that

$$\delta L \propto R_w \left(\frac{Dr}{R_w{}^2}\right)^{|l|} . \tag{2.103}$$

Note that this agrees with the scaling given by Eq. (2.5) for a flat end ($l = \pm 1$). In the experiments D/R_w and r/R_w are typically very small and therefore the terms in Eq. (2.98) with $l \neq \pm 1$ are negligible despite the coefficient $\sim l^6$. Keeping only the

$n = 1$ and $l = \pm 1$ terms and changing variables from (p_θ, θ) to (r, θ) the transport equation can be written as

$$\frac{\partial N(p_\theta)}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}r\left\{4\left(\frac{1}{64}\sqrt{\pi/2}\frac{\omega_R^6}{\bar{\omega}_B^5}\exp\left[-\frac{\omega_R^2}{8\bar{\omega}_B^2}\right]\right)\frac{\langle\delta L^2\rangle}{L_0^2}\frac{T}{-e\partial\Phi/\partial r}N\langle p_\theta\rangle\right\}.$$

(2.104)

This result may be understood physically by considering the orbit of a single resonant particle. In one reflection the particle takes a radial step as given by Eq.(2.75) (recall $p_\theta = \frac{eB}{2c}r^2$). The particle takes approximately ω_B/ν of these steps before being converted to a non-resonant particle so that the fundamental step size governing the transport is

$$\Delta r = \frac{\omega_B}{\nu}\left(\frac{c}{eBr}mv_z\frac{\partial\delta L}{\partial\theta}\right)$$

(2.105)

One can estimate the size of the diffusion coefficient as the average of the step size squared times the rate at which particles take steps, that is,

$$D = \nu\langle(\Delta r)^2\rangle_\theta$$

(2.106)

The radial particle flux is given by

$$\Gamma = -D\left(\frac{\partial f_0}{\partial r}\right)_H\Delta v,$$

(2.107)

where the distribution function and the diffusion coefficient are to be evaluated at the resonant velocity. Δv is the width of the resonance in velocity space and (along with the distribution function) indicates the relative number of particles which participate in the resonant interaction. For the case $n = 1$, $l = \pm 1$, the resonance condition is

$$2\omega_B \pm \omega_R + i\nu = 0$$

(2.108)

so the width of the resonance may be estimated to be

$$\Delta v = \frac{L_0}{\pi}\Delta\omega_B = \frac{L_0}{2\pi}\nu.$$

(2.109)

After some algebra the estimate for the flux can be written as

$$\Gamma = \left(\frac{1}{64} \sqrt{2\pi^3} \frac{\omega_R{}^6}{wBb^5} \exp\left[-\frac{\omega_R{}^2}{8\bar{\omega}_B{}^2} \right] \right) \frac{\langle \delta L^2 \rangle}{L_0^2} \frac{T}{-e\partial\Phi/\partial r} N. \qquad (2.110)$$

Except for a numerical coefficient this expression agrees with the flux given in Eq. (2.101).

To find the damping rate of the $m = 1$ diocotron mode, we would again use conservation of angular momentum (Eq. 2.14). The ratio of the resonant particle damping rate to the adiabatic damping rate is given by

$$\frac{\gamma^r}{\gamma^a} = \frac{1}{64} \sqrt{\pi/2} \frac{\omega_R{}^6}{\bar{\omega}_B{}^5 \nu_{\perp,\parallel}} \exp\left[-\frac{\omega_R{}^2}{8\bar{\omega}_B{}^2} \right] . \qquad (2.111)$$

When this ratio becomes larger than 1, we expect that resonant particle damping will dominate. In typical experiments, $\bar{\omega}_B \gg \omega_R$ and therefore resonance effects are negligible. Even when $\gamma^r/\gamma^a > 1$, we must be sure that the frequency ordering $\nu \ll \bar{\omega}_B$ strictly holds. Even moderate collisionality will destroy the resonance effect. In the limit $\nu > \bar{\omega}_B$ a fluid-like treatment is more appropriate.

This chapter, in part, is a reprint of the material as it appears in Physics of Plasmas **2**, 355 (1995). The dissertation author was the primary investigator and author and the co-author listed in that publication directed and supervised the research which forms the basis for this chapter.
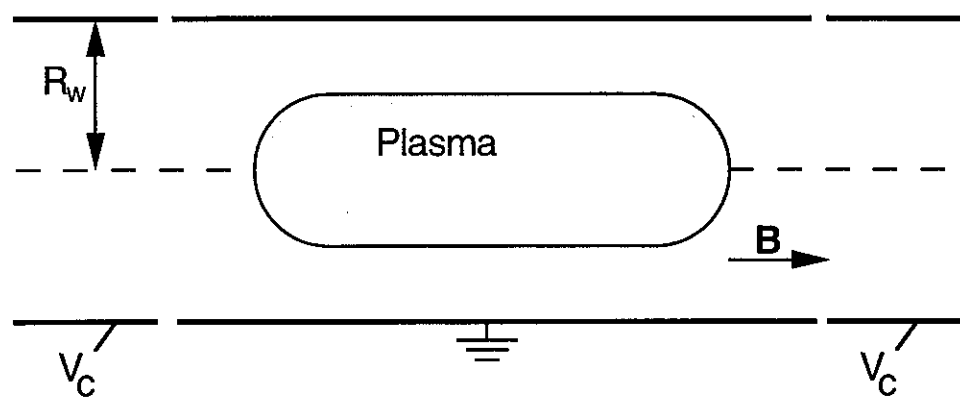
Figure 2.1: The confinement geometry.
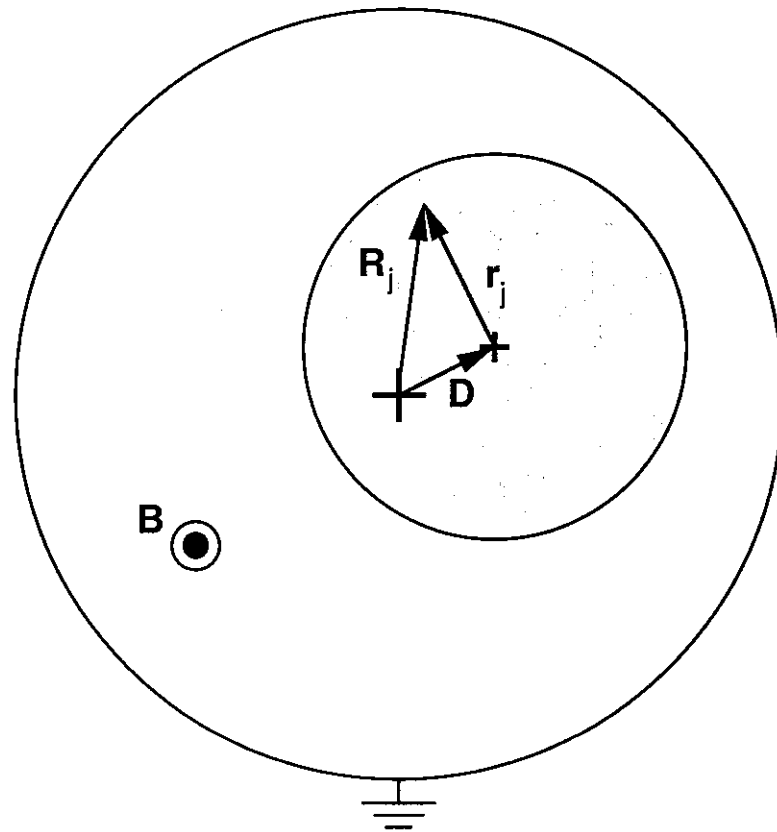
Figure 2.2: Coordinate system for the off-axis plasma (end view).
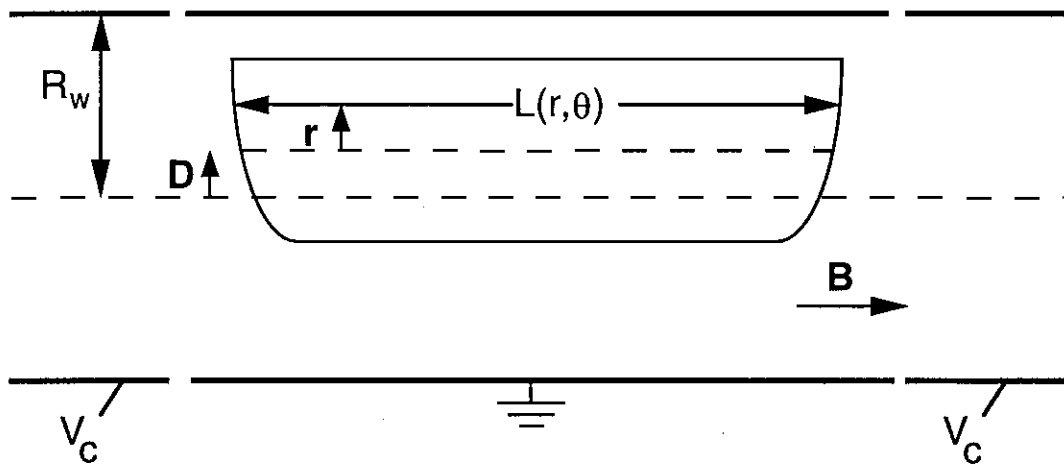
Figure 2.3: Length of the off-axis plasma.
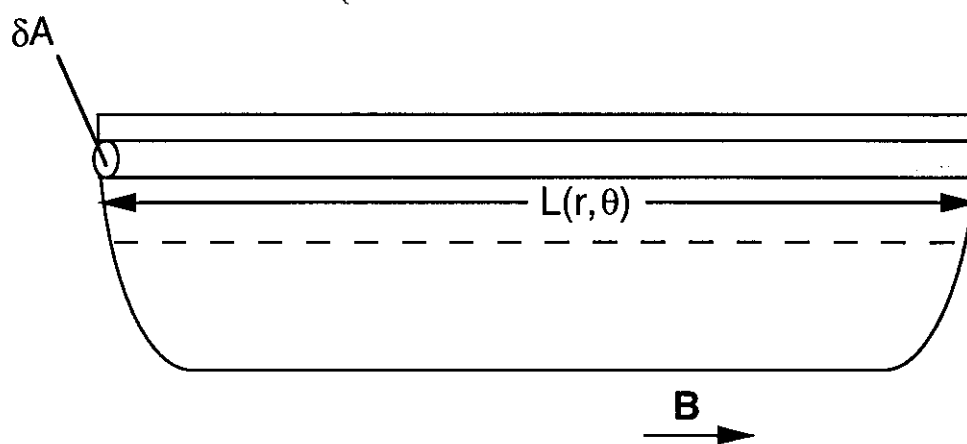
Figure 2.4: Side view of flux tube.
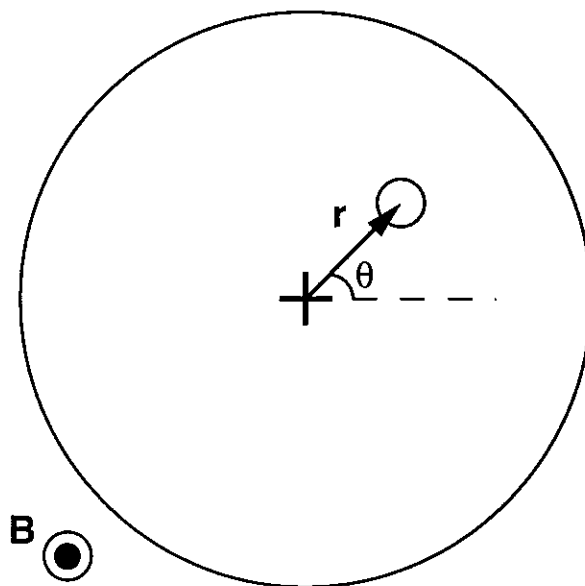
Figure 2.5: End view of flux tube.
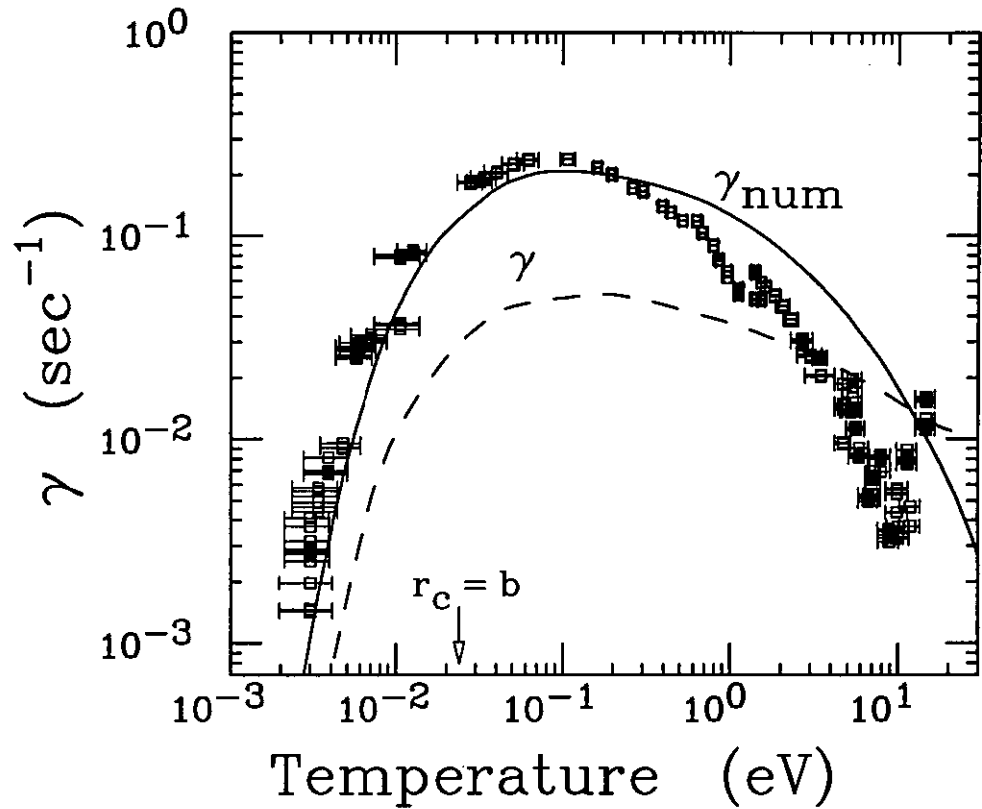
Figure 2.6: Damping rate of the $m = 1$ diocotron mode vs. temperature. Experimental measurements and numerical calculations (γ_{num}) by Cluggish and Driscoll[9]
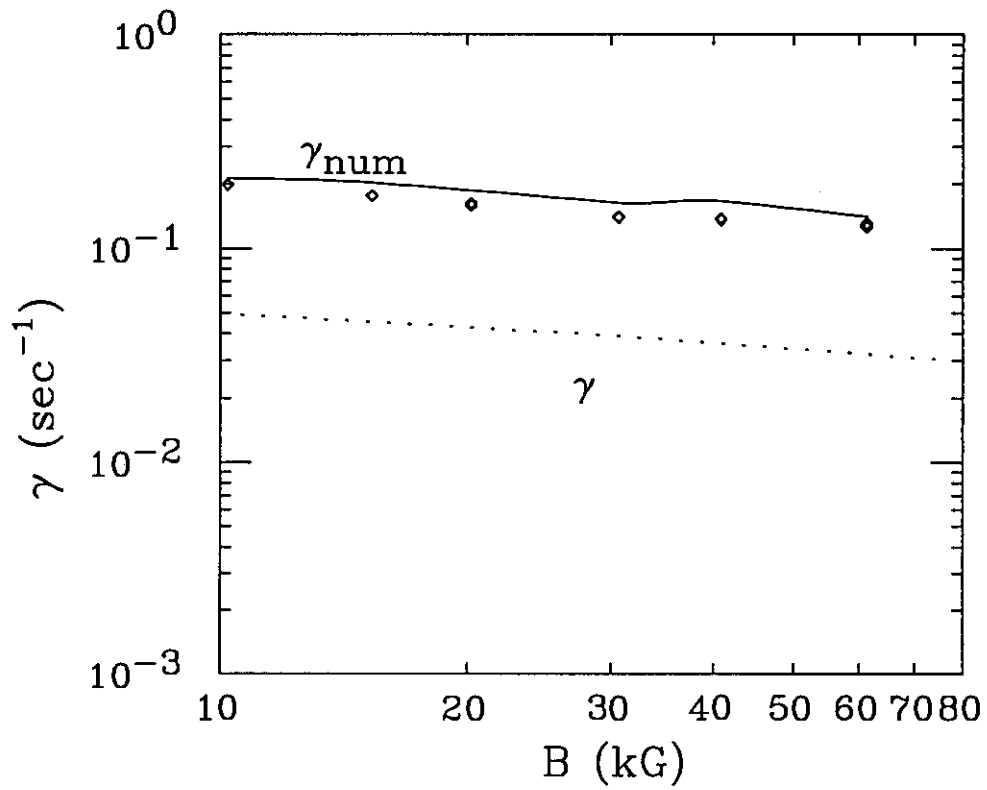
Figure 2.7: Damping rate of the $m = 1$ diocotron mode vs. magnetic field strength. Experimental measurements and numerical calculations (γ_{num}) by Cluggish and Driscoll[9]

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Chapter 3

Rotational Pumping and the "Rotating Wall"

3.1   Abstract

In the guiding center limit, the canonical angular momentum of a non-neutral plasma is proportional to its mean square radius. While static asymmetries in Penning traps exert a drag on the plasma and cause it to expand radially, a rapidly rotating asymmetry is shown to exert a torque in the same direction that the plasma rotates and cause it to be compressed. The torque and the inward particle flux are calculated for case that the rotating asymmetry is located at the end of the plasma. In realistic experiments the plasma is also influenced by a background torque causing the plasma to expand. Stationary states are possible when these torques balance and a cooling mechanism exists to dissipate the heating that is produced by the applied time-dependent asymmetry. The stability of stationary states of pure ion plasmas is investigated for the case that the background torque and cooling is due to cold neutrals.

3.2   Introduction

In experiments with non-neutral plasmas confined in Penning traps, time-dependent voltages applied to a sectored ring have been observed to cause inward

radial transport and plasma compression[1 − 5]. The typical confinement geometry for these experiments is shown schematically in Figure 3.1. A conducting cylinder is divided axially into rings, one of which is also divided azimuthally. Axial confinement is provided by negatively biased end rings (positive for an ion plasma) and radial confinement by a uniform magnetic field directed along the axis of the cylinder. Voltages applied to the azimuthally sectored ring cause radial particle transport.

The most important concepts necessary for understanding transport in these plasmas are angular momentum and torque. In the guiding center limit the total canonical angular momentum of the plasma is approximately given by

$$P_\theta = \frac{eB}{2c} \sum_{j=1}^{N} r_j^2 = \frac{eB}{2c} N \langle r^2 \rangle \tag{3.1}$$

where r_j is the position of the jth particle as measured from the axis of the trap and e carries a sign. In the absence of external torques, P_θ is a conserved quantity and $\langle r^2 \rangle$, the mean square radius of the plasma, is constant. External torques cause radial particle transport. At high neutral pressures the dominant torque arises from neutrals which act as stationary scattering centers in the lab frame[6 − 8]. At lower neutral pressures the dominant torque is believed to be due to time independent azimuthal asymmetries in the confining fields[6, 8]. Both of these effects oppose the direction of plasma rotation and cause the plasma to expand. In experiments with pure ion plasmas laser beams have been used to exert a torque in the same direction that the plasma rotates and cause inward transport of the plasma[10]. In this paper we consider the torque and transport arising from time-dependent voltages applied to an azimuthally sectored ring. We show that this technique may be used to compress a non-neutral plasma. If the compression rate is larger than the transport due to field errors or neutrals, one can confine a non-neutral plasma for an arbitrary length of time. This may be of practical importance in experiments with pure electron and pure positron plasmas where lasers can not be used to exert a torque.

Of particular interest is the special case where the voltages vary sinusoidally, but are phased so that the electrostatic asymmetry appears to rotate, i.e.,

$$\Phi_c = \Phi_0 + \delta\Phi \cos[m\theta - \omega t] \tag{3.2}$$

We call this a "rotating wall". An Asymmetry that rotates more slowly than the plasma exerts a drag on the plasma and causes $\mathbf{F} \times \mathbf{B}$ drifts that are directed radially outward. An asymmetry that rotates more rapidly than the plasma exerts a drag in the same direction that the plasma rotates and causes $\mathbf{F} \times \mathbf{B}$ drifts that are directed radially inward.

This result also follows from more formal thermodynamic arguments. For a sufficiently weak rotating asymmetry, the plasma remains near the thermal equilibrium state which is specified by the instantaneous values of the energy E, the canonical angular momentum P_θ , and the particle number N. The entropy of the plasma is a function of these variables. In a time interval dt the torques cause differential changes in E and P_θ ; N is constant. Using the thermodynamic relations $T = (\partial E/\partial S)_{N,P_\theta}$ and $\omega_R = (\partial E/\partial P_\theta)_{S,N}$ [11], we find that the entropy changes according to

$$TdS = dE - \omega_R dP_\theta, \tag{3.3}$$

where ω_R is the $\mathbf{E} \times \mathbf{B}$ drift rotation frequency of the plasma.

In this expression, E is the energy of the plasma in the lab frame. In a frame that rotates with the asymmetry, the energy is given by [11]

$$E' = E - \frac{\omega}{m}P_\theta. \tag{3.4}$$

In this frame the confinement potentials are time independent , and therefore E' is constant during the evolution. It follows that $dE = (\omega/m)dP_\theta$. Eq. (3.3) may then be expressed as

$$dP_\theta = \frac{TdS}{\omega/m - \omega_R}. \tag{3.5}$$

Since dS must be positive (second law of thermodynamics), the sign of dP_θ is determined solely by the sign of $\omega/m - \omega_R$. For an electron plasma, P_θ is negative and ω_R is positive. When $\omega/m > \omega_R$ the angular momentum of the electron plasma increases towards zero and the mean square radius of the plasma decreases. For a positron plasma or an ion plasma, P_θ is positive and ω_R is negative. To compress a positron plasma one would set $\omega/m < \omega_R$ (more negative) forcing P_θ and $\langle r^2 \rangle$ to decrease as the entropy increases. As the plasma is compressed its rotation frequency approaches the frequency of the rotating wall. Eventually the plasma relaxes to thermal equilibrium in the frame that rotates with the asymmetry and $dS \rightarrow 0$.

The preceeding results apply to any transport which is driven by a pure rotating asymmetry. In the remainder of this paper we consider transport driven by time dependent voltages positioned at one end of the plasma (See Fig. 3.2). We choose this geometry for two reasons. In experiments with pure electron plasmas it is often the case that the bounce frequency for a thermal electron is much larger than its $\mathbf{E} \times \mathbf{B}$ drift rotation frequency. In this limit, the plasma responds adiabatically. If the voltages are placed along the side of the plasma as in Fig. 3.1, the potential is screened out on the scale of a Debye length and the transport only occurs at the edge of the plasma. When the sectored ring is placed at one end of the plasma, all of the electrons interact with the asymmetry as they are reflected and the entire plasma participates in the transport. The other reason we choose this geometry is that the transport due to static end asymmetries is well understood theoretically. This transport is called rotational pumping [12]. The theory has been used successfully to explain the transport associated with damping of the $m = 1$ diocotron mode[13]. Motivated by the good agreement between theory and experiment found in this case, we generalize the theory to include time-dependent asymmetries.

In section 3.4. we outline a calculation of the particle flux based on a solution

of the drift-kinetic Boltzmann equation. The calculations are somewhat tedious and are not included in their entirety. See reference [12] for the details of similar calculations. In section 3.5 we consider an ion plasma which is influenced by a background torque due to neutrals as well as an applied torque and look at the possibility of a achieving a stable stationary plasma. In section 3.4 we estimate the transport due to aliasing and show that in practical experiments these effects can often be neglected.

3.3   Calculation of the Flux

In this section we outline formal solutions to the drift-kinetic Boltzmann equation. These calculations are closely related to those found in reference [12] which are valid for the case of static asymmetries in the end confinement potentials. When the end confinement potentials take the form of a pure rotating asymmetry, the static results are recovered in a frame that rotates with the asymmetry.

In this analysis, we assume the following frequency ordering:

$$\Omega_c \gg |\omega_R - w|, \omega_B \gg \nu \gg \tau^{-1}, \tag{3.6}$$

where Ω_c is the cyclotron frequency, ω_B is the axial bounce frequency, ω_R is the rotation frequency, ω is the frequency of the applied asymmetry, ν is the collision frequency, and τ is the transport timescale. Since Ω_c is the largest frequency, we may describe the collisionless single particle dynamics with a guiding center Hamiltonian of the form[15]

$$H = \frac{P_z^2}{2m} + \mu B + e\Phi(p_\theta) + e\Phi_e(\theta, p_\theta, z, t), \tag{3.7}$$

where $p_\theta = eB/2cr^2$ is the canonical angular momentum conjugate to θ. We break up the potential into two parts: $\Phi(p_\theta)$ is the space charge potential and $\Phi_e(\theta, p_\theta, z, t)$

is the Debye-screened end potential. Since the Debye length is small we let

$$e\Phi_e(\theta, p_\theta, z, t) = \begin{cases} 0; & z_{min}(\theta, p_\theta, t) < z < z_{max}(p_\theta) \\ \infty; & \text{otherwise} \end{cases} \qquad (3.8)$$

where z_{min} and z_{max} are the axial positions at which the particles are reflected. In this expression we assume that Φ_e oscillates at a frequency ω which is small compared to the frequency of plasma modes. The term $\mu B = 1/2mv_\perp^2$ in the expression for H is the perpendicular kinetic energy of the particle. In the guiding center limit, $\mu = CONST.$, and since the magnetic field is assumed to be uniform, μB enters the Hamiltonian as an additive constant. We retain this term in the Hamiltonian because it is useful to write Maxwellian distribution functions as a function of H.

3.3.1 The Adiabatic Limit

For the case that the bounce frequency $\omega_B = 2\pi |v_z|/2L$ is large compared to ω and ω_R the bounce action

$$I = \frac{1}{2\pi} \oint p_z dz = \frac{1}{2\pi} \oint \sqrt{2m (H - B\mu - e\Phi - e\Phi_e)}, \qquad (3.9)$$

is a good adiabatic invariant. As noted by J.B. Taylor,[15] an equation of this form implicitly defines H in terms of I, θ, p_θ, and t. Given the simple form of the end potential, this equation is easily inverted to give

$$H(I, \theta, p_\theta, t) = \frac{\pi^2 I^2}{2mL^2(\theta, p_\theta, t)} + B\mu + e\Phi(P_\theta), \qquad (3.10)$$

where $L(\theta, p_\theta, t) = z_{max} - z_{min}$.

We represent the plasma with a distribution of guiding centers,

$$f = f(I, \psi, p_\theta, , \theta, \mu, t), \qquad (3.11)$$

where ψ is the angle conjugate to I and indicates the phase of a particle in its bounce motion (i.e. its position along the magnetic field). This distribution function evolves

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according to the drift-kinetic Boltzmann equation,

$$\frac{\partial f}{\partial t} + [f, H] = C(f), \tag{3.12}$$

where $C(\cdot)$ is the collision operator, and the Poisson-bracket is given by

$$[f, H] = \frac{\partial f}{\partial \psi}\frac{\partial H}{\partial I} + \frac{\partial f}{\partial \theta}\frac{\partial H}{\partial p_\theta} - \frac{\partial f}{\partial p_\theta}\frac{\partial H}{\partial \theta}. \tag{3.13}$$

In the adiabatic limit, $\omega_B = \partial H/\partial I$ is large and so $\partial f/\partial \psi$ must be small. Otherwise, $\partial f/\partial t$ would be large and the distribution would evolve rapidly along the magnetic field. Physically, this corresponds to the fact that any initially large ψ variations are rapidly phase mixed by the bounce motion. The small ψ variations are uninteresting from the standpoint of cross-field transport and may be eliminated by integrating Eq. (3.12) over ψ. The result is

$$\frac{\partial \bar{f}}{\partial t} + \frac{\partial \bar{f}}{\partial \theta}\frac{\partial H}{\partial p_\theta} - \frac{\partial \bar{f}}{\partial p_\theta}\frac{\partial H}{\partial \theta} = C(\bar{f}), \tag{3.14}$$

where

$$\bar{f}(I, p_\theta, \theta, \mu, t) = \int_0^{2\pi} d\psi\, f(I, \psi, p_\theta, \theta, \mu, t). \tag{3.15}$$

Rewriting Eq. (3.14) as

$$\frac{\partial \bar{f}}{\partial t} + \frac{\partial}{\partial \theta}\left[\frac{\partial H}{\partial p_\theta}\bar{f}\right] - \frac{\partial}{\partial p_\theta}\left[\frac{\partial H}{\partial \theta}\bar{f}\right] = C(\bar{f}) \tag{3.16}$$

and integrating over I, μ, θ, and the short time interval $2\pi/\omega$, yields the transport equation

$$\frac{\partial n(p_\theta)}{\partial t} = \frac{\partial}{\partial p_\theta}\left[\int \frac{d\theta}{2\pi}\int_t^{t+2\pi/\omega} dt\, \frac{\omega}{2\pi}\int dI d\mu \frac{\partial H}{\partial \theta}\bar{f}\right], \tag{3.17}$$

where

$$n(p_\theta) = \int \frac{d\theta}{2\pi}\int_t^{t+2\pi/\omega} dt\, \frac{\omega}{2\pi}\int dI d\mu \bar{f}. \tag{3.18}$$

The integral over the collision operator vanishes because collisions conserve the number of particles.

To obtain a transport equation accurate to second order in $\partial H / \partial \theta$ we must obtain $\bar{f}$ accurate to first order in $\partial H / \partial \theta$. We look for solution to Eq. (3.14) of the form

$$\bar{f} = \bar{f}_0(H, p_\theta) + \delta \bar{f}^{(0)}(I, p_\theta, \theta, t, \mu) + \delta \bar{f}^{(1)}(I, p_\theta, \theta, t, \mu), \qquad (3.19)$$

where

$$\bar{f}_0 = n(p_\theta)(2\pi T/m)^{-3/2} e^{e\Phi/T} e^{-H/T} \qquad (3.20)$$

is a Maxwellian, δ indicates terms first order in $\partial H / \partial \theta$ and the superscripts indicate an ordering in collisions. In solving perturbatively in collisions we are assuming the frequency ordering $|\omega_R - w| \gg \nu$.

Inserting the Fourier series

$$\delta \bar{f}^{(0)} = \sum_{l, \pm w} \delta \bar{f}^{(0)}_{l,w} e^{i(l\theta - wt)}, \qquad (3.21)$$

$$\delta \bar{f}^{(1)} = \sum_{l, \pm w} \delta \bar{f}^{(1)}_{l,w} e^{i(l\theta - wt)}, \qquad (3.22)$$

and

$$\delta L = \sum_{l, \pm w} \delta L_{l,w} e^{i(l\theta - wt)} \qquad (3.23)$$

into the Boltzmann equation we find the fluctuations

$$\delta \bar{f}^{(0)}_{l,w} = -\frac{1}{T} \left(\frac{\pi^2 I^2}{2mL^2} \right) \frac{2}{L} \delta L_{l,w} \bar{f}_0 \qquad (3.24)$$

and

$$\delta \bar{f}^{(1)}_{l,w} = \frac{-i}{l\omega_R - w} C \left(\bar{f}_0 - \frac{1}{T} \left(\frac{\pi^2 I^2}{2mL^2} \right) \frac{2}{L} \delta L_{l,w} \bar{f}_0 \right). \qquad (3.25)$$

Plugging these results back into the transport equation and performing the indicated integrations gives

$$\frac{\partial n(p_\theta)}{\partial t} = -\frac{\partial}{\partial p_\theta} \left[-4\nu_{\perp, \parallel} \frac{T}{L_0^2} n(p_\theta) \sum_{l, \pm w} \frac{l|\delta L_{l,w}|^2}{l\omega_R - w} \right] \qquad (3.26)$$

where $\nu_{\perp, \parallel}$ is the equipartition rate defined by

$$\nu_{\perp,\parallel} = -\frac{n}{T^2} \int d^3v \left(\frac{1}{2}mv_z^2\right) \int d^3v_1 d\sigma |v_{rel}| \left[\left(\frac{1}{2}mv_{z1}'^2 + \frac{1}{2}mv_z'^2\right) f_{M1}' f_M' \right.$$
$$\left. - \left(\frac{1}{2}mv_{z1}^2 + \frac{1}{2}mv_z^2\right) f_{M1} f_M\right]. \tag{3.27}$$

For details of a similar calculation see reference [12]. Changing variables from (p_θ, θ) to (r, θ) yields

$$\frac{\partial n(r)}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}r \left[4\nu_{\perp,\parallel}\frac{T}{L_0^2}n(r)\sum_{l,\pm w} \frac{l|\delta L_{l,w}|^2}{-\frac{eBr}{c}(l\omega_R - w)}\right] \tag{3.28}$$

The quantity in brackets is the radial particle flux.

Consider the limit of a pure rotating asymmetry. Here we take

$$L(\theta, p_\theta, t) = L_0 + |\delta L|\cos(m\theta - wt) \tag{3.29}$$

and the only nonvanishing Fourier components are

$$\delta L_{m,+w} = \delta L_{-m,-w} = \frac{1}{2}|\delta L|. \tag{3.30}$$

Plugging this into Eq. (3.28) and recognizing that $\langle \delta L^2 \rangle = 1/2\,|\delta L|^2$ yields the radial particle flux

$$\Gamma_r = 4\nu_{\perp,\parallel}n(r)\frac{T}{\frac{-eBr}{c}[m\omega_R - \omega]}\frac{m\langle \delta L^2\rangle}{L_0^2}, \tag{3.31}$$

Since the $\mathbf{E} \times \mathbf{B}$ drift rotation frequency of the plasma in a frame that rotates with the asymmetry is $\omega_R - \omega/m$, this transport flux is identical to the adiabatic flux derived in reference [12].

The transport has a simple physical interpretation. Rotational pumping causes the plasma to heat. Since energy is conserved in the frame that rotates with the asymmetry, the electrostatic energy in the frame that rotates with the asymmetry must decrease as the plasma temperature increases. The transformation to the rotating frame gives a $1/c$ $\mathbf{v} \times \mathbf{B}$ contribution to the electric field so that

$$-\frac{\partial \Phi'}{\partial r} = -\frac{\partial \Phi}{\partial r} + \frac{Br}{c}\frac{\omega}{m} \tag{3.32}$$

where Φ is the potential in the lab frame and is related to the density through Poisson's equation. If ω is sufficiently large, the direction of the electric field is reversed in the rotating frame. To change electrostatic energy into heat, the plasma must be transported radially inward rather than outward.

When $|\omega_R - \omega/m|$ becomes small the electric field in the rotating frame becomes weak and large radial transport accompanies the heating. It is important to note, however, that the rotational pumping calculation is only valid when the azimuthal $\mathbf{E} \times \mathbf{B}$ drift is larger than the drifts associated with reflections off the end potentials. These drifts are proportional to the temperature of the plasma and therefore the electrostatic potential in the frame that rotates with the asymmetry must be larger than the kinetic energy, i.e.,

$$\frac{e\left[\Phi - \frac{1}{2}\frac{Br^2}{c}\frac{\omega}{m}\right]}{T} \gg 1. \tag{3.33}$$

In terms of rotation frequencies this condition can be expressed as

$$\frac{|\omega_R - \omega/m|}{\omega_R} \gg \frac{\lambda_D{}^2}{r_p{}^2}, \tag{3.34}$$

where we have used $T/(e\Phi) \sim \lambda_D{}^2/r_p{}^2$ and have taken the $\mathbf{E} \times \mathbf{B}$ drift rotation frequency for a constant density plasma.

Our solution to the drift-kinetic Boltzmann equation assumes that the drifts associated with reflections are a small perturbation on the $\mathbf{E} \times \mathbf{B}$ drift orbits. When the condition given in Eq. (3.34) fails, the $\mathbf{E} \times \mathbf{B}$ drift becomes a small perturbation on orbits determined by the end shape of the plasma and our transport calculation is no longer valid. Fortunately, the Debye length is typically much smaller than the radius of the plasma so this regime is rather narrow.

Another limitation is that $|\omega_R - \omega/m|$ is assumed to be much larger than $\nu_{\perp,\parallel}$. When $|\omega_R - w/m| < \nu_{\perp,\parallel}$ two new effects must be considered. The heating of the plasma depends on a phase shift between the parallel and perpendicular temperature

fluctuations. In the regime of strong collisionality $(\nu_{\perp,\parallel} \gg |\omega_R - \omega/m|)$, $T_\parallel$ and $T_\perp$ remain equal to one another and the rotational pumping process is reversible. As the net heating rate decreases the transport decreases. A more subtle transport process probably dominates in this regime. In the adiabatic limit, the drift orbits of the particles are determined by the constraints of constant energy and constant bounce action. For large $|\omega_R - \omega/m|$ the drift orbits are nearly circular. This is assumed in the rotational pumping calculation. When $|\omega_R - \omega/m|$ is small the drift orbits change dramatically and the particles follow "banana orbits". This regime was discussed by Ryutov and Stupakov in their work on transport in tandem mirrors[14].

3.3.2   Resonant Particle Transport

In the previous section we neglected the effect of particles satisfying the resonance condition $l\omega_R - \omega - n\omega_B = 0$ where l and n are integers. When a particle is reflected off the non-axisymmetric end potential it experiences a force in the $\hat{\theta}$ direction causing its angular momentum, $p_\theta = \frac{eB}{2c}r^2$, and radial position to change. Particles that satisfy the resonance condition take radial steps in the same direction for many bounces. In the adiabatic limit the contribution from these particles is negligible, but when $\bar{\omega}_B$ is comparable to $lwR - \omega$ the resonant particle transport may be the dominate effect.

In this section we solve the drift-kinetic Boltzmann equation,

$$\frac{\partial f}{\partial t} + [f, H] = C(f), \qquad (3.35)$$

where the Hamiltonian is taken to be a function of z and p_z as given by Eq. (3.7). Writing the Poisson bracket as

$$[f, H] = \frac{\partial}{\partial z}\left(f\frac{\partial H}{\partial p_z}\right) - \frac{\partial}{\partial p_z}\left(f\frac{\partial H}{\partial z}\right) + \frac{\partial}{\partial \theta}\left(f\frac{\partial H}{\partial p_\theta}\right) - \frac{\partial}{\partial p_\theta}\left(f\frac{\partial H}{\partial \theta}\right) \qquad (3.36)$$

and integrating over all variables except p_θ yields the result

$$\frac{\partial n(p_\theta)}{\partial t} = \frac{\partial}{\partial p_\theta}\left[\int \frac{d\theta}{2\pi}\int_t^{t+2\pi/\omega}dt\,\frac{\omega}{2\pi}\int d^2v_\perp\int dv_z\int dzf\frac{\partial H}{\partial \theta}\right], \qquad (3.37)$$

where

$$n(p_\theta) = \int \frac{2\theta}{2\pi}\int_t^{t+2\pi/\omega}dt\,\frac{\omega}{2\pi}\int d^2v_\perp\int dv_z\int dzf. \qquad (3.38)$$

In the adiabatic limit we solved for f perturbatively in both $\partial H/\partial\theta$ and in the effective collision frequency, ν. Here we cannot solve perturbatively in ν because we are looking for an effect which is independent of the collision frequency. Instead, we approximate the collision operator by

$$C(f_0 + \delta f) = -\nu\delta f, \qquad (3.39)$$

and let $\nu \to 0$ at the end of the calculation. The effective collision frequency then drops out. This corresponds to transport in the "resonant-plateau" regime[16, 17].

The calculation is rather lengthy and is nearly identical to that found in reference [12]. Essentially one just replaces $l\omega_R$ with $l\omega_R - \omega$. The final result for the resonant particle transport equation is

$$\begin{aligned}\frac{\partial n(r)}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}r\left[4\left(\frac{1}{2}\sqrt{\pi/2}\sum_{l,n,\pm w}\frac{(l\omega_R - w)^6}{(n\bar\omega_B)^5}\frac{l|\delta L_{l,w}|^2}{L_0^2}\exp\left[-\frac{(l\omega_R - w)^2}{2(n\bar\omega_B)^2}\right]\right)\right.\\&\quad\left.\times n(r)\frac{T}{-\frac{eBr}{c}(l\omega_R - w)}\right]\end{aligned} \qquad (3.40)$$

where $\bar\omega_B = \pi\sqrt{T/m}/L_0$ is the mean bounce frequency in the plasma. As might be expected from the thermodynamic arguments, terms in the sum with $\omega/l > \omega_R$ give rise to an inward particle flux while terms with $\omega/l > \omega_R$ give rise to an outward particle flux.

In the "rotating wall" limit (Eqs. (3.29) and (3.30)) the flux is

$$\begin{aligned}\Gamma_r &= 4\left(\frac{1}{2}\sqrt{\pi/2}\sum_n\frac{(m\omega_R - \omega)^6}{(n\bar\omega_B)^5}\exp\left[-\frac{(m\omega_R - \omega)^2}{2(n\bar\omega_B)^2}\right]\right)\\&\quad\times n(r)\frac{T}{-\frac{eBr}{c}(m\omega_R - \omega)}\frac{m\langle\delta L^2\rangle}{L_0^2}.\end{aligned} \qquad (3.41)$$

In a frame that rotates with the asymmetry, the static rotational pumping transport flux[12] is again recovered. Note, however, that the coefficient of $\bar{\omega}_B$ in this expression differs by a factor of 2 from that found in reference [12]. The reason for this is that we have taken the asymmetry to be located at one end of the plasma only, so the radial steps occur with frequency ω_B. For the diocotron damping calculation in [12] the particles take steps at both ends of the plasma with frequency $2\omega_B$.

The quantity in parenthesis in Eq. (3.41) replaces $\nu_{\perp,\parallel}$ in the adiabatic transport calculated in the previous section. Written this way, one can easily see that in the adiabatic limit, $\bar{\omega}_B \gg \omega_R, \omega$ the resonant particle transport is negligible. It is interesting to note, however, that even when $\omega_R \simeq \bar{\omega}_B$ the adiabatic transport will still dominate provided that the plasma rotation frequency is close to the frequency of the rotating wall, i.e., $|l\omega_R - w|/\bar{\omega}_B \ll 1$. The important quantity is the $\mathbf{E} \times \mathbf{B}$ drift rotation frequency in a frame that rotates with the asymmetry. When this is small the resonance is located at a relatively small velocity and the resonant particles take small and infrequent steps.

3.4 Stationary States

In this section we consider an ion plasma which is subjected to an applied torque due to a rotating wall and an ambient torque due to collisions with neutrals. We look for a stationary plasma state where the net torque is zero and where the heating due to the applied torque is balanced by collisional cooling on the neutrals. This section is motivated by recent experiments in which pure ion plasmas have been maintained in a stationary state for a period of days[1].

The external torques are assumed to be weak so the plasma remains near thermal equilibrium. That is, the transport time scales associated with the external torques individually are assumed to be long in comparison to the time for coulomb

interactions (internal torques) to bring the plasma into a state of thermal equilibrium. In the limit where the Debye length is small, the thermal equilibrium density profile may be approximated by a constant density out to the plasma radius, r_p . Of course, the temperature is also uniform in thermal equilibrium.

The angular momentum evolves according to

$$\dot{P_\theta} = \tau_{wall} + \tau_N \tag{3.42}$$

where τ_{wall} is the torque due to the applied rotating asymmetry and τ_N is the ambient torque due to neutrals. The internal torques act to keep the plasma near thermal equilibrium, but they cannot change the total angular momentum. For an ion plasma P_θ is positive and therefore negative torques cause inward radial transport and positive torques cause outward radial transport.

To calculate the temperature evolution of the plasma, we first consider the heating caused by the rotating wall. In a frame that rotates with the asymmetry the energy is given by

$$E' = E - \frac{\omega}{m} P_\theta \tag{3.43}$$

where E is the energy in the lab frame and ω/m is the rotation frequency of the asymmetry (phase velocity). In the frame that rotates with the asymmetry, the perturbation is time independent and therefore $\dot{E'} = 0$. This allows one to write the rate of change of the energy in the lab frame due to the rotating wall as

$$\dot{E} = \frac{\omega}{m} \tau_{wall}. \tag{3.44}$$

In equilibrium we expect $\omega < \omega_R < 0$ and $\tau_{wall} < 0$ so the rotating wall acts to increase the energy.

Collisions with a background of cold neutrals cause the plasma energy to decrease. We denote this as

$$\dot{E} = -\gamma_c \frac{3}{2} NT, \tag{3.45}$$

where N is the total number of particles in the plasma. Here γ_c is the cooling rate and is of order (m_n/m_i) ν where ν is the ion neutral collision frequency, m_n is the mass of the neutrals, and m_i is the mass of the ions (we assume that $m_n/m_i < 1$). Summing the right hand sides of Eqs. (3.44) and (3.45) yields

$$\dot{E} = \frac{\omega}{m}\tau_{wall} - \gamma_c\frac{3}{2}NT. \tag{3.46}$$

To relate $\dot{E}$ to $\dot{T}$ we express the total energy as the sum of the thermal energy and the electrostatic energy. For a long plasma ($L/R_w \gg 1$) with constant density the energy is given by

$$E = \frac{3}{2}NT + \frac{(Ne)^2}{L}\left[\frac{1}{4} - \ln\left(\frac{r_p}{R_w}\right)\right], \tag{3.47}$$

where r_p is the radius of the plasma and R_w is the radius of the conducting cylinder. Differentiating this equation with respect to time yields

$$\dot{E} = \frac{3}{2}N\dot{T} - \frac{(Ne)^2}{L}\frac{\dot{r_p}}{r_p}. \tag{3.48}$$

The total torque acting on the plasma is related to $\dot{r_p}$ by

$$\tau_{wall} + \tau_N = \frac{\partial}{\partial t}\left(\frac{eB}{2c}N\frac{r_p{}^2}{2}\right) = \frac{eB}{2c}Nr_p\dot{r_p} \tag{3.49}$$

and the $\mathbf{E} \times \mathbf{B}$ drift rotation frequency of the plasma is given by

$$\omega_R = -\frac{2c}{B}\frac{eN}{r_p{}^2 L}. \tag{3.50}$$

From these two equations one can show that

$$\frac{(Ne)^2}{L}\frac{\dot{r_p}}{r_p} = -\omega_R(\tau_{wall} + \tau_N), \tag{3.51}$$

and therefore

$$\dot{E} = \frac{3}{2}N\dot{T} + \omega_R(\tau_{wall} + \tau_N). \tag{3.52}$$

Combining this equation with Eq. (3.46) then yields an expression for the temperature evolution in the plasma due to the applied torque and the neutral cooling

$$\frac{3}{2}N\dot{T} = -\omega_R(\tau_{wall} + \tau_N) + \frac{\omega}{m}\tau_{wall} - \frac{3}{2}N\gamma_c T. \qquad (3.53)$$

The first term in this equation represents a transformation of electrostatic energy into heat, the second term represents heating from the applied torque and the last term represents neutral cooling. In equilibrium the torques balance and the first term is zero. The equilibrium temperature in the plasma is then determined by a balance between the neutral cooling and the heating from the applied torque. It is interesting to note that since $\tau_{wall} = -\tau_N$ in equilibrium, the temperature may be found without specifying the transport due to the rotating wall (i.e. $T = -2\omega\tau_N/(3N\gamma_c m)$).

To find an equilibrium and analyze it for stability it is convenient to convert Eqs. (3.42) and (3.53) into equations for $\dot{\omega}_B$ and $\dot{\omega}_R$. The bounce frequency for a thermal particle is defined by

$$\bar{\omega}_B = \frac{\pi}{L}\sqrt{\frac{T}{m_i}}, \qquad (3.54)$$

and therefore

$$\dot{T} = 2m_i\left(\frac{L}{\pi}\right)^2 \bar{\omega}_B\dot{\omega}_B. \qquad (3.55)$$

The total angular momentum of the plasma is

$$P_\theta = \frac{eB}{2c}N\frac{r_p^2}{2}, \qquad (3.56)$$

and the $\mathbf{E} \times \mathbf{B}$ drift rotation frequency is

$$\omega_R = -\frac{2c}{B}\frac{eN}{r_p^2 L}. \qquad (3.57)$$

Eliminating r_p^2 from these two equations yields

$$P_\theta = -\frac{(eN)^2}{2L}\frac{1}{\omega_R}, \qquad (3.58)$$

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from which one finds

$$\dot{P}_\theta = \frac{(eN)^2}{2L} \frac{\dot{\omega}_R}{\omega_R{}^2}.$$ (3.59)

Eqs. (3.42) and (3.53) can then be expressed as

$$\dot{\omega}_R = F(\omega_R, \bar{\omega}_B),$$ (3.60)

$$\dot{\bar{\omega}}_B = G(\omega_R, \bar{\omega}_B),$$ (3.61)

where

$$F(\omega_R, \bar{\omega}_B) = \left(\frac{2L}{N^2 e^2}\right) \omega_R{}^2 \left(\tau_{wall} + \tau_N\right),$$ (3.62)

$$G(\omega_R, \bar{\omega}_B) = -\left(\frac{1}{3N} \frac{\pi^2}{m_i L^2}\right) \left[\frac{\omega_R}{\bar{\omega}_B} \left(\tau_{wall} + \tau_N\right) - \frac{\omega/m}{\bar{\omega}_B} \tau_{wall}\right] - \frac{1}{2} \gamma_c \bar{\omega}_B.$$ (3.63)

Equilibria are determined by the coupled equations

$$F(\omega_R, \bar{\omega}_B) = 0,$$ (3.64)

$$G(\omega_R, \bar{\omega}_B) = 0.$$ (3.65)

Linear stability may be analyzed in the usual way by Taylor expanding the right hand sides of Eqs. (3.60) and (3.61) about the equilibrium as

$$\dot{\delta\omega}_R = \delta\omega_R \frac{\partial F}{\partial \omega_R} + \delta\bar{\omega}_B \frac{\partial F}{\partial \bar{\omega}_B},$$ (3.66)

$$\dot{\delta\bar{\omega}}_B = \delta\omega_R \frac{\partial G}{\partial \omega_R} + \delta\bar{\omega}_B \frac{\partial G}{\partial \bar{\omega}_B},$$ (3.67)

where the derivatives are understood to be evaluated at the equilibrium values of ω_R and $\bar{\omega}_B$. The stability condition for this linear system is

$$\left(\frac{\partial F}{\partial \omega_R} + \frac{\partial G}{\partial \bar{\omega}_B}\right) \pm \sqrt{\left(\frac{\partial F}{\partial \omega_R} + \frac{\partial G}{\partial \bar{\omega}_B}\right)^2 - 4\left(\frac{\partial F}{\partial \omega_R} \frac{\partial G}{\partial \bar{\omega}_B} + \frac{\partial F}{\partial \bar{\omega}_B} \frac{\partial G}{\partial \omega_R}\right)} < 0.$$ (3.68)

When this condition fails the equilibrium is unstable. Often, the question of stability may be answered if the signs of the derivatives of F and G are known, even if their

magnitudes are unknown. For example, if $\partial F/\partial \omega_R$ and $\partial G/\partial \bar{\omega}_B$ are both positive then the equilibrium is clearly unstable. Similarly if

$$\frac{\partial F}{\partial \omega_R}\frac{\partial G}{\partial \bar{\omega}_B} + \frac{\partial F}{\partial \bar{\omega}_B}\frac{\partial G}{\partial \omega_R} < 0 \tag{3.69}$$

then the equilibrium is also unstable.

In order to evaluate the derivatives of F and G, one needs to find expressions for the torques in terms of ω_R and $\bar{\omega}_B$. Since the angular momentum of the plasma is

$$P_\theta = \frac{eB}{2c} \int d^3r \; r^2 n(r), \tag{3.70}$$

an external torque may be expressed as

$$\tau = \dot{P}_\theta = \frac{eB}{2c} \int d^3r \; r^2 \frac{\partial n}{\partial t} = \frac{eB}{2c} \int d^3r \; r^2 (-\frac{1}{r}\frac{\partial}{\partial r}(r\Gamma_r)), \tag{3.71}$$

where Γ_r is the transport flux associated with the external torque. Integrating by parts yields the result

$$\tau = \frac{eB}{c}(2\pi L) \int_0^{r_p} dr \; r^2 \Gamma_r, \tag{3.72}$$

where L is the length of the plasma. Given a transport flux, Eq. (3.72) provides a means of calculating the torque. Note that all internal torques cancel in this integral.

The radial particle flux due to collisions with neutrals is [6]

$$\Gamma_r^{neutrals} = \frac{\partial}{\partial r}(\nu r_L^2 n) + \nu r_L^2 \frac{eE}{T}n, \tag{3.73}$$

where ν is the ion-neutral collision frequency and r_L is the Larmor radius. Since the Debye length is small compared to the radius of the plasma, the diffusive term in this equation is negligible and Eq. (3.72) gives a neutral torque of

$$\tau_N = \frac{(eN)^2}{L}\frac{\nu}{\Omega_c}, \tag{3.74}$$

where Ω_c is the cyclotron frequency.

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When one takes derivatives of the torques with respect to ω_R it is the radius of the plasma, or equivalently the plasma density, which is allowed to vary. The magnetic field is not a dynamical variable. Therefore,

$$\frac{\partial \tau_N}{\partial \omega_R} = 0. \tag{3.75}$$

To calculate $\partial \tau_N / \partial \bar{\omega}_B$, we choose a simple hard sphere model for the ion-neutral collisions so that

$$\nu \propto \bar{v} \propto \bar{\omega}_B, \tag{3.76}$$

and thus

$$\frac{\partial \tau_N}{\partial \bar{\omega}_B} = \frac{\tau_N}{\bar{\omega}_B} = -\frac{\tau_{wall}}{\bar{\omega}_B}. \tag{3.77}$$

In this expression we have explicitly used the fact that the derivatives are to be evaluated at an equilibrium where $\tau_{wall} = -\tau_N$.

This result allows one to write the derivatives of F and G in terms of τ_{wall} only:

$$\frac{\partial F}{\partial \omega_R} = \left(\frac{2L}{N^2 e^2}\right) \omega_R^2 \frac{\partial \tau_{wall}}{\partial \omega_R}, \tag{3.78}$$

$$\frac{\partial F}{\partial \bar{\omega}_B} = \left(\frac{2L}{N^2 e^2}\right) \omega_R^2 \bar{\omega}_B \frac{\partial}{\partial \bar{\omega}_B} \left(\frac{\tau_{wall}}{\bar{\omega}_B}\right), \tag{3.79}$$

$$\frac{\partial G}{\partial \omega_R} = -\left(\frac{1}{3N} \frac{\pi^2}{m_i L^2}\right) \left(\frac{\omega_R - \omega/m}{\bar{\omega}_B}\right) \frac{\partial \tau_{wall}}{\partial \omega_R}, \tag{3.80}$$

$$\frac{\partial G}{\partial \bar{\omega}_B} = -\left(\frac{1}{3N} \frac{\pi^2}{m_i L^2}\right) \left(\frac{\omega_R - \omega/m}{\bar{\omega}_B}\right) \bar{\omega}_B \frac{\partial}{\partial \bar{\omega}_B} \left(\frac{\tau_{wall}}{\bar{\omega}_B}\right) - \gamma_c. \tag{3.81}$$

In the last two equations we have also used the fact that $\gamma_c \propto \bar{\omega}_B$ which follows from Eq. (3.76). In equilibrium $\omega_R - \omega/m > 0$ because the "wall" must be rotating faster than the plasma (for an ion plasma $\omega_R < 0$).

If one can show that

$$\bar{\omega}_B \frac{\partial}{\partial \bar{\omega}_B} \left(\frac{\tau_{wall}}{\bar{\omega}_B}\right) > 0, \tag{3.82}$$

then the stability is determined by the sign of $\partial \tau_{wall}/\partial \omega_R$. If

$$\frac{\partial \tau_{wall}}{\partial \omega_R} < 0 \tag{3.83}$$

then

$$\frac{\partial F}{\partial \omega_R} < 0, \tag{3.84}$$

$$\frac{\partial F}{\partial \bar{\omega}_B} > 0, \tag{3.85}$$

$$\frac{\partial G}{\partial \omega_R} > 0, \tag{3.86}$$

$$\frac{\partial G}{\partial \bar{\omega}_B} < 0. \tag{3.87}$$

One can see that Eq. (3.68) is then satisfied and the equilibrium is stable. Similarly if

$$\frac{\partial \tau_{wall}}{\partial \omega_R} > 0 \tag{3.88}$$

then

$$\frac{\partial F}{\partial \omega_R} > 0, \tag{3.89}$$

$$\frac{\partial F}{\partial \bar{\omega}_B} > 0, \tag{3.90}$$

$$\frac{\partial G}{\partial \omega_R} < 0, \tag{3.91}$$

$$\frac{\partial G}{\partial \bar{\omega}_B} < 0, \tag{3.92}$$

and the equilibrium is unstable.

First consider the torque due to resonant particle transport. We expect this to be the dominant effect when $|\omega_R - \omega/m| \sim \bar{\omega}_B$. Substituting the particle flux form Eq. (3.41) into Eq. (3.72) yields

$$\tau_{wall} = -Nm_i \left(\frac{L}{\pi}\right)^2 \bar{\omega}_B{}^2 \epsilon \sum_{n=1}^{\infty} \frac{(m\omega_R - \omega)^5}{(n\bar{\omega}_B)^5} \exp\left(-\frac{(m\omega_R - \omega)^2}{2(n\bar{\omega}_B)^2}\right) \tag{3.93}$$

where N is the total number of particles in the plasma and

$$\epsilon = \frac{\sqrt{8\pi}}{r_p{}^2} m \int_0^{r_p} dr \; r \frac{\langle \delta L^2 \rangle}{L^2} \tag{3.94}$$

is a measure of the strength of the rotating wall. On general grounds one should expect ϵ to be proportional to the square of the applied voltage.

The evaluation of the derivatives of τ_{wall} can be simplified somewhat if one expresses the torque as

$$\tau_{wall} = \left[Nm_i \left(\frac{L}{\pi} \right)^2 \right] \bar{\omega_B}^2 \epsilon f(x), \tag{3.95}$$

where

$$f(x) = -\sum_{n=1}^{\infty} \left(\frac{x}{n} \right)^5 \exp\left[-\frac{1}{2} \left(\frac{x}{n} \right)^2 \right] \tag{3.96}$$

and

$$x = \frac{m\omega_R - \omega}{\bar{\omega_B}}. \tag{3.97}$$

One then finds that

$$\bar{\omega_B} \frac{\partial}{\partial \bar{\omega_B}} \left(\frac{\tau_{wall}}{\bar{\omega_B}} \right) = \left[Nm_i \left(\frac{L}{\pi} \right)^2 \right] \epsilon \bar{\omega_B} \left(f(x) - x \frac{\partial f}{\partial x} \right) \tag{3.98}$$

and that

$$\frac{\partial \tau_{wall}}{\partial \omega_R} = \left[Nm_i \left(\frac{L}{\pi} \right)^2 \right] \bar{\omega_B}^2 \epsilon \left(\frac{m}{\bar{\omega_B}} \frac{\partial f}{\partial x} + \frac{1}{\epsilon} \frac{\partial \epsilon}{\partial \omega_R} f(x) \right) \tag{3.99}$$

In Fig. 3.3 we plot $f(x)$, $\partial f/\partial x$ and the $f - x\partial f/\partial x$. When $f - x\partial f/\partial x > 0$ the quantity in Eq. (3.98) is positive and the stability of the equilibrium may be determined from the sign of $\partial \tau_{wall}/\partial \omega_R$. In Fig. 3.3 it is clear that $\partial f/\partial x < 0$ when $f - x\partial f/\partial x > 0$. Although the functional dependence of ϵ on ω_R may be quite complicated, we can determine the sign of $\partial \epsilon/\partial \omega_R$ with a simple physical argument. As r_p increases, the plasma moves closer to the applied rotating asymmetry and we expect ϵ, the effective strength of the rotating asymmetry, to increase. That is, $\partial \epsilon/\partial r_p > 0$. Since $\omega_R \sim -r_p^{-2}$ we expect

$$\frac{\partial \epsilon}{\partial \omega_R} = \frac{\partial r_p}{\partial \omega_R} \frac{\partial \epsilon}{\partial r_p} > 0 \tag{3.100}$$

and the quantity $\epsilon^{-1} f \partial \epsilon/\partial \omega_R < 0$. Therefore, in the region where $f - x\partial f/\partial x > 0$, $\partial \tau_{wall}/\partial \omega_R < 0$ and the equilibrium is stable. In recent experiment[1] a stable ion

plasma has been observed with $x \simeq 2$ which is consistent with the analysis presented here. When $f - x \partial f / \partial x < 0$ the stability analysis is more complicated. In the region where $\partial f / \partial x > 0$, for example, one would need to determine the magnitude of $\partial \epsilon / \partial \omega_R$.

Now consider the case where $|\omega_R - \omega/m|/\bar{\omega}_B$ is small. In this limit, the resonant particle transport is negligible (as can be seen from the behavior of $f(x)$ for small x) and the adiabatic transport flux is the dominant effect. Substituting the adiabatic flux form Eq. (3.31) into Eq. (3.72) yields

$$\tau_{wall} = -\left[\sqrt{\frac{8}{\pi}} N \left(\frac{L}{\pi} \right)^2 m_i \right] \epsilon \bar{\omega}_B{}^2 \frac{\nu_{\perp,\,\parallel}}{m\omega_R - \omega}. \qquad (3.101)$$

In the limit where the plasma is weakly magnetized,

$$\nu_{\perp,\,\parallel} \propto nT^{-\frac{3}{2}} \propto -\frac{\omega_R}{\bar{\omega}_B{}^3}. \qquad (3.102)$$

Therefore,

$$\tau_{wall} \propto \epsilon \frac{1}{\bar{\omega}_B} \frac{\omega_R}{m\omega_R - \omega}, \qquad (3.103)$$

and it is easy to show that

$$\bar{\omega}_B \frac{\partial}{\partial \bar{\omega}_B} \left(\frac{\tau_{wall}}{\bar{\omega}_B} \right) > 0. \qquad (3.104)$$

Once again, the sign of $\partial \tau_{wall} / \partial \omega_R$ is sufficient to determine stability.

In the regime where the adiabatic transport is large, the behavior of τ_{wall} is dominated by the small quantity $(\omega_R - \omega/m)/\omega_R$ and one may treat ϵ as a constant. For fixed $\bar{\omega}_B$ the torque can be expressed as

$$\tau_{wall} \simeq -\kappa/y, \qquad (3.105)$$

where $y = (\omega_R - \omega/m)/|\omega_R|$ and κ is a positive constant. Note that for small y

$$\frac{\partial \tau_{wall}}{\partial \omega_R} \simeq \frac{1}{|\omega_R|} \frac{\partial \tau}{\partial y}. \qquad (3.106)$$

In Fig. 3.4 we plot τ_{wall} vs. y. For positive values of y (where an equilibrium is possible) $\partial \tau_{wall}/\partial y > 0$ and the equilibrium is unstable.

In Fig. 3.4 the torque diverges as $|y| \to 0$. As argued in section 3.3.1 the assumptions underlying our transport theory break down when $|y| < \lambda_D{}^2/r_p{}^2$. On physical grounds, one should expect the torque curve to turn over and go to zero as $|y| \to 0$. This behavior is illustrated in Fig. 3.5 where the dashed curve indicates the expected behavior of the torque.

Along the dashed curve, $\partial \tau_{wall}/\partial \omega_R < 0$ and one would expect the equilibrium to be stable. In this regime, one must also assume that the torque satisfies the condition

$$\bar{\omega}_B \frac{\partial}{\partial \bar{\omega}_B} \left(\frac{\tau_{wall}}{\bar{\omega}_B} \right) > 0. \tag{3.107}$$

To formally derive the transport flux in this region one could solve the drift-kinetic Boltzmann equation in the limit were the electrostatic energy in the frame that rotates with the asymmetry is small in comparison to the temperature. Provided that $|\omega_R - \omega/m| > \nu_{\perp, \parallel}$ one could still solve perturbatively in the effective collision frequency and one would expect to find a transport flux which is proportional to ν. Since $\nu \sim T^{-3/2} \sim \bar{\omega}_B{}^{-3}$ we expect the magnitude of the torque to decrease as $\bar{\omega}_B$ increases. Therefore, the assumption given in Eq. (3.107) seems to be a reasonable one.

3.5 Aliasing

In this section we consider the confinement geometry shown in Fig. 3.2 and estimate the transport due to aliasing. One of the end confinement cylinders is divided azimuthally into N sectors. The voltage on each sector oscillates with frequency ω, but the phases are set independently. The voltage on the confinement

cylinder may be expressed as

$$\Phi_c(\theta, t) = V_c + \delta V \cos[\phi_j - \omega t], \qquad \frac{2\pi}{N}(j-1) < \theta < \frac{2\pi}{N}j \qquad (3.108)$$

Here V_c is a constant confinement voltage and ϕ_j is the phase of the voltage on the jth section.

Simulating a rotating wall amounts to a careful choice of the phases, ϕ_j. The azimuthal position in the center of the jth sector is

$$\theta_j = \frac{2\pi}{N}j - \frac{\pi}{N}. \qquad (3.109)$$

In order to simulate a rotating wall as given in Eq. (3.5) one would set $\phi_j = m\theta_j$ so that

$$\Phi_c(\theta, t) = V_c + \delta V \cos\left[m\left(\frac{2\pi}{N}j - \frac{\pi}{N}\right) - \omega t\right], \qquad \frac{2\pi}{N}(j-1) < \theta < \frac{2\pi}{N}j. \qquad (3.110)$$

Using

$$\delta V_{l,\pm w} = \int_t^{t+2\pi/\omega} dt\, \frac{\omega}{2\pi}\, e^{\pm i\omega t} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-il\theta} \Phi_c(\theta, t) \qquad (3.111)$$

the Fourier components are found to be

$$\delta V_{l,+w} = \delta V_{-l,-w} = \begin{cases} \frac{1}{2\pi l}\delta V \sin(m\frac{\pi}{N})N; & l = m - pN \\ 0; & \text{otherwise} \end{cases} \qquad (3.112)$$

where p is an integer. The largest term is the $p = 0$ term which corresponds to a traveling wave. In the limit $N \to \infty$ this is the only component that survives and is equal to

$$\delta V_{m,+w} = \delta V_{-m,-w} = \frac{1}{2}\delta V. \qquad (3.113)$$

This is the rotating wall limit.

The most important aliased component that contributes to outward radial particle transport is the $p = 1$ component. Taking only terms for $p = 0$ and $p = 1$ we write the potential on the confinement cylinder as

$$\Phi_c(\theta, t) = V_c + \delta V \cos(m\theta - \omega t) - \frac{\delta V}{N - m} \cos((N - m)\theta + \omega t). \qquad (3.114)$$

Expressed in this way one can easily see the aliased component traveling in the opposite direction.

To accurately find the length fluctuations that result from the confinement potentials in this equation, one can follow the procedure of Cluggish and Driscoll and use a three dimensional Poisson solver[13]. Here we estimate how the length fluctuations scale by assuming that δL is proportional to the potential fluctuations evaluated at the end of the plasma. This idea was used by Peurrung and Fajans [18] to estimate the shape of a displaced plasma column.

Solving Laplaces equation with the boundary conditions of Eq. (3.88) we find that the potential at the end of the plasma is approximately given by A solution to Laplace's equation with the boundary conditions given in Eq. (3.114) is given by

$$
\begin{aligned}
\delta\Phi \quad \propto \quad & J_m(\chi_{m,1} r/R_w)\exp[-\chi_{m,1}\Delta z/R_w]\cos(m\theta - \omega t) - \\
& J_{N-m}(\chi_{(N-m),1} r/R_w)\exp[-\chi_{(N-m),1}\Delta z/R_w] \\
& \times \cos((N-m)\theta + \omega t)
\end{aligned}
\tag{3.115}
$$

where Δz is the distance from the end of the plasma to the confinement cylinder and $\chi_{l,1}$ is the first zero of the Bessel function J_l. The length fluctuations may then be estimated as

$$
\delta L(\theta, t) \simeq \delta L \cos(m\theta - \omega t) + \delta L^A \cos((N-m)\theta + \omega t)
\tag{3.116}
$$

where

$$
\frac{|\delta L^A|}{|\delta L|} \sim \frac{1}{N-m}\frac{J_{N-m}(\chi_{(N-m),1} r/R_w)}{J_m(\chi_{m,1} r/R_w)}\exp[-(\chi_{(N-m),1} - \chi_{m,1})\Delta z/R_w].
\tag{3.117}
$$

One might have expected the aliased component to be small by a factor of $1/(N-m)$. In fact, the aliased component is exponentially small. This is simply due to the fact that the aliased components have a large azimuthal mode number and these do not penetrate very far into a cylinder. The potential is smoothed with distance and

the low azimuthal mode numbers dominate. Since the transport scales as $|\delta L^2|$, the outward radial transport due to aliasing is typically exponentially small. Of course, this conclusion assumes that $N > 2m$ so that the aliased component of the potential does have a larger azimuthal mode number.
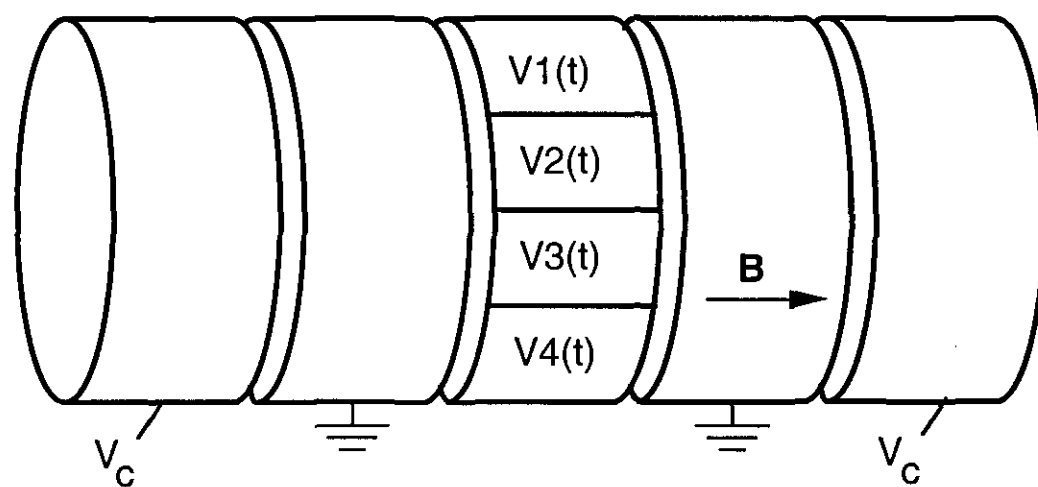
82

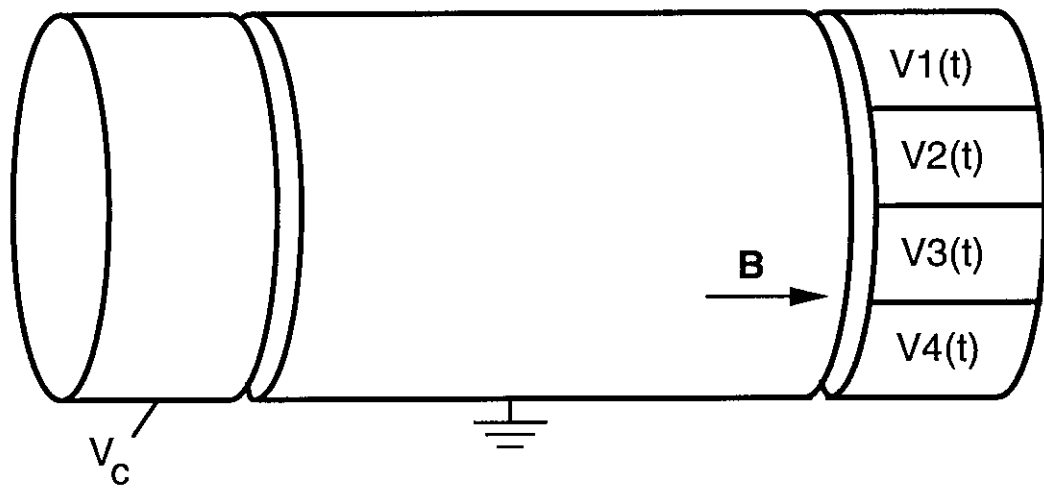**Figure 3.1:** The confinement geometry.

Figure 3.2: The confinement geometry with time-dependent voltages positioned at one end of the plasma
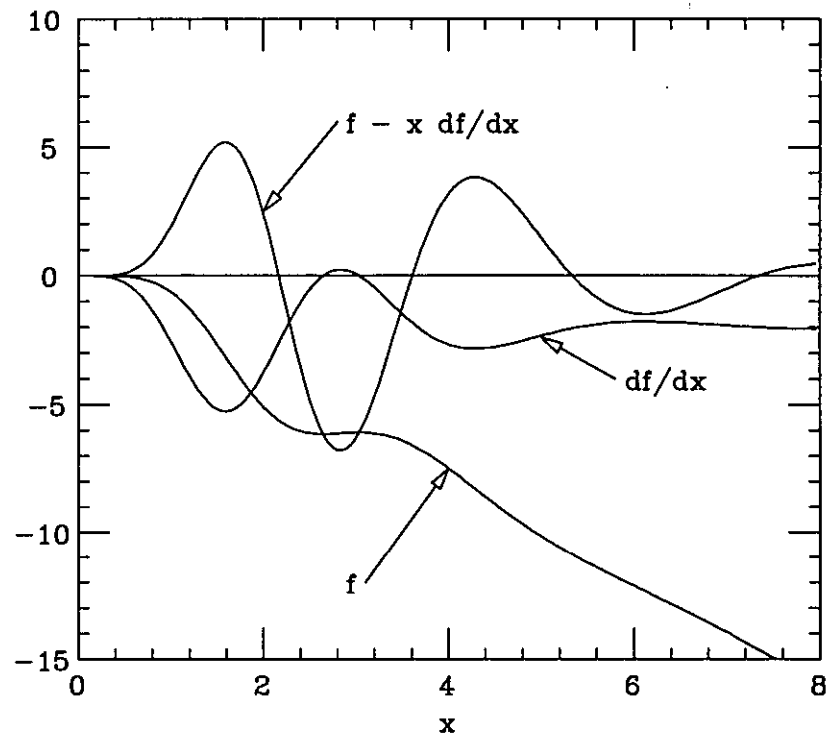
Figure 3.3: $f(x)$, $\partial f/\partial x$ and $f(x) - x\partial f/\partial x$ where $x = (m\omega_R - \omega)/\bar{\omega}_B$
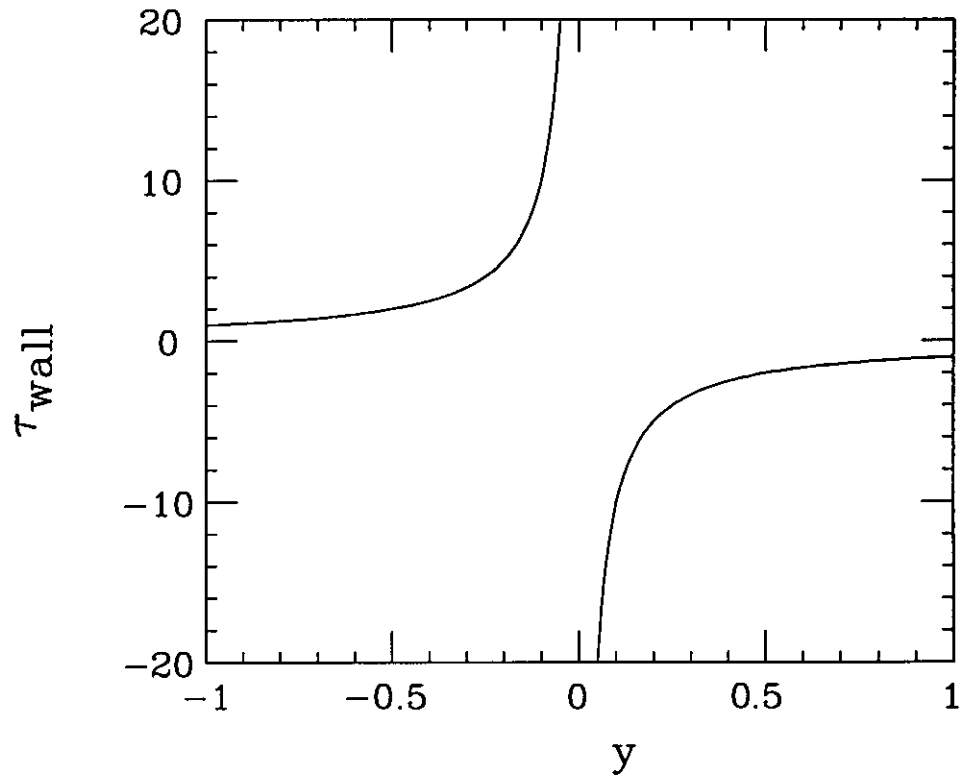
Figure 3.4: The adiabatic torque plotted as a function $y = (\omega_R - \omega/m)/|\omega_R|$
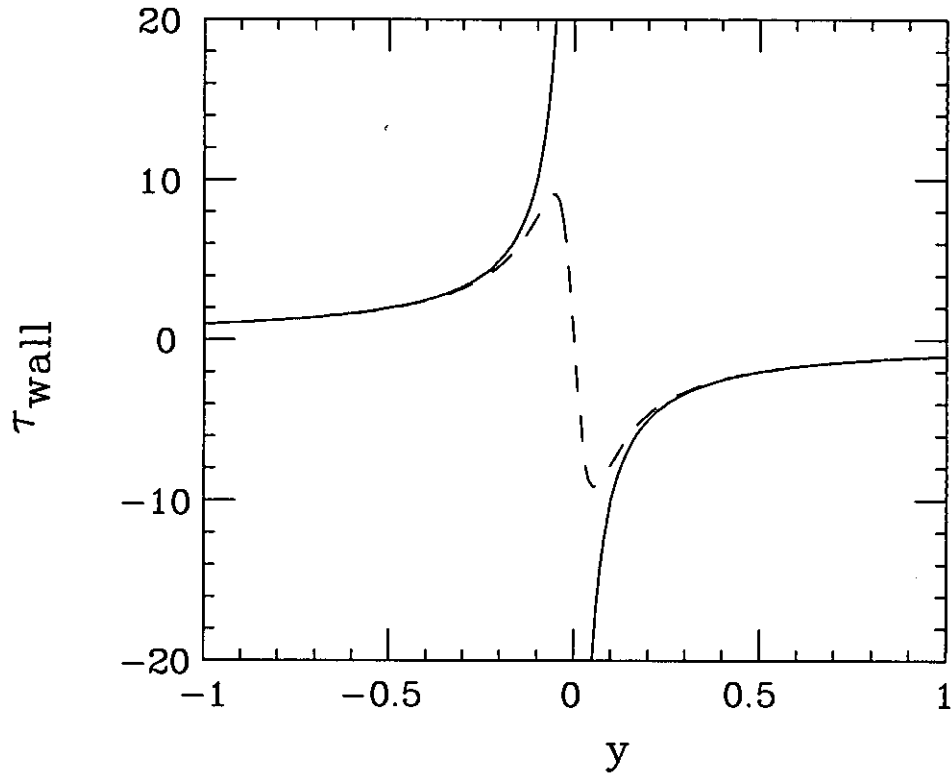
Figure 3.5: The adiabatic torque plotted as a function $y = (\omega_R - \omega/m)/|\omega_R|$ and the expected behavior of the torque for small values of $|y|$ (the dashed curve).

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Chapter 4

Transport in a Toroidally Confined Pure Electron Plasma

4.1 Abstract

O'Neil and Smith have argued that a pure electron plasma can be confined stably in a toroidal magnetic field configuration. This paper shows that the toroidal curvature of the magnetic field of necessity causes slow cross-field transport. The transport mechanism is similar to magnetic pumping and may be understood by considering a single flux tube of plasma. As the flux tube of plasma undergoes poloidal $\mathbf{E} \times \mathbf{B}$ drift rotation about the center of the plasma, the length of the flux tube and the magnetic field within the flux tube oscillate, and this produces corresponding oscillations in $T_{\parallel}$ and $T_{\perp}$. The collisional relaxation of $T_{\parallel}$ toward $T_{\perp}$ produces a slow dissipation of electrostatic energy into heat and a consequent expansion (cross-field transport) of the plasma. In the limit where the cross-section of the plasma is nearly circular the radial particle flux is given by $\Gamma_r = \frac{1}{2}\nu_{\perp,\parallel}T(r/\rho_0)^2 n/(-e\partial\Phi/\partial r)$, where $\nu_{\perp,\parallel}$ is the collisional equipartition rate, ρ_0 is the major radius at the center of the plasma, and r is the minor radius measured from the center of the plasma. The transport flux is first calculated using this simple physical picture and then is calculated by solving the drift-kinetic Boltzmann equation. This latter calculation is not limited to a plasma with a circular cross section.

4.2 Introduction

Pure electron plasmas confined by a toroidal magnetic field have been studied both experimentally and theoretically since the 1960's[1 − 3] and have received renewed attention in recent years[4, 5]. The equilibria of pure electron plasmas confined by a toroidal magnetic field were studied by Dougherty and Levy[2]. The equilibria exist due to the strong space charge electric fields that arise because the plasma is nonneutral. These fields cause particle drift orbits to be closed. One can think of the poloidal $\mathbf{E} \times \mathbf{B}$ drifts as providing the rotational transform.

Recently, O'Neil and Smith [5] argued that a pure electron plasma can be confined stably in such a configuration when the frequencies are ordered so that the cross-field motion may be described by toroidal-averaged drift dynamics. They found equilibria for which the energy is a maximum relative to neighboring states. The system point evolves on a contour of constant energy in the space of accessible states, and when the energy is a maximum, the contour shrinks to a point and no further change in the state is possible.

In this paper we obtain a collisional transport equation for a pure electron plasma that is confined in this geometry. We assume the same frequency ordering that was used to analyze stability [5]

$$\Omega_c \gg \bar{\omega}_T \gg \omega_E \gg \nu \gg \tau^{-1}, \tag{4.1}$$

where Ω_c is the cyclotron frequency, $\bar{\omega}_T = \bar{v}/\rho$ is the toroidal frequency for a thermal particle, $\omega_E \sim \omega_p^2/\Omega_c$ is the characteristic $\mathbf{E} \times \mathbf{B}$ drift frequency in the poloidal direction, ν is the collision frequency, and τ is the transport timescale. The length scale ordering is

$$\rho_0 \gg r \gg \lambda_D \tag{4.2}$$

where ρ_0 is the major radius at the center of the plasma, r is the minor radius

measured from the center of the plasma, and λ_D is the Debye length. We also assume that $r^2 \omega_E{}^2/c^2 \ll 1$ so that the diamagnetic corrections to the magnetic field are negligible. These conditions are well satisfied in typical experiments.

There are two ways to understand the transport. One can focus on a flux tube and note that the length of the tube and the magnetic field strength in the tube vary as the tube undergoes poloidal $\mathbf{E} \times \mathbf{B}$ drift rotation. The constancy of the adiabatic invariants $\mu = m v_\perp{}^2/2B$ and $I = (2\pi)^{-1} \oint dl\, m v_\parallel$ then imply a cyclic variation in $T_\parallel$ and $T_\perp$. The variations are unequal, and collisional relaxation between $T_\parallel$ and $T_\perp$ produces a slow heating of the plasma. This heating comes about at the expense of electrostatic energy, so the plasma must expand in minor radius. In section 4.3, we use this viewpoint to calculate the radial flux for the simple case where the plasma has circular cross section.

Alternatively, one can focus directly on the drift orbits as determined by the particle energy H and the adiabatic invariants μ and I. When a particle undergoes velocity scattering in a collision, these quantities change value and the drift orbit changes, allowing the particle to step in radius. In section 4.4, the drift kinetic Boltzmann equation is used to calculate th flux. This calculation does not require the plasma cross section to be circular but reduces to the result of section 4.3 when a circular cross section is specified.

4.3 Heating and Transport

A schematic diagram for a toroidal trap is shown in Figure 4.1. The confinement region is bounded by a toroidal conductor and the magnetic field is purely toroidal. Here (ρ, Θ, z) is a cylindrical coordinate where ρ is the major radius, Θ is the toroidal angle, and the z-axis is the axis of symmetry of the torus.

In this section we assume for simplicity that the toroidal conductor and the

plasma both have a circular cross section. This assumption is not necessary and will be relaxed in section 4.4. For the case of a circular cross section it is useful to introduce a polar coordinate system (r, θ) which is centered on the plasma and is locally oriented perpendicular to the magnetic field. Here θ is the poloidal angle and r is the minor radius measured from the center of the plasma, The (ρ, Θ, z) coordinate system and the (r, θ) coordinate system are related through the relations

$$\rho = \rho_0 + r\cos\theta \qquad (4.3)$$

$$z = r\sin\theta, \qquad (4.4)$$

where ρ_0 is the major radius at the center of the plasma.

We derive an expression for the radial particle flux by considering a single flux tube of plasma as shown in Fig. 4.2. The flux tube has length $L(r, \theta) = 2\pi\rho(r, \theta)$, cross sectional area δA, and contains δN particles, where δN is a constant. Using Ampere's law, the toroidal magnetic field can be expressed as $\mathbf{B} = \hat{\Theta}B_0\rho_0/\rho$ where B_0 and ρ_0 are constants. Thus, the field strength in the flux tube is $B_0\rho_0/\rho(r, \theta)$, where $\rho(r, \theta)$ is given by Eq. (4.1). The dominant motion of the flux tube is the $\mathbf{E} \times \mathbf{B}$ drift. Under the assumption of a small inverse aspect ratio ($r/\rho_0 \ll 1$) the electric field is nearly radial and the flux tube drifts in a circular orbit with frequency

$$\omega_E = \frac{c}{Br}\frac{\partial\Phi(r)}{\partial r}. \qquad (4.5)$$

As the flux tube drifts toward the inside of the torus its length decreases and the magnetic field inside the flux tube increases. Setting $\theta = \omega_E t$ and using $r/\rho_0 \ll 1$ yields

$$L(t) = 2\pi\rho_0 + 2\pi r\cos\omega_E t$$

$$B(t) = B_0 - B_0\frac{r}{\rho_0}\cos\omega_E t. \qquad (4.6)$$

Since the magnetic moment, $\mu = mv_\perp^2/2B$, is a constant, the perpendicular velocity of each particle increases as the magnetic field increases. Of course,

the average magnetic moment is also constant and is related to the perpendicular temperature by

$$CONST. = \frac{1}{\delta N} \sum_{i=1}^{\delta N} \frac{\frac{1}{2}mv_{\perp i}^2}{B} = \frac{1}{B}T_\perp. \tag{4.7}$$

Differentiating this equation with respect to time yields an equation for the perpendicular temperature evolution

$$\frac{\partial T_\perp}{\partial t} = \frac{1}{B}\frac{\partial B}{\partial t}T_\perp. \tag{4.8}$$

Similarly, the average of the square of the individual toroidal actions is a constant and is related to the parallel temperature by

$$CONST. = \frac{1}{\delta N} \sum_{i=1}^{\delta N} (Lmv_i)^2 = \frac{2}{m}L^2 T_\parallel. \tag{4.9}$$

Differentiating this expression with respect to time yields

$$\frac{\partial T_\parallel}{\partial t} = -\frac{2}{L}\frac{\partial L}{\partial t}T_\parallel. \tag{4.10}$$

The parallel and perpendicular temperatures also couple collisionally so that the full temperature evolution is more accurately described by

$$\frac{\partial T_\parallel}{\partial t} = -\frac{2}{L}\frac{\partial L}{\partial t}T_\parallel + 2\nu_{\perp,\parallel}(T_\perp - T_\parallel) \tag{4.11}$$

and

$$\frac{\partial T_\perp}{\partial t} = \frac{1}{B}\frac{\partial B}{\partial t}T_\perp - \nu_{\perp,\parallel}(T_\perp - T_\parallel) \tag{4.12}$$

where $\nu_{\perp,\parallel}$ is the equipartition rate. The factor of two difference in the collisional coupling term for Eq. (4.12) relative to Eq. (4.11) simply reflects the fact that there are two perpendicular degrees of freedom and one parallel.

A two time scale analysis of these equations based on the smallness of r/ρ_0 and on the frequency ordering $\omega_E \gg \nu$ yields the result

$$\frac{d\left[\frac{1}{2}\langle T_\parallel \rangle + \langle T_\perp \rangle\right]}{dt} = \frac{1}{2}\nu_{\perp,\parallel}\left(\frac{r}{\rho_0}\right)^2 T \tag{4.13}$$

where $\langle \cdot \rangle$ indicates an average over the fast time scale, that is, over a poloidal $\mathbf{E} \times \mathbf{B}$ drift time. The heating of the plasma arises because the parallel and perpendicular temperature fluctuations are unequal. Collisions cause a small phase shift in the fluctuations and to second order in r/ρ_0 there is a net heating in the plasma. This effect is very similar to rotational pumping calculated in [4]. The difference here is that the perpendicular temperature also fluctuates because the magnetic field is not constant. To lowest order in ν, $T_{\|}$ and $T_{\perp}$ fluctuate in phase and therefore this has the effect of reducing the heating rate.

Since the confinement potentials are time independent, the total energy in the plasma is conserved and the increase in thermal energy must be balanced by a corresponding decrease in the electrostatic energy. The particle flux is found by equating the increase in thermal energy to local Joule heating

$$n\frac{d}{dt}\left(\frac{1}{2}\langle T_{\|}\rangle + \langle T_{\perp}\rangle\right) = -e\frac{\partial \Phi}{\partial r}\Gamma_r, \qquad (4.14)$$

where Γ_r is the radial particle flux and n is the density. The right hand side of this equation is the Joule heating per unit volume. Equations (4.13) and (4.14) are solved for the flux and yield

$$\Gamma_r = \frac{1}{2}\nu_{\perp,\|}n(r)\frac{T}{-e\partial\Phi/\partial r}\left(\frac{r}{\rho_0}\right)^2. \qquad (4.15)$$

Note that the flux depends on magnetic field strength only through $\nu_{\perp,\|}$. This rather surprising result is due to an accidental cancellation. The net heating in each poloidal rotation is proportional to the phase shift in the temperature fluctuations which scales as $\nu_{\perp,\|}/\omega_E$. The heating rate is equal to the heating per poloidal rotation times the poloidal rotation frequency. Therefore, ω_E drops out of the calculation. In the regime of weak magnetization (i.e., $r_c >> b$, where $r_c = \bar{v}/\Omega_c$ and $b = e^2/m\bar{v}^2$), the dependence on the magnetic field strength is very weak, $\nu_{\perp,\|} \propto \ln(r_c/b)$. In the regime of strong magnetization (i.e., $r_c << b$) $\nu_{\perp,\|}$ becomes exponentially small[5, 6] and our theory predicts that Γ_r becomes exponentially small.

4.4 Kinetic Treatment

One can also understand this transport process in terms of single particle drift orbits. The frequency ordering $\Omega_c \gg \bar{\omega}_T \gg \omega_E$ insures that the cyclotron action $\mu = mv_\perp^2/2B$ and the toroidal action

$$P_\Theta = \frac{1}{2\pi} \oint dP_\Theta = \frac{1}{2\pi} \oint mv_\parallel \, dl \qquad (4.16)$$

are good adiabatic invariants. Since the confinement potentials are time independent, the energy is also conserved. The closed single particle drift orbits are determined by these three constants. As illustrated in Fig. 4.3, when a particle undergoes a collision its parallel and perpendicular velocity change and it begins to move on a new drift surface. This event constitutes the fundamental step underlying the transport.

The guiding center Hamiltonian in toroidal geometry is given by [7]

$$H = \frac{P_\Theta^2}{2m\rho^2(p_z)} + \frac{\mu B_0 \rho_0}{\rho(p_z)} + e\Phi(z, \rho(p_z)) \qquad (4.17)$$

where z and p_z are canonically conjugate, $p_z = (eB_0\rho_0/c)\ln(\rho_0/\rho)$, and ρ_0 and B_0 are constants. The first term gives the curvature drift, the second the gradient $|B|$ drift and the third term the $\mathbf{E} \times \mathbf{B}$ drift. In a pure electron plasma the $\mathbf{E} \times \mathbf{B}$ drift is the dominant drift. Equivalently, one may say that the the poloidal drift surfaces differ only slightly from the equipotential surfaces.

It is useful to introduce a canonical transformation

$$
\begin{aligned}
z &= z(p_\psi, \psi), \\
p_z &= p_z(p_\psi, \psi)
\end{aligned}
\qquad (4.18)
$$

which is chosen such that

$$\Phi(z, \rho(p_z)) = \Phi(p_\psi). \qquad (4.19)$$

The new momentum, p_ψ, is nearly constant during the evolution except for small curvature and gradient $|B|$ drifts. The new Hamiltonian is given by

$$H = \frac{P_\Theta^2}{2m\rho^2(p_\psi,\psi)} + \frac{\mu B_0 R}{\rho(p_\psi,\psi)} + e\Phi(z,\rho(p_z)). \qquad (4.20)$$

The gradient $|B|$ and curvature drifts normal to the $p_\psi = CONST$ surfaces are proportional to $\partial H/\partial \psi$.

We represent the plasma with a distribution of guiding centers

$$f = f(p_\Theta, \Theta, p_\psi, \psi, \mu, t). \qquad (4.21)$$

This distribution evolves according to the drift-kinetic Boltzmann equation

$$\frac{\partial f}{\partial t} + [f,H] = C(f), \qquad (4.22)$$

where $C(\cdot)$ is the collision operator and the Poisson bracket is given by

$$[f,H] = \frac{\partial f}{\partial \Theta}\frac{\partial H}{\partial p_\Theta} + \frac{\partial f}{\partial \psi}\frac{\partial H}{\partial p_\psi} - \frac{\partial f}{\partial p_\psi}\frac{\partial H}{\partial \psi}. \qquad (4.23)$$

In the limit $\bar{\omega}_T \gg \omega_E$, $\partial H/\partial p_\Theta$ is large and so $\partial f/\partial \Theta$ must be small. Physically this corresponds to the fact that any initially large Θ variations are rapidly mixed by the toroidal streaming along the magnetic field. The small Θ variations are uninteresting from the standpoint of cross-field transport and may be eliminated by integrating Eq. (4.22) over Θ, that is averaging over the toroidal motion. The result is

$$\frac{\partial \bar{f}}{\partial t} + \frac{\partial \bar{f}}{\partial \psi}\frac{\partial H}{\partial p_\psi} - \frac{\partial \bar{f}}{\partial p_\psi}\frac{\partial H}{\partial \psi} = C(\bar{f}) \qquad (4.24)$$

where

$$\bar{f}(p_\Theta, p_\psi, \psi, \mu, t) = \int_0^{2\pi} d\Theta f(p_\Theta, \Theta, p_\psi, \psi, \mu, t). \qquad (4.25)$$

Rewriting Eq. (4.23) as

$$\frac{\partial \bar{f}}{\partial t} + \frac{\partial}{\partial \psi}\left[\frac{\partial H}{\partial p_\psi}f\right] - \frac{\partial}{\partial p_\psi}\left[\frac{\partial H}{\partial \psi}f\right] = C(\bar{f}) \qquad (4.26)$$

and integrating over p_Θ, μ, and ψ yields a transport equation

$$\frac{\partial N(p_\psi)}{\partial t} = \frac{\partial}{\partial p_\psi} \left[\int \frac{d\psi}{2\pi} \int dp_\Theta d\mu \frac{\partial H}{\partial \psi} \bar{f} \right] \tag{4.27}$$

where

$$N(p_\psi) = \int \frac{d\psi}{2\pi} \int dp_\Theta d\mu \bar{f}. \tag{4.28}$$

The integral over the collision operator vanishes because collisions conserve the number of particles.

To obtain a transport equation accurate to second order in the small quantity, $\partial H/\partial \psi$, we must obtain $\bar{f}$ accurate to first order in $\partial H/\partial \psi$. Thus we look for a solution to Eq. (4.23) in the form

$$\bar{f} = \bar{f}_0(H, p_\psi) + \delta\bar{f}(p_\Theta, p_\psi, \psi, \mu) \tag{4.29}$$

where $\delta\bar{f}/\bar{f}_0 \sim (1/H)\,(\partial H/\partial \psi)$ and

$$\bar{f}_0 = N(p_\psi)(2\pi T/m)^{-3/2} e^{-H/T} e^{e\Phi/T}. \tag{4.30}$$

Written in velocity variables, $\bar{f}_0$ is just a maxwellian times a density distribution which coincides with the equipotential contours.

$\delta\bar{f}$ is obtained from Eq. (4.23) written to first order in $\partial H/\partial \psi$,

$$\omega_E \frac{\partial \delta\bar{f}}{\partial \psi} + \frac{1}{\rho} \frac{\partial \rho}{\partial \psi} \left[2\frac{P_\Theta^2}{2m\rho^2} + \frac{\mu B_0 R}{\rho} \right] \frac{\omega_E}{T} \bar{f}_0 = C(\bar{f}_0 + \delta\bar{f}) \tag{4.31}$$

where $\omega_E = \partial e\Phi/\partial p_\psi$ is the poloidal $\mathbf{E} \times \mathbf{B}$ drift frequency. Here we have neglected terms of order $1/N\ \partial N/\partial p_\psi$ relative to ω_E/T because they are smaller by a factor λ_D^2/r^2.

Given the frequency ordering $\nu \ll \omega_E$, this equation may be solved perturbatively in the effective collision frequency. Dropping the collision operator and integrating yields

$$\delta\bar{f}^{(0)} = -\frac{\delta\rho}{\rho} \left[2\frac{P_\Theta^2}{2m\rho^2} + \frac{\mu B_0 R}{\rho} \right] \frac{1}{T} \bar{f}_0 \tag{4.32}$$

where the superscript indicates the ordering in collisions and $\delta\rho$ is the ψ-dependent part of $\rho(p_\psi, \psi)$. To find the collisional response we insert $\delta\bar{f}^{(0)}$ into the collision operator on the right hand side of Eq. (4.30) and obtain

$$\frac{\partial}{\partial\psi}\delta\bar{f}^{(1)} = \frac{1}{\omega_E}C\left[\bar{f}_0\left(1 - \frac{\delta\rho}{\rho}\left[2\frac{P_\Theta^2}{2m\rho^2} + \frac{\mu B_0 R}{\rho}\right]\frac{1}{T}\right)\right]. \tag{4.33}$$

Substituting $\bar{f}_0 + \delta\bar{f}^{(0)} + \delta\bar{f}^{(1)}$ into the transport equation yields

$$\frac{\partial N(p_\psi)}{\partial t} = \frac{\partial}{\partial p_\psi}\left[\int\frac{d\psi}{2\pi}\int dp_\Theta d\mu\left[2\frac{P_\Theta^2}{2m\rho^2} + \frac{\mu B_0 R}{\rho}\right]\left(-\frac{1}{\rho}\frac{\partial\rho}{\partial\psi}\right)\delta\bar{f}^{(1)}\right]. \tag{4.34}$$

The collisionless terms vanish in the integral over ψ. Integrating by parts and substituting from Eq. (4.32) yields the result

$$\begin{aligned}
\frac{\partial N(p_\psi)}{\partial t} &= \frac{\partial}{\partial p_\psi}\left\{\int\frac{d\psi}{2\pi}\int dp_\Theta d\mu\left[2\frac{P_\Theta^2}{2m\rho^2} + \frac{\mu B_0 R}{\rho}\right]\left(\frac{\delta\rho}{\rho}\right)\right. \\
&\quad\left.\times\frac{1}{\omega_E}C\left[\bar{f}_0\left(1 - \frac{\delta\rho}{\rho}\left[2\frac{P_\Theta^2}{2m\rho^2} + \frac{\mu B_0 R}{\rho}\right]\frac{1}{T}\right)\right]\right\}.
\end{aligned} \tag{4.35}$$

After changing variables of integration from (p_Θ, μ) to $(v_\parallel, v_\perp)$ this may be written as

$$\begin{aligned}
\frac{\partial N(p_\psi)}{\partial t} &= \frac{\partial}{\partial p_\psi}\left\{\int\frac{d\psi}{2\pi}\int dv_\parallel d^2 v_\perp\left[mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right]\left(\frac{\delta\rho}{\rho}\right)\right. \\
&\quad\left.\times\frac{1}{\omega_E}C\left[\bar{f}_0\left(1 - \frac{\delta\rho}{\rho}\left[mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right]\frac{1}{T}\right)\right]\right\}.
\end{aligned} \tag{4.36}$$

where $\bar{f}_0 = N(p_\psi)f_M$ and f_M is a maxwellian.

We take the collision operator in the general form

$$C(f) = \int d^3 v_1 d\sigma |v_{rel}| \left(f(v_1')f(v') - f(v_1)f(v)\right) \tag{4.37}$$

where $d\sigma$ is the differential cross section and $v_{rel} = v - v_1$. Using this form we obtain

$$\begin{aligned}
\frac{\partial N(p_\psi)}{\partial t} &= -\frac{\partial}{\partial p_\psi}\left\{\int\frac{d\psi}{2\pi}\left(\frac{\delta\rho}{\rho}\right)^2\frac{N}{\omega_E T}N\int d^3 v\left(mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right)\int d^3 v_1 d\sigma |v_{rel}|\right. \\
&\quad\times\left[\left(mv_{\parallel 1}'^2 + \frac{1}{2}mv_{\perp 1}'^2 + mv_\parallel'^2 + \frac{1}{2}mv_\perp'^2\right)f_{M1}'f_M'\right. \\
&\quad\left.\left.- \left(mv_{\parallel 1}^2 + \frac{1}{2}mv_{\perp 1}^2 + mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right)f_{M1}f_M\right]\right\}.
\end{aligned} \tag{4.38}$$

Energy is conserved in the binary collisions. That is,

$$\frac{1}{2}mv_1^2 + \frac{1}{2}mv^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv'^2. \tag{4.39}$$

This fact may be used to simplify Eq. (4.37) to the form

$$\frac{\partial N(p_\psi)}{\partial t} = -\frac{\partial}{\partial p_\psi}\left\{ \langle\left(\frac{\delta\rho}{\rho}\right)^2\rangle_\psi \frac{N}{\omega_E} TN \int d^3v \left(\frac{1}{2}mv_\parallel^2\right) \int d^3v_1 d\sigma |v_{rel}| \right.$$
$$\left. \times \left[\left(\frac{1}{2}mv_\parallel'^2_1 + \frac{1}{2}mv_\parallel'^2\right) f'_{M1}f'_M - \left(\frac{1}{2}mv_{\parallel1}^2 + \frac{1}{2}mv_\parallel^2\right) f_{M1}f_M\right]\right\} \tag{4.40}$$

To evaluate the velocity integral in this equation, it is instructive to consider the collisional relaxation of an anisotropic maxwellian distribution

$$f_A(v) = \left(\frac{2\pi T_\parallel}{m}\right)^{-1/2} \left(\frac{2\pi T_\perp}{m}\right)^{-1} \exp\left[\frac{\frac{1}{2}mv_\parallel^2}{T_\parallel} - \frac{\frac{1}{2}mv_\perp^2}{T_\perp}\right]. \tag{4.41}$$

The change in the parallel temperature due to collisions is given by

$$\frac{d}{dt}\left(\frac{T_\parallel}{2}\right) = N \int d^3v \left(\frac{1}{2}mv_\parallel^2\right) \int d^3v_1\, d\sigma |v_{rel}| \left(f_A(v_1')f_A(v') - f_A(v)f_A(v_1)\right)$$
$$= \nu_{\perp,\parallel}\left(T_\perp - T_\parallel\right), \tag{4.42}$$

and may be used as a definition of the equipartition rate, $\nu_{\perp,\parallel}$. Consider the case $T_\perp = T$ and $T_\parallel = (1-\alpha)T$. Substituting this into Eq. (4.41) and taking the limit $\alpha \to 0$, one can easily show

$$N \int d^3v \left(\frac{1}{2}mv_\parallel^2\right) \int d^3v_1 d\sigma |v_{rel}| \left[\left(\frac{1}{2}mv_\parallel'^2_1 + \frac{1}{2}mv_\parallel'^2\right) f'_{M1}f'_M\right.$$
$$\left. - \left(\frac{1}{2}mv_{\parallel1}^2 + \frac{1}{2}mv_\parallel^2\right) f_{M1}f_M\right] = -T^2\nu_{\perp,\parallel}. \tag{4.43}$$

This is precisely the integral that appears in Eq. (4.39). The transport equation can now be written in the relatively simple form

$$\frac{\partial N(p_\psi)}{\partial t} = -\frac{\partial}{\partial p_\psi}\left\{\nu_{\perp,\parallel}\langle\left(\frac{\delta\rho}{\rho}\right)^2\rangle_\psi \left(\frac{T}{-\omega_E}\right) N(p_\psi)\right\}. \tag{4.44}$$

This equation describes transport in poloidal action-angle variables. To make contact with section 4.3, we consider the simple case where the plasma cross section is circular. In this case, $\psi = \theta$, $p_\psi = p_\theta = (eB/2c)r^2$, and

$$\langle \left(\frac{\delta\rho}{\rho}\right)^2 \rangle_\psi = \int \frac{d\theta}{2\pi} \frac{r^2}{\rho^2} \cos^2\theta = \frac{r^2}{2\rho^2}. \qquad (4.45)$$

The transport equation then becomes

$$\frac{\partial N(r)}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r} r \left\{ \frac{1}{2}\nu_{\perp,\parallel} \frac{T}{-e\partial\Phi/\partial r} \left(\frac{r}{\rho_0}\right)^2 \right\}. \qquad (4.46)$$

The quantity in brackets is the same radial particle flux that was found in section 4.3.
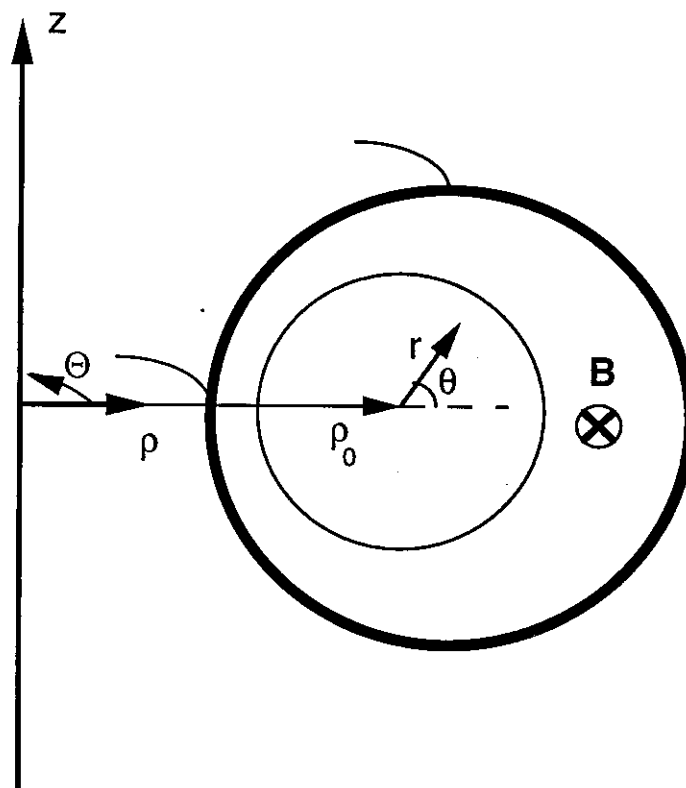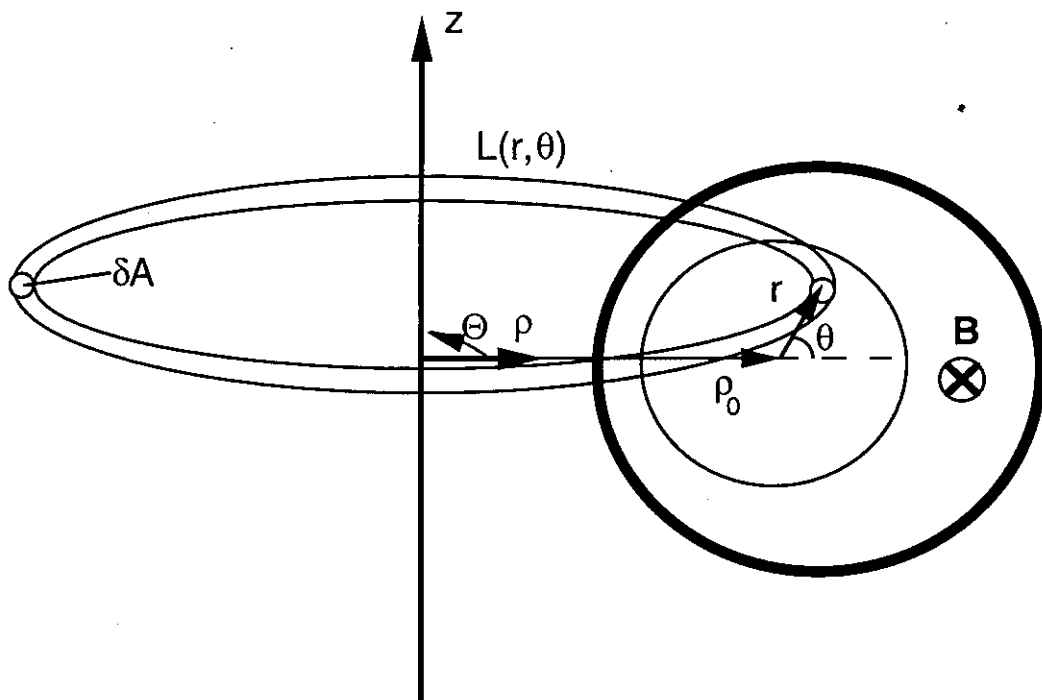
Figure 4.1: Toroidal confinement geometry.

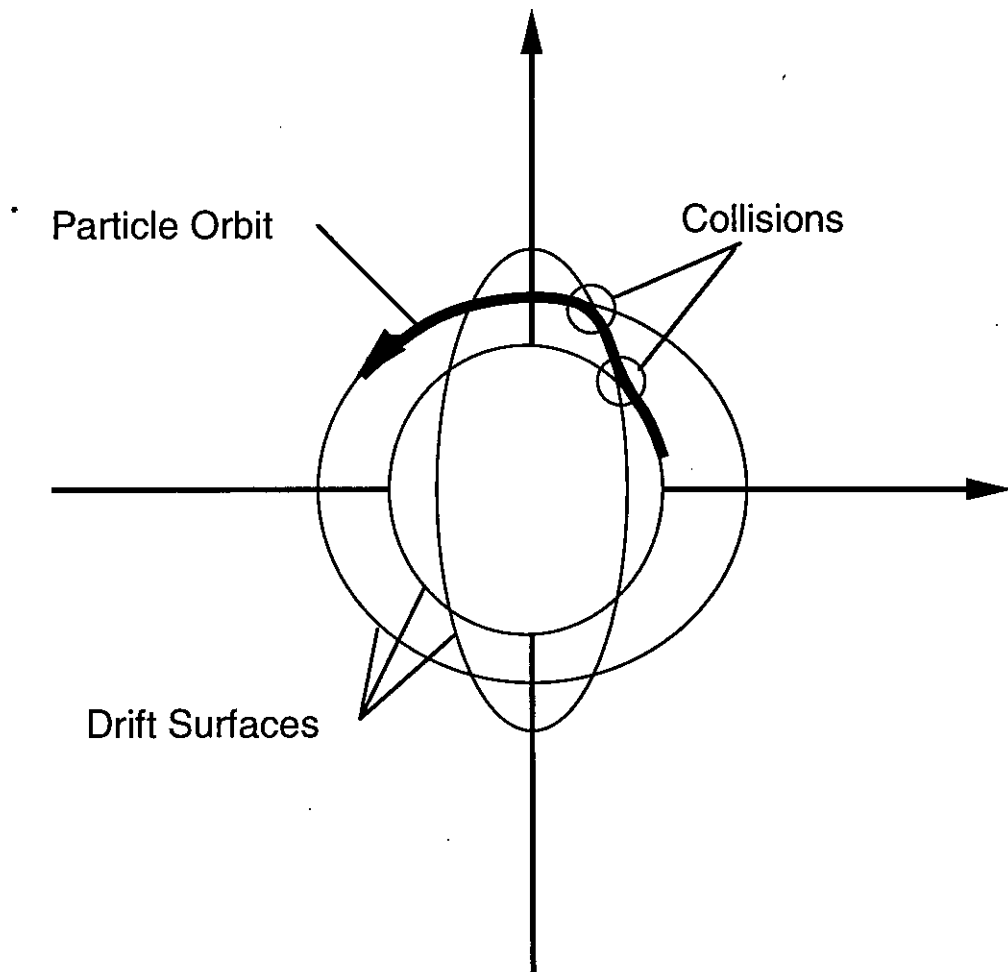**Figure 4.2:** A flux tube of plasma

Figure 4.3: An Illustration of how collisions allow a particle to change drift surfaces.

4.5 References

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