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A mechanistic interpretation of the resonant wave-particle interaction

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This paper provides a simple mechanistic interpretation of the resonant wave-particle interaction of Landau. For the simple case of a Langmuir wave in a Vlasov plasma, the non-resonant electrons satisfy an oscillator equation that is driven resonantly by the bare electric field from the resonant electrons, and in the case of wave damping, this complex driver field is of a phase to reduce the oscillation amplitude. The wave-particle resonant interaction also occurs in waves governed by 2D $E \times B$ drift dynamics, such as a diocotron wave. In this case, the bare electric field from the resonant electrons causes $E \times B$ drift motion back in the core plasma, reducing the amplitude of the wave. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4948480]

I. INTRODUCTION

This paper provides a re-interpretation of the resonant wave-particle interaction of Landau.[1] There are two halves to this interaction: first there is the influence of the wave on the resonant particles and second the influence of the resonant particles back on the wave. The mechanisms for the two halves of the interaction are usually described differently. For the first half, the mechanism is obvious; the wave electric field acts on the resonant particles and produces a perturbation in the resonant particle charge density. The mechanism for the second half of the interaction is usually described through Poisson's equation, or equivalently, a dispersion relation that follows from Poisson's equation; the perturbed charge density from the resonant particles makes a small correction to the dispersion relation, and this correction yields a small imaginary frequency shift, which is the damping decrement for the wave. In contrast, here we provide a mechanical interpretation of the second half of the interaction that is similar to the interpretation of the first half.

Consider the simple case of a Langmuir wave that is excited in a collisionless, Maxwellian plasma, with the wave phase velocity well out on the tail of the velocity distribution. We will see that the wave induced displacement of the non-resonant electrons, that is, the electrons in the main part of the Maxwellian, satisfies an oscillator equation that is driven by the bare electric field from the perturbed charge density of the resonant electrons. This field drives the oscillator resonantly, since the resonant electrons travel at the phase velocity of the wave. From this perspective, the wave damping simply results from the action of the driver field from the resonant electrons back on the oscillator.

The interpretation does not specify the perturbed charge density of the resonant particles, so the interpretation applies equally well to the cases of linear Landau damping and growth and to the case of a large amplitude wave with nonlinear, trapped particle orbits. In general, the portion of the drive field that is 90° out of phase with the oscillator produces damping or growth and the portion that is in phase produces a frequency shift.

Because Landau's analysis of the damping was rather formal and did not offer a physical interpretation,[2] other authors have provided physical interpretations. Here, we find a particularly simple interpretation by focusing on only half of the wave-particle interaction: namely, the influence of the resonant particles back on the wave.

One usually thinks of Landau resonances in connection with waves in a collisionless plasma, that is, waves that are described by Vlasov dynamics, but such resonances also occur for waves that are described by 2D $\mathbf{E} \times \mathbf{B}$ drift dynamics. A simple example is a diocotron wave that is excited on a nonneutral plasma column in a Penning-Malmberg trap.[3–6] The analysis is simplest for the case where the plasma column consists of a high-density core surrounded by a relatively low-density halo. The diocotron wave can be thought of as a surface wave that propagates azimuthally around the core. At some critical radius in the halo, the azimuthal $\mathbf{E} \times \mathbf{B}$ drift rotation velocity of the halo fluid elements matches the phase velocity of the wave potential, and the resonant interaction of the wave potential and fluid elements gives rise to Landau damping.

In the standard analysis, the linearized continuity equation for the $\mathbf{E} \times \mathbf{B}$ drift flow is combined with Poisson's equation to obtain a dispersion relation. When the resonant region is in the low density halo, the perturbed charge density of the resonant electrons makes a small correction to the dispersion relation, yielding a small imaginary frequency shift, which is the wave damping decrement. To understand more clearly how the resonant particles act back on the wave, we focus on the equation of motion for the surface ripple on the plasma core. As we will see, the bare electric field from the perturbed charge density of the resonant electrons acts back on the core, causing $\mathbf{E} \times \mathbf{B}$ drifts that reduce the amplitude of the surface ripple, that is, damp the wave. Again, we find a simple mechanistic description of the manner in which the resonant electrons act back on the wave.

II. LANGMUIR WAVE

First, we consider the case of a Langmuir wave that propagates in the x-direction, writing the perturbed electric field in the form

$$\delta E(x,t) = \delta E_k(t)\exp(ikx) + \text{c.c.}, \tag{1}$$

where c.c. stands for the complex conjugate. It is convenient to write the field as the sum

$$\delta E_k(t) = \delta E_k^{\text{non-res}}(t) + \delta E_k^{\text{res}}(t), \tag{2}$$

where

$$\delta E_k^{\text{non-res}}(t) = -\frac{4\pi e \delta n_k^{\text{non-res}}(t)}{ik}, \tag{3}$$

$$\text{and } \delta E_k^{\text{res}}(t) = -\frac{4\pi e \delta n_k^{\text{res}}(t)}{ik}, \tag{4}$$

are the fields produced by the perturbed charge densities of the non-resonant and resonant electrons, $-e\delta n_k^{\text{non-res}}(t)$ and $-e\delta n_k^{\text{res}}(t)$, following Gauss's law. The non-resonant electrons are those in the bulk of the velocity distribution, and the resonant electrons are assumed to be well out on the tail of the distribution.

For the non-resonant electrons, it is convenient to introduce a displacement $\delta x(x,t)$ defined through the relation $\partial\delta x(x,t)/\partial t = \delta v$, where $\delta v(x,t)$ is the velocity perturbation. The spatial Fourier transform of this relation is the equation $\partial\delta x_k/\partial t = \delta v_k(t)$, which allows the continuity equation to be written in the form

$$0 = \frac{\partial n_k^{\text{non-res}}}{\partial t} + ikn\delta v_k = \frac{\partial}{\partial t}\left[\delta n_k^{\text{non-res}} + ikn\delta x_k\right], \tag{5}$$

where n is the unperturbed density of the non-resonant electrons. The last form yields the solutions

$$\delta n_k^{\text{non-res}}(t) = -ikn\delta x_k(t), \tag{6}$$

$$\text{and } \delta E_k^{\text{non-res}}(t) = 4\pi en\delta x_k(t). \tag{7}$$

The linearized Euler equation for the non-resonant electrons governed by fluid theory takes the form

$$nm\frac{\partial \delta v_k}{\partial t} = -ne\delta E_k - ik\gamma T\delta n_k^{\text{non-res}}, \tag{8}$$

where m is the electron mass, T is the electron temperature, and γ has the value 3 for a one-dimensional adiabatic compression.[7] By using the definition $\partial\delta x_k/\partial t = \delta v_k$ and Eqs. (1), (6), and (7), Eq. (8) can be rewritten as the driven oscillator equation

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2 + 3k^2\bar{v}^2\right)\delta x_k(t) = -\frac{e}{m}\delta E_k^{\text{res}}(t), \tag{9}$$

where $\omega_p^2 = 4\pi ne^2/m$ is the square of the plasma frequency, $\bar{v}^2 = T/m$ is the square of the thermal velocity, and the quantity $k^2\bar{v}^2/\omega_p^2 = k^2\lambda_D^2$ is assumed to be small. Here, λ_D is the Debye length. Physically, Eq. (9) states that the non-resonant electrons moving in the wave field may be thought as an oscillator that is driven by the bare electric field from the resonant electrons. Of course, Eq. (9) also can be obtained from the coupled Vlasov and Poisson equation.

To understand the effect of the driver field on the amplitude of the oscillations, we look for a solution to Eq. (9) of the form $\delta x_k(t) = \delta\tilde{x}_k(t)\exp(-i\omega_0 t)$, where $\omega_0^2 = \omega_p^2 + 3k^2\bar{v}^2$ is the original Langmuir wave frequency squared and $\delta\tilde{x}_k(t)$ is a slowly varying complex amplitude. This solution yields the expected form for a Langmuir wave traveling in the positive x-direction. Since the resonant particles travel at the wave phase velocity ω_0/k, the driving field due to these particles can be written as $\delta E_k^{\text{res}}(t) = \delta\tilde{E}_k^{\text{res}}(t)\exp(-i\omega_0 t)$, where $\delta\tilde{E}_k^{\text{res}}(t)$ again is a slowly varying complex amplitude. Substituting these forms into Eq. (9) and neglecting $|\delta\ddot{\tilde{x}}_k/\delta\tilde{x}_k|$ compared to ω_0^2, yields the reduced equation

$$-2i\omega_0\frac{d\delta\tilde{x}_k}{dt} = -\frac{e}{m}\delta\tilde{E}_k^{\text{res}}(t). \tag{10}$$

Thus, when the ratio $\delta\tilde{E}_k^{\text{res}}/\delta\tilde{x}$ is imaginary, the driver produces damping or growth, and when the ratio is real the driver produces a frequency shift.

As noted in the Introduction, this paper focuses on only half of the wave-particle interaction, namely, the influence of the resonant particles back on the wave, and Eq. (10) solves that problem for the case of a Langmuir wave. The other half of the problem determines the influence of the wave on the resonant particles, that is, determines the perturbed charge density of the resonant particles. As a simple application of Eq. (10), we use the well-known perturbed charge density for resonant particles in a weakly damped, linear Langmuir wave[2]

$$\begin{aligned}\delta\tilde{n}_k^{\text{res}}(t) &= n\int_{\text{res}} dv\frac{e}{m}\frac{\delta\tilde{E}_k}{i(kv-\omega_0)}\frac{\partial f_0}{\partial v}\\ &\simeq n\int_{\text{res}} dv\frac{e}{m}\pi\delta(kv-\omega_0)\delta\tilde{E}_k\frac{\partial f_0}{\partial v}\\ &= \frac{\pi ne}{m}\frac{\delta\tilde{E}_k(t)}{k}\frac{\partial f_0}{\partial v}\bigg|_{\omega_0/k},\end{aligned} \tag{11}$$

where $f_0(v)$ is the unperturbed velocity distribution, and the Plemelj formula has been used in the second step.[8]

Since $\partial f_0/\partial v|_{\omega_0/k}$ is first order in the small number of resonant particles, $\delta\tilde{E}_k(t)$ need only be accurate to zero order, and we can use Eq. (7) to obtain the relation

$$\delta\tilde{E}_k(t) \simeq \delta\tilde{E}_k^{\text{non-res}}(t) = 4\pi ne\delta\tilde{x}_k(t). \tag{12}$$

Eq. (4) then yields the equation

$$-\frac{e}{m}\delta\tilde{E}_k^{\text{res}}(t) = -\pi i\frac{\omega_p^4}{k^2}\frac{\partial f_0}{\partial v}\bigg|_{\omega_0/k}\delta\tilde{x}_k(t). \tag{13}$$

Eq. (10) then implies the oscillator damping rate

$$\begin{aligned}\gamma_k &= \frac{d\delta\tilde{x}_k/dt}{\delta\tilde{x}_k(t)} = \frac{\pi}{2\omega_0}\frac{\omega_p^4}{k^2}\frac{\partial f_0}{\partial v}\bigg|_{\omega_0/k}\\ &= -\sqrt{\frac{\pi}{8}}\frac{\omega_p}{k^3\lambda_D^3}\exp\left[-\frac{1}{2k^2\lambda_D^2}\left(1+3k^2\lambda_D^2\right)\right],\end{aligned} \tag{14}$$

where the last form is the well-known form of the damping rate for a Maxwellian velocity distribution.[2]

Of course, the use of Eq. (10) is not limited to the case where the resonant particle density perturbation is determined by the linearized Vlasov equation. For a large amplitude wave where trapping of resonant particles in wave troughs is important,[9,10] Eq. (10) can still be used to determine the influence of the resonant particles back on the wave.

III. DIOCOTRON WAVE

To illustrate the wave-particle interaction that can occur in 2D $\mathbf{E}\times\mathbf{B}$ dynamics, we consider a diocotron wave that is excited on a pure electron plasma column in a Malmberg-Penning trap.[3,6,11] An analytic treatment is possible for the case where the electron column consists of a uniform density central core surrounded by a relatively low-density halo. Such a density profile often is said to be of the "top hat" form. We assume that the unperturbed density has the constant value $n(r) = n_c$ out to the radius $r = R_c$, and there drops abruptly to the much lower density $n(R_c^+) = n_h$, where the subscripts c and h refer to the core and halo, respectively. Consistent with the standard trap configuration, we assume that the electron column is immersed in a uniform, axial magnetic field $\mathbf{B} = B\hat{z}$, where (r, θ, z) is a cylindrical coordinate system with the z-axis coincident with the axis of the trap.

Since the 2D $\mathbf{E}\times\mathbf{B}$ drift flow is incompressible and since the unperturbed density profile for the core is uniform with an abrupt fall off at the surface, the diocotron wave can be characterized by specifying the ripple on the surface of the core. For a diocotron wave of azimuthal wave number m, the θ- and t-dependent radial position of the core surface can be written as

$$r_s(\theta, t) = R_c + D(t)\exp[i(m\theta - \omega_m t)] + \text{c.c.}, \tag{15}$$

where ω_m is the still-to-be-determined wave frequency and $D(t)$ is a complex wave amplitude. The slow time dependence in the complex amplitude is due to the interaction with the resonant particles.

The total time derivative of $r_s(\theta, t)$ is given by the equation

$$\begin{aligned}\frac{dr_s(\theta,t)}{dt} &= \left[\frac{\partial}{\partial t} + \omega_E(R_c)\frac{\partial}{\partial\theta}\right] r_s(\theta, t) \\ &= \{\dot{D}(t) + i[m\omega_E(R_c) - \omega_m]D(t)\} \\ &\quad \times \exp[i(m\theta - \omega_m t)] + \text{c.c.},\end{aligned} \tag{16}$$

where $\omega_E(r)$ is the $\mathbf{E}\times\mathbf{B}$ drift rotation frequency at radius r.

Since the motion of the surface is due to $\mathbf{E}\times\mathbf{B}$ drifts caused by the mode potential, we also can write the time derivative as the drift velocity

$$\frac{dr_s(\theta,t)}{dt} = -\frac{c}{BR_c}\frac{\partial\delta\phi(R_c,\theta,t)}{\partial\theta}, \tag{17}$$

where $\delta\phi = \delta\phi(r, \theta, t)$ is the mode potential.

The m-th Fourier components of the potential and the density perturbation are related by the Green's function integral[6]

$$\delta\phi_m(r,t) = 4\pi e \int_0^{R_w} 2\pi r' dr' G_m(r, r')\delta n_m(r', t), \tag{18}$$

where

$$G_m(r, r') = \frac{1}{4\pi m}\begin{cases} \dfrac{r^m}{r'^m}\left(\dfrac{r'^{2m}}{R_w^{2m}} - 1\right) & \text{for } r < r' \\ \dfrac{r'^m}{r^m}\left(\dfrac{r^{2m}}{R_w^{2m}} - 1\right) & \text{for } r' < r, \end{cases} \tag{19}$$

is the Green's function and $-e$ is the electron charge. Here, R_w is the radius of a conducting wall that bounds the confinement region, and the Green's function vanishes at $r = R_w$ in accord with the boundary condition on the wave potential.

It is convenient to write the perturbed density as the sum of a term from the non-resonant region and a term from the resonant region, $\delta n_m^{\text{non-res}}(r,t)$ and $\delta n_m^{\text{res}}(r,t)$, and to write the potential as the sum of the corresponding terms $\delta\phi_m(r,t) = \delta\phi_m^{\text{non-res}}(r,t) + \delta\phi_m^{\text{res}}(r,t)$. Because the unperturbed core density is uniform out to the core surface and because the halo density is relatively low, the dominant contribution to $\delta n^{\text{non-res}}(r,\theta,t)$ comes from the surface of the core and is given by the expression

$$\begin{aligned}\delta n^{\text{non-res}}(r,\theta,t) &= -D(t)\exp[i(m\theta - \omega_m t)]\frac{\partial n}{\partial r} + \text{c.c.} \\ &= D(t)\exp[i(m\theta - \omega_m t)] \\ &\quad \times (n_c - n_h)\delta(r - R_c) + \text{c.c.},\end{aligned} \tag{20}$$

where $\delta(r - R_c)$ is a delta function.

The Green's function integral then implies the non-resonant potential

$$\begin{aligned}\delta\phi^{\text{non-res}}(R_c,\theta,t) &= 8\pi^2 e R_c(n_c - n_h)G_m(R_c, R_c) \\ &\quad \times D(t)\exp[i(m\theta - \omega_m t)] + \text{c.c.} \\ &= -\frac{2\pi e}{m}R_c(n_c - n_h)\left(1 - \frac{R_c^{2m}}{R_w^{2m}}\right) \\ &\quad \times D(t)\exp[i(m\theta - \omega_m t)] + \text{c.c.}\end{aligned} \tag{21}$$

Combining Eqs. (16) and (17) and substituting Eq. (21) for the non-resonant potential yield the relation

$$\begin{aligned}&-\frac{c}{BR_c}\frac{\partial\delta\phi^{\text{res}}(R_c,\theta,t)}{\partial\theta} + \frac{2i\pi ec(n_c - n_h)}{B}\left(1 - \frac{R_c^{2m}}{R_w^{2m}}\right) \\ &\quad \times D(t)\exp[i(m\theta - \omega_m t)] + \text{c.c.} \\ &= \{\dot{D}(t) + i[m\omega_E(R_c) - \omega_m]D(t)\}\exp[i(m\theta - \omega_m t)] + \text{c.c.}\end{aligned} \tag{22}$$

It is instructive to examine Eq. (22) in the limit where there is no resonance, and $\delta\phi^{\text{res}}$ and $\dot{D}(t)$ are zero. The equation then implies the dispersion relation for a diocotron wave on a "top-hat" density profile

$$\omega_m - m\omega_E(R_c) = -\omega_E(R_c)\left(1 - \frac{n_h}{n_c}\right)\left(1 - \frac{R_c^{2m}}{R_w^{2m}}\right), \tag{23}$$

using that fact that $\omega_E(R_c) = 2\pi e c n_c/B$ at the surface of the core. This dispersion relation is well-known in the limit $n_h = 0$.[3,4] By using this dispersion relation, Eq. (22) reduces to the form

$$-\frac{c}{BR_c}\frac{\partial\delta\phi^{\text{res}}}{\partial\theta} = \dot{D}(t)\exp[i(m\theta - \omega_m t)] + \text{c.c.} \tag{24}$$

Thus, we obtain the rate of change of the complex wave amplitude

$$\dot{D}(t) = -\int_0^{2\pi} \frac{d\theta}{2\pi} \frac{c}{BR_c} \frac{\partial \delta\phi^{\rm res}(R_c,\theta,t)}{\partial\theta} \exp[-i(m\theta - \omega_m t)] = -\frac{imc}{BR_c}\delta\phi_m^{\rm res}(R_c,t)\exp[i\omega_m t]. \quad (25)$$

Physically, the electric field from the resonant particles acts back on the core causing $\mathbf{E}\times\mathbf{B}$ drift motion, and this motion produces a slow rate of change of the complex wave amplitude.

Since the resonant particles travel at the wave phase speed, the perturbed density $\delta n_m^{\rm res}(R_c,t)$ can be written in the form $\delta\tilde{n}_m^{\rm res}(R_c,t)\exp[-i\omega_m t]$, where $\delta\tilde{n}_m^{\rm res}(R_c,t)$ is slowly varying. Likewise, the perturbed potential $\delta\phi_m^{\rm res}(R_c,t)$ can be written in the form $\delta\tilde{\phi}_m^{\rm res}(R_c,t)e^{-i\omega_m t}$, where $\delta\tilde{\phi}_m^{\rm res}(R_c,t)$ is slowly varying. Eqs. (18) and (19) then imply the relationship

$$\delta\tilde{\phi}_m^{\rm res}(R_c,t) = \frac{e}{m}\int_{\rm res} 2\pi r'dr' \frac{R_c^m}{r'^m}\left(\frac{r'^{2m}}{R_w^{2m}} - 1\right)\delta\tilde{n}_m^{\rm res}(r',t), \quad (26)$$

and Eq. (25) reduces to the result

$$\dot{D}(t) = -\frac{icm}{BR_c}\delta\tilde{\phi}_m^{\rm res}(R_c,t). \quad (27)$$

To obtain Eq. (25), we projected out the m-th Fourier component of Eq. (24), but one may worry about other Fourier components in the potential $\delta\phi^{\rm res}(r,\theta,t)$. When the perturbed resonant particle density, $\delta n^{\rm res}(r,\theta,t)$, is obtained by linear theory, as is the case in linear Landau damping, there is only the m-th Fourier component, so there is no issue. However, when the resonant particle dynamics is nonlinear, say, when particle trapping is involved, higher harmonics typically are present in $\delta n^{\rm res}(r,\theta,t)$ and correspondingly in $\delta\phi^{\rm res}(r,\theta,t)$. Why are these harmonic terms not balanced by such terms on the right hand side of Eq. (24)? The reason is that we neglected small harmonic terms in the surface ripple of Eq. (15). These higher harmonic ripples are indeed small because the higher harmonics in $\delta\phi^{\rm res}(r,\theta,t)$ do not drive the core surface resonantly. One can easily show that the harmonic ripple amplitudes are smaller than $D(t)$ by the factor $n_h/n_c \ll 1$.

As a simple application of Eq. (27), we evaluate $\delta\tilde{\phi}_m^{\rm res}(R_c,t)$ for the case of a diocotron mode that experiences a linear Landau resonance in the low-density halo.[6,12] We work only to first order in the small quantity $n_h/n_c \ll 1$. The resonant radius then need only be calculated to zero order in n_h/n_c. To this order, the $\mathbf{E}\times\mathbf{B}$-drift rotation frequency in the halo region ($r > R_c$) is $\omega_E(r) = \omega_E(R_c)R_c^2/r^2$. Substituting this expression and dispersion relation (23) into the resonance condition $\omega_m = m\omega_E(r_{\rm res})$ and dropping first order terms in n_h/n_c yield the expression for the resonant radius

$$\frac{R_c^2}{r_{\rm res}^2} = 1 - \frac{1}{m}\left(1 - \frac{R_c^{2m}}{R_w^{2m}}\right). \quad (28)$$

Note that $r_{\rm res} > R_c$ for all m.

From the linearized continuity equation and the Plemelj formula,[8] one finds the expression for the perturbed density at the resonance

$$\delta\tilde{n}_m^{\rm res}(r,t) = \frac{mc}{Br}\frac{\partial n}{\partial r}\delta\tilde{\phi}_m^{\rm non-res}(r,t)i\pi\delta[\omega_m - m\omega_E(r)], \quad (29)$$

where $\delta\tilde{\phi}_m^{\rm non-res}(r,t)$ is the potential due to the perturbed charge density on the surface of the core. Here we ignore $\delta\tilde{\phi}_m^{\rm res}(r,t)$ set up by the fewer resonant particles, similar to the case in Section II. For $r > R_c$, this latter potential can be written as

$$\delta\tilde{\phi}_m^{\rm non-res}(r,t) = \delta\tilde{\phi}_m^{\rm non-res}(R_c,t)\frac{G_m(r,R_c)}{G_m(R_c,R_c)}, \quad (30)$$

where $\delta\tilde{\phi}_m^{\rm non-res}(R_c,t)$ is easily extracted from Eq. (21). Substituting Eqs. (29) and (30) into Eq. (26) and evaluating the Green's function with Eq. (19) yield the result

$$\delta\tilde{\phi}_m^{\rm res}(R_c,t) = \frac{(2\pi e)^2}{m}\frac{cR_c}{B}\frac{R_c^{2m}}{r_{\rm res}^{2m}}\left(1 - \frac{r_{\rm res}^{2m}}{R_w^{2m}}\right)^2\frac{n'(r_{\rm res})}{m|\omega_E'(r_{\rm res})|}i\pi D(t)n_c. \quad (31)$$

Substituting into Eq. (27) then yields the well-known damping rate[6,12]

$$\frac{\dot{D}(t)}{D(t)} = \omega_E(R_c)\frac{n'(r_{\rm res})R_c}{n_c}\frac{\pi}{2m}\left(\frac{R_c}{r_{\rm res}}\right)^{2m-3}\left(1 - \frac{r_{\rm res}^{2m}}{R_w^{2m}}\right)^2. \quad (32)$$

The case of an $m=1$ diocotron wave provides a particularly clear illustration of this mechanical approach to the wave-particle interaction.[13] First, note that the $m=1$ wave is special in that an analytic description of the wave is not limited to the case of a "top-hat" density profile, but also is possible for any monotonically decreasing density profile, $n(r)$, that vanishes at the conducting wall. For many years, it was thought that there can be no resonant wave-particle resonance for the $m=1$ wave since the resonant radius is at the wall, and the unperturbed density is zero at the wall. However, recent experiments have observed a novel algebraic damping of the $m=1$ wave when transport sweeps a low density halo of particles out from a central core to the wall.[14] The damping begins when the halo reaches the wall and is thought to be due to a nonlinear wave particle interaction in the region of the wall.

In the absence of a wave-particle interaction, the self-consistent density perturbation and wave potential for the $m=1$ wave are given by the expressions

$$\delta n^{\rm non-res}(r,\theta,t) = -\frac{\partial n}{\partial r}[D\exp[i(\theta - \omega_1 t)] + {\rm c.c.}] = -\frac{\partial n}{\partial r}A\cos(\theta - \omega_1 t - \alpha), \quad (33)$$

and

$$\delta\phi^{\rm non-res}(r,\theta,t) = -\frac{rB}{c}[-\omega_1 + \omega_E(r)]A\cos(\theta - \omega_1 t - \alpha). \quad (34)$$

Here, we have set $D = (A/2)\exp(-i\alpha)$, where A and α are real. By using the Green's function integral in Eq. (18), one can easily show that the density perturbation and potential are self-consistent, that is, substituting the density perturbation into the Green's function integral yields potential. The wave frequency is given by $\omega_1 = \omega_E(R_w)$, so the wave potential vanishes at the conducting wall.

Physically, such a density perturbation results when the plasma column is displaced off the trap axis by the amount A in the instantaneous direction $\theta = \omega_1 t + \alpha$. The displaced column produces an image in the conducting wall, and for small displacement (i.e., $A \ll R_w$) the image is well outside the wall, producing an image electric field that is nearly uniform in the region of the column. The uniform field produces a uniform $\mathbf{E}\times\mathbf{B}$ drift velocity of the column transverse to the instantaneous displacement off axis, and in turn this produces a rotation of the column around the trap axis at the mode frequency ω_1. In the wave potential, the term proportional to ω_1 is the potential due to the uniform image electric field, and the term proportional to $\omega_E(r)$ is the correction to the radial space charge potential due the shift of the column off axis.

We postulate that the non-resonant density perturbation still can be described as a displacement of the column off the trap axis even when the potential due to the resonant electrons acts back on the column. The reason for this simplification is easy to understand. The resonant particles are near the wall, so the field from these particles in the non-resonant region is a vacuum field, and the dipole component of such a field is uniform, as will be explained shortly. Thus, the field due to the resonant particles simply produces an increment to the uniform $\mathbf{E}\times\mathbf{B}$ drift motion produced by the non-resonant potential, and we will see that the increment can be accommodated simply by allowing a slow time dependence in $A(t)$ and $\alpha(t)$.

Formally, the condition that the postulate be satisfied is that continuity equation in the non-resonant region

$$\left[\frac{\partial}{\partial t} + i\omega_E(r)\right]\delta n_1^{\text{non-res}}(r,t) = \frac{ic}{Br}\left[\delta\phi_1^{\text{non-res}}(r,t) + \delta\phi_1^{\text{res}}(r,t)\right]\frac{\partial n}{\partial r}, \tag{35}$$

be satisfied when the Fourier components $\delta\phi_1^{\text{non-res}}(r,t)$ and $\delta n_1^{\text{non-res}}(r,t)$ are evaluated using the functional forms for the potential and density perturbation in Eqs. (33) and (34), allowing only that D, or equivalently A and α, are time-dependent. Substituting the Fourier components yields the equation

$$\dot{A}(t) - iA(t)\dot{\alpha}(t) = 2\dot{D}(t)e^{i\alpha(t)} = \frac{-2ic}{Br}\delta\phi_1^{\text{res}}(r,t)e^{i\omega_1 t + i\alpha(t)}. \tag{36}$$

Since the left hand side of the equation is independent of r, it is necessary that the right hand side be independent of r, or equivalently that $\delta\phi_1^{\text{res}}(r,t)$ be proportional to r, and the Green's function solution

$$\delta\phi_1^{\text{res}}(r,t) = -er\int_{\text{res}} 2\pi r' dr'\left(1 - \frac{r'^{2m}}{R_w^{2m}}\right)\delta n_1^{\text{res}}(r,t), \tag{37}$$

does imply the required proportionality. In choosing the correct form of the Green's function from Eq. (18), we used the fact that $r < r'$ in the non-resonant region. Proper choice of the time-dependence in $A(t)$ and $\alpha(t)$ then allows both the real and imaginary parts of the equation to be satisfied.

Since $\delta\phi^{\text{res}}(r,\theta,t)$ is a vacuum potential in the non-resonant region, the dipole portion of the potential can be written in the form

$$\delta\phi^{\text{res}}(r,\theta,t) = -\delta E_x^{\text{res}}(t) r\cos(\theta - \omega_1 t - \alpha) - \delta E_y^{\text{res}}(t) r\sin(\theta - \omega_1 t - \alpha), \tag{38}$$

where a rotating (x, y) coordinate system has been introduced, with the x-axis directed along the instantaneous displacement of the plasma column. The Fourier component of this expression is simply

$$\delta\phi_1^{\text{res}}(r,t) = \left[\frac{-\delta E_x^{\text{res}}(t)r}{2} + i\frac{\delta E_y^{\text{res}}(t)r}{2}\right]\exp[-i(\omega_1 t + \alpha)], \tag{39}$$

so the real and imaginary parts of Eq. (36) take the form

$$\dot{A}(t) = \frac{c\delta E_y^{\text{res}}(t)}{B}, \tag{40}$$

$$\dot{\alpha}(t)A(t) = \Delta\omega_1 A(t) = -\frac{c\delta E_x^{\text{res}}(t)}{B}. \tag{41}$$

Here, we have identified $\dot{\alpha} \equiv \Delta\omega_1$ as a frequency shift. Physically, the uniform field that is transverse to the instantaneous displacement of the column (i.e., δE_y^{res}) produces an $\mathbf{E}\times\mathbf{B}$ drift motion of the plasma column parallel to the displacement, that is a damping or growth of the wave amplitude, and the component that is parallel to the displacement (i.e., δE_x^{res}) produces an increment to the rotation velocity of the column around the trap-axis, that is, a wave frequency shift.

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